CHAPTER 8

CORE-ENVELOPE MODELS ON PSEUDO SPHEROIDAL SPACE-TIMES

1. INTRODUCTION

The exact nature of matter when its density exceeds nuclear regime is not precisely known. So assumptions of general nature become a necessity in the studies of superdense compact fluid configurations. One such assumption is distributions with two densities. Durgapal and Gehlot (1969) have obtained exact internal solutions for massive fluid spheres with two density distributions. They assumed two densities to surmount the difficulty due to the existence of infinite density and pressure at the centre in their earlier solution (Durgapal and Gehlot-1968). The models with two density distributions are generally called core-envelope models. They consist of two regions: an inner core where the equation of state is not known and an outer envelope where the equation of state is known. Hartle (1978) has discussed such models in detail. Iyer and Vishveshwara (1984) have discussed the application of core-envelope models for describing ultra compact objects. Patel and Kopper (1987) have given a model for two density distributions whose inner core is described by Vaidya -Tikekar (1982) metric for superdense star and the surrounding envelope by Durgapal Gehlot (1968)metric. The matter content in all these models is a distribution of perfect fluid in equilibrium. Discussions of Core-envelope models incorporating anisotropic matter distributions are not found in literature.

In this chapter we have investigated this possibility and discussed a core-envelope model with core consisting of an anisotropic fluid surrounded by an envelope of a perfect fluid.

2. DISTRIBUTIONS WITH ANISOTROPIC CORE

A common feature of the core-envelope models reported in literature is that their core and envelope regions contain distributions of perfect fluids in equilibrium with different density distributions. We shall consider here a core- envelope model in which the matter distribution in the core region is in the form anisotropic fluid in equilibrium with the metric of the space-time

$$ds^2 = \sqrt{1+\frac{2r^2}{R^2}}\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)^2 dt^2 - \frac{1+2\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2, \tag{8.1}$$

obtained in Chapter-7.The matter density ρ , the radial pressure P_r and anisotropy S have the respective expressions

$$8\pi\rho = \frac{3+2\frac{r^2}{R^2}}{R^2\left(1+2\frac{r^2}{R^2}\right)^2}, \tag{8.2}$$

$$8\pi P_r = \frac{C\sqrt{1+\frac{r^2}{R^2}}\left(3+4\frac{r^2}{R^2}\right)+D}{R^2\left(1+2\frac{r^2}{R^2}\right)^2\left[C\sqrt{1+\frac{r^2}{R^2}}+D\right]}, \tag{8.3}$$

$$8\pi\sqrt{3}S = \frac{\frac{r^2}{R^2}\left(2-\frac{r^2}{R^2}\right)}{R^2\left(1+2\frac{r^2}{R^2}\right)^3} \,. \tag{8.4}$$

The anisotropic fluid core is surrounded by the envelope of perfect fluid in equilibrium with the metric of its space-time as

$$ds^2 = \left\{A\sqrt{1+\frac{r^2}{R^2}} + B\left[\sqrt{1+\frac{r^2}{R^2}}\ln\left(\sqrt{2}\sqrt{1+\frac{r^2}{R^2}}+\sqrt{1+2\frac{r^2}{R^2}}\right)-\frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}^2 dt^2$$

$$-\frac{1+2\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2. \tag{8.5}$$

The fluid distribution is characterized by matter density

$$\rho = \frac{3+2\frac{r^2}{R^2}}{8\pi R^2\left(1+2\frac{r^2}{R^2}\right)^2}, \tag{8.6}$$

and fluid pressure

$$P = \frac{A\sqrt{1+\frac{r^2}{R^2}} + B\left[\sqrt{1+\frac{r^2}{R^2}}\ln\left(\sqrt{2}\sqrt{1+\frac{r^2}{R^2}}+\sqrt{1+2\frac{r^2}{R^2}}\right)+\frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]}{8\pi R^2\left(1+2\frac{r^2}{R^2}\right)\left\{A\sqrt{1+\frac{r^2}{R^2}} + B\left[\sqrt{1+\frac{r^2}{R^2}}\ln\left(\sqrt{2}\sqrt{1+\frac{r^2}{R^2}}+\sqrt{1+2\frac{r^2}{R^2}}\right)-\frac{1}{\sqrt{2}}\sqrt{1+2\frac{r^2}{R^2}}\right]\right\}} \tag{8.7}$$

where the arbitrary constants A and B are given by

$$A = \frac{\sqrt{1+\frac{a^2}{R^2}}\ln\left(\sqrt{2}\sqrt{1+\frac{a^2}{R^2}}+\sqrt{1+2\frac{a^2}{R^2}}\right)+\frac{1}{\sqrt{2}}\sqrt{1+2\frac{a^2}{R^2}}}{\sqrt{2}\left(1+\frac{2a^2}{R^2}\right)}, \tag{8.8}$$

$$B = -\frac{\sqrt{1+\frac{a^2}{R^2}}}{\sqrt{2}\left(1+2\frac{a^2}{R^2}\right)}. \tag{8.9}$$

The space-time outside the core-envelope configuration is uniquely described by the Schwarzschild exterior metric:

$$ds^2 = \left(1-\frac{2m}{r}\right)dt^2 - \left(1-\frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{8.10}$$

At the boundary $r = b$ separating the core and the envelope we stipulate the following boundary conditions:

(i) The metric coefficients g_{oo} and g_{11} of (8.1) be continuous a cross $r = b$ with the corresponding metric coefficients of (8.5)

(ii) The radial pressure P_r should be continuous across $r = b$.

(iii) The tangential pressure $P_\perp$ should be equal to the radial pressure P_r at $r = b$.

At the boundary $r = a$ of the envelope we stipulate the following boundary conditions:

(iv) The metric coefficients g_{00} and g_{11} of (8.5) be continuos across $r=a$ with Schwarzschild exterior metric (8.10).

(v) The pressure P should vanish at $r = a$.

It may be observed from expressions (8.2) and (8.6) for the matter densities in the core and envelope that the density distribution will be continuous at $r=b$ for the above choice. The boundary condition (iii) fixes up the size of the core

$$b=\sqrt{2}R. \tag{8.11}$$

The conditions (i) and (ii) lead to the equations

$$5^{\frac{1}{4}}\left(\sqrt{3}C + D\right) = \sqrt{3}A + \left[\sqrt{3}\ln\left(\sqrt{5}+\sqrt{6}\right) - \sqrt{2.5}\right]B, \tag{8.12}$$

and

$$5^{\frac{1}{4}}\left(11\sqrt{3}C + D\right) = 5\sqrt{3}A + 5\left[\sqrt{3}\ln\left(\sqrt{5}+\sqrt{6}\right) + \sqrt{2.5}\right]B. \tag{8.13}$$

The constants C and D are now determined in the form

$$C=\frac{2H(a)+\sqrt{2}\sqrt{1+2\frac{a^2}{R^2}}-\sqrt{7.5}\sqrt{1+\frac{a^2}{R^2}}}{\sqrt{2}\,5^{5/4}\left(1+\frac{2a^2}{R^2}\right)}, \tag{8.14}$$

$$D=\frac{3\sqrt{3}H(a)+\sqrt{4.5}\sqrt{1+2\frac{a^2}{R^2}}+8\sqrt{2.5}\sqrt{1+\frac{a^2}{R^2}}}{\sqrt{2}\,5^{5/4}\left(1+\frac{2a^2}{R^2}\right)}, \tag{8.15}$$

where

$$H(a)=\sqrt{1+\frac{a^2}{R^2}}\left[\ln\left(\sqrt{2}\sqrt{1+\frac{a^2}{R^2}}+\sqrt{1+2\frac{a^2}{R^2}}\right)-\ln(\sqrt{5}+\sqrt{6})\right],$$

on using (8.8) and (8.9) to eliminate A and B.

Substituting the values of C and D in the expression (8.3) for radial pressure,we get

$$8\pi\ P_r = \frac{H(a)\left[2\sqrt{1+\frac{r^2}{R^2}}\left(3+4\frac{r^2}{R^2}\right)+3\sqrt{3}\right]+G(r)+\sqrt{25}\sqrt{1+\frac{a^2}{R^2}}\left[8-\sqrt{3}\left(3+4\frac{r^2}{R^2}\right)\sqrt{1+\frac{r^2}{R^2}}\right]}{R^2\left(1+2\frac{r^2}{R^2}\right)^2\left\{H(a)\left[2\sqrt{1+\frac{r^2}{R^2}}+3\sqrt{3}\right]+F(r)+\sqrt{25}\sqrt{1+\frac{a^2}{R^2}}\left[8-\sqrt{3}\sqrt{1+\frac{r^2}{R^2}}\right]\right\}}$$

(8.17)

where

$$G(r)=\sqrt{1+\frac{a^2}{R^2}}\left[\sqrt{2}\sqrt{1+\frac{r^2}{R^2}}\left(3+4\frac{r^2}{R^2}\right)+\sqrt{4.5}\right]\ , \qquad (8.18)$$

and

$$F(r)=\sqrt{1+2\frac{a^2}{R^2}}\left[\sqrt{2}\sqrt{1+\frac{r^2}{R^2}}+\sqrt{4.5}\right]\ , \qquad (8.19)$$

Since $\frac{b^2}{R^2}<\frac{a^2}{R^2}$,we must have $H(a)>0$.

We can also show that when $2=\frac{b^2}{R^2}<\frac{a^2}{R^2}$,the expression

$$G(r)+\sqrt{2.5}\sqrt{1+\frac{a^2}{R^2}}\left[8-\sqrt{3}\left(3+4\frac{r^2}{R^2}\right)\sqrt{1+\frac{r^2}{R^2}}\right]>0,$$

at the core boundary $r=b$.

Therefore P_r will be positive throughout the core. The verification of positivity of tangentical pressure, the strong energy condition and the causality requirement in general is highly tedious as it is evident from the nature of expression involved. However we have verified the above requirements numerically for a specific model with $\lambda=0.05$ of

Table-7.1. Numerical estimates of $\bar{\rho} = 8\pi\rho$, $\bar{P}_r = 8\pi P_r$, $\bar{P}_\perp = 8\pi P_\perp$, $\bar{\rho} - \bar{P}_r - 2\bar{P}_\perp$, $\frac{dP_r}{d\rho}$

and $\frac{dP_\perp}{d\rho}$ are shown in Table -8.1.

3.DISCUSSION

The boundary radius a of the core-envelope model should be greater than b $(= \sqrt{2}\tilde{R})$. This restricts the density variation parameter $\lambda = \rho(a) / \rho(0)$ as $\lambda < 0.093$. Thus the core-envelope models discussed here will have high density variation as one moves from centre to boundary.

It is observed from the expressions of P_r and $P_\perp$ that $P_r > P_\perp$ throughout the core. They are equal at the centre and at the core boundary. Numerical estimates of P_r and $P_\perp$ show that they are decreasing functions of *r*. Further $\frac{dP_r}{d\rho} \leq 1$ and $\frac{dP_\perp}{d\rho} \leq 1$ ruling out causality violations throughout the core. Moreover they are decreasing radially outward . The strong energy condition $\rho - P_r - 2P_\perp$ is also satisfied throughout the core.

It is observed that the matter density and fluid pressure in the envelope are positive and they are decreasing radially outward. Numerical estimates indicate that $\frac{dP}{d\rho} < 1$ and $\rho - 3P > 0$.The variation of fluid pressure from centre to the boundary of the of the distribution is shown graphically in Figure-8.1.

The density and radial pressure are continuous throughout the distribution and there is no discontinuity for tangential pressure also at the core boundary. The distribution retains the same geometry for its physical-3 space, which is not the case with or Patel-Koppar models.

Thus the core-envelope model on a pseudo spheroidal space-time with core consisting of an anisotropic fluid distributions surrounded by an envelope of a perfect fluid will be a physically viable model of a compact fluid sphere.

TABLE-8.1

Numerical estimates of $\bar{\rho} = 8\pi\rho$, $\bar{P}_r = 8\pi P_r$, $\bar{P}_\perp = 8\pi P_\perp$, $\bar{\rho} - \bar{P}_r - 2\bar{P}_\perp$, $\frac{dP_r}{d\rho}$ and $\frac{dP_\perp}{d\rho}$ for a model with λ = 0.05 of Table-7.1.

C	D	$\frac{r^2}{R^2}$	$\bar{\rho}$	$\bar{p}_r$	$\bar{p}_\perp$	$\bar{\rho}-\bar{P}_r-2\bar{P}_\perp$	$\frac{dp_r}{d\rho}$	$\frac{dp_\perp}{d\rho}$
0.036	1.781	0.0	0.0745	0.0238	0.0238	0.0030	0.3938	0.5938
		0.2	0.0431	0.0119	0.0026	0.0140	0.3621	0.3673
		0.4	0.0291	0.0070	0.0043	0.0136	0.3345	0.2541
		0.6	0.0215	0.0045	0.0026	0.0118	0.3104	0.1902
		0.8	0.0169	0.0031	0.0018	0.0102	0.2891	0.1516
		1.0	0.0138	0.0023	0.0014	0.0088	0.2701	0.1273
		1.2	0.0116	0.0017	0.0011	0.0077	0.2530	0.1116
		1.4	0.0100	0.0013	0.00091	0.0068	0.2376	0.1014
		1.6	0.0087	0.0010	0.00079	0.0061	0.2236	0.0947
		1.8	0.0077	0.0008	0.00070	0.0055	0.2108	0.0905
		2.0	0.0070	0.0006	0.0006	0.0050	0.1991	0.0880

FIGURE-8.1

The variation of pressure against $x = \frac{r^2}{R^2}$ for the core-envelope model with $\lambda = 0.05$ of Table- 7.1