CHAPTER 7

ANISOTROPIC FLUID DISTRIBUTIONS ON PSEUDO SPHEROIDAL SPACE-TIMES

1. INTRODUCTION

The study of spherical distributions of matter in the form of perfect fluid with isotropic pressure is an idealization rather than reality when we consider situations wherein high densities of matter are involved. A large number of solutions of Einstein's field equations for perfect fluid distributions are available in literature (Kramer, Stephani, Mac Callam and Herlt-1980). Theoretical investigations of Ruderman (1972) and Canuto (1974) suggest that matter in superdense state with densities exceeding their values corresponding to nuclear matter regime is likely to develop anisotropy in pressure.The pressure anisotropy is possible in different cases like - the existance of a solid core, the presence of type-*P* super fluid, the complexity of interactions or the existance of an external field . These observations provide motivation for studying models of spherical distributions of matter with radial pressure differing from tangential pressure and of the superdense star models based on them.

Bowers and Liang (1974) showed that anisotropy may have non-negligible effect on maximum equilibrium mass and surface red shift of the distribution. Bayin (1982) studied solutions for anisotropic fluid spheres as well as for slowly rotating anisotropic distribution. Maharaj and Maartens (1989) have reported a model for anisotropic fluid distributions with uniform density. Gokhroo and Mehra (1994) have solved the Einstein's field equations for anisotropic distributions with variable density. Recently Patel and Mehta (1995) have discussed solutions for anisotropic distributions on a pseudo spheroidal space-time.

In this chapter,after formulating Einstein's field equations for spherical distributions of matter with anisotropic pressure on the background of pseudo spheroidal space-time in Section-2, in Section-3 we have solved them to obtain a class of their solutions different from that given by Patel and Mehta (1995). A specific solution of this class, obtained for the particular choice of the parameter K, is discussed in detail and examined its physical plausibility in Section-4.

2. THE FIELD EQUATIONS

We reproduce the metric of the space-time whose physical space obtained as t = constant is pseudo spheroidal.

$$ds^2 = e^{\nu(r)}dt^2 - \frac{1+K\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2. \tag{7.1}$$

The geometrical and physical variables as usual are related through Einstein's field equations ($c = G = 1$):

$$\Re_{ij} - \frac{1}{2}\Re g_{ij} = -8\pi T_{ij}. \tag{7.2}$$

Following Maharaj and Maartens (1989), we adopt the expression for energy-momentum tensor for anisotropic fluid distribution in equilibrium as

$$T_{ij} = (\rho + P)u_i u_j - P g_{ij} + \pi_{ij}, \quad u^i u_i \quad = 1, \tag{7.3}$$

where u_i is the four velocity field of matter and ρ, P respectively denote the energy density and isotropic pressure of matter and the anisotropic stress tensor π_{ij} is given by the expression:

$$\pi^{ij} = \sqrt{3}S\left[C^i C^j - \frac{1}{3}\left(u^i u^j - g^{ij}\right)\right] . \tag{7.4}$$

For radially symmetric anisotropic distributions of matter, the magnitude of the stress tensor is $S = S(r)$ and $C^i = \left(0, e^{-\lambda/2}, 0, 0\right)$ is a radial vector. Accordingly the energy-momentum tensor has as its non-vanishing components

$$T_0^0 = \rho \qquad , \quad T_1^1 = -\left(P + \frac{2S}{\sqrt{3}}\right) \ , \quad T_2^2 = T_3^3 = -\left(P - \frac{S}{\sqrt{3}}\right) \tag{7.5}$$

The pressure along the radial direction

$$P_r = -T_1^1 = P + \frac{2S}{\sqrt{3}}, \tag{7.6a}$$

will be different from the pressure

$$P_\perp = -T_2^2 = P - \frac{S}{\sqrt{3}} , \tag{7.6b}$$

along the tangential direction.The difference between these pressures

$$S = \frac{P_r - P_\perp}{\sqrt{3}} \tag{7.7}$$

characterizes the measure of anisotropy of the fluid.

The field equations (7.2) are equivalent to the following set of three equations:

$$8\pi\rho = \frac{3(K-1)}{R^2}\left(1 + \frac{K}{3}\frac{r^2}{R^2}\right)\left(1 + K\frac{r^2}{R^2}\right)^{-2} > 0, \tag{7.8}$$

$$8\pi P_r = \left[\left(1+\frac{r^2}{R^2}\right)\frac{\nu'}{r} - \frac{(K-1)}{R^2}\right]\left(1+K\frac{r^2}{R^2}\right)^{-1}, \tag{7.9}$$

$$8\pi\sqrt{3}S = -\left(\frac{\nu''}{2}+\frac{\nu'^2}{4}-\frac{\nu'}{2r}\right)\left(1+\frac{r^2}{R^2}\right)\left(1+K\frac{r^2}{R^2}\right)^{-1} - \frac{K-1}{R^2}\left(1+K\frac{r^2}{R^2}\right)^{-1}$$
$$+\frac{K-1}{R^2}\left(1+K\frac{r^2}{R^2}\right)^{-2} + \frac{K-1}{2R^2}\left(1+K\frac{r^2}{R^2}\right)^{-2} r\nu'$$

(7.10)

From (7.8) ,which is the law of variation of density of matter,it follows that the density gradient

$$\frac{d\rho}{dr} = -\frac{2K(K-1)r}{8\pi R^4}\left(5+K\frac{r^2}{R^2}\right)\left(1+K\frac{r^2}{R^2}\right)^{-3} \tag{7.11}$$

is negative and density is decreasing radially outward throughout.

3. SOLUTION OF FIELD EQUATIONS

Equations (7.8), (7.9) and (7.10) constitute a set of three equations in four variables ν, ρ, P_r , and S. Specific solutions of this system can be obtained when one more relation connecting them is specified a priori .This is usually provided in the form of an equation of state $P = P(\rho)$ for perfect fluids.

Making a suitable choice for the variation of anisotropy *S(r)* throughout the distribution Patel and Mehta have solved the system of equations (7.8) through (7.10). We have shown here that the system of Einsteins field equations (7.8) through (7.10) admits a class of solutions different from that of Patel and Mehta (1995) and investigated the physical plausibility and appropriateness of this new class of solutions to describe interiors of superdense stars.

We adopt the usual independent variable z introduced earlier as

$$z = \sqrt{1+\frac{r^2}{R^2}}\ , \tag{7.12}$$

and the new dependent variable

$$\Psi = \frac{e^{\nu/2}}{\left(1-K+Kz^2\right)^{1/4}}\ , \tag{7.13}$$

in terms of which the equation (7.10) reduces to

$$\frac{d^2\Psi}{dz^2}+\left[\frac{2K(2K-1)\left(1-K+Kz^2\right)-5K^2z^2}{4\left(1-K+Kz^2\right)^2}+\frac{8\sqrt{3}\pi R^2 S\left(1-K+Kz^2\right)}{z^2-1}\right]\Psi = 0\ , \tag{7.14}$$

an ordinary differential equation in which the first order derivative term is absent.

We solve the equation (7.14) under the simplifying assumption

$$8\pi\sqrt{3}S = -\frac{\left(z^2-1\right)\left[2K(2K-1)\left(1-K+Kz^2\right)-5K^2z^2\right]}{4R^2\left(1-K+Kz^2\right)^3}, \tag{7.15}$$

which facilitates its integration and leads to a solution which is easily surveyable. Equation (7.14) admits

$$\Psi = Cz + D, \tag{7.16}$$

as its general solution, where C and D are arbitrary constants of integration. Subsequently it leads to

$$e^{\nu/2} = \left(1-K+Kz^2\right)^{1/4}(Cz+D), \tag{7.17}$$

with $S(r)$ as specified in (7.15) as solution of Einstein's equation (7.10).

The space-time metric of this solution has the explicit form

$$ds^2 = \sqrt{1+K\frac{r^2}{R^2}}\left(C\sqrt{1+\frac{r^2}{R^2}}+D\right)^2 dt^2 - \frac{1+K\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2$$

(7.18)

The radial pressure P_r and the anisotropy $S(r)$ are found to have the following explicit expressions

$$8\pi P_r = \frac{C\sqrt{1+\frac{r^2}{R^2}}\left[3+2K\frac{r^2}{R^2}+K(2-K)\frac{r^2}{R^2}\right]+D\left[1+K(2-K)\frac{r^2}{R^2}\right]}{R^2\left(1+K\frac{r^2}{R^2}\right)^2\left[C\sqrt{1+\frac{r^2}{R^2}}+D\right]}$$

(7.19)

$$8\pi\sqrt{3}S = -\frac{\frac{r^2}{R^2}\left[2K(2K-1)\left(1+K\frac{r^2}{R^2}\right)-5K^2\left(1+\frac{r^2}{R^2}\right)\right]}{4R^2\left(1+K\frac{r^2}{R^2}\right)^3} \quad (7.20)$$

Since we have not assumed any equation of state for matter distribution,the fulfilment of the physical requirements is to be examined carefully.

A physically plausible solution is expected to fulfil the following requirements throughout its region of validity.

(i) The matter density and fluid presssure should be positive throughout .

i.e. $\rho > 0$, $P_r \geq 0$, $P_\perp \geq 0$ (7.21a)

(ii) The matter density and radial pressure must be decreasing radially outward.

i.e. $\frac{d\rho}{dr} < 0$, $\frac{dp_r}{dr} < 0$ (7.21b)

(iii) The matter distribution must comply with the strong energy condition.

$$i.e. \quad \rho - P_r - 2P_\perp \geq 0 \tag{7.21c}$$

(iv) It must satisfy the causality requirements.

$$i.e. \quad \frac{dP_r}{d\rho} \leq 1, \quad \frac{dP_\perp}{d\rho} \leq 1 \tag{7.21d}$$

(v) If metric (7.18) is to describe space-time inside an anisotropic distribution of finite extent, it should further continuously match with the Schwarzschild exterior metric

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \tag{7.22}$$

across the boundary $r=a$ where $P_r(a) = 0$.

The appropriate boundary conditions ensuring the continuity of metric coefficients of (7.18) and (7.22) read

$$e^{\nu(a)} = \frac{1 + \frac{a^2}{R^2}}{1 + K\frac{a^2}{R^2}} = 1 - \frac{2m}{a}. \tag{7.23}$$

These conditions together with the continuity of radial pressure across the boundary

i.e. $P_r(a)=0$

$$C\sqrt{1 + \frac{a^2}{R^2}}\left[3 + 2K\frac{a^2}{R^2} + K(2-K)\frac{a^2}{R^2}\right] + D\left[1 + K(2-K)\frac{a^2}{R^2}\right] = 0 \quad , \tag{7.24}$$

determine the constants m, C and D as

$$m = \frac{(K-1)a^3}{2R^2\left(1 + K\frac{a^2}{R^2}\right)}, \tag{7.25}$$

$$C = -\frac{1+K(2-K)\dfrac{a^2}{R^2}}{2\left(1+K\dfrac{a^2}{R^2}\right)^{7/4}}, \tag{7.26}$$

$$D = \frac{\sqrt{1+\dfrac{a^2}{R^2}}\left[3+K(4-K)\dfrac{a^2}{R^2}\right]}{2\left(1+K\dfrac{a^2}{R^2}\right)^{7/4}} . \tag{7.27}$$

In the next section we discuss some particular easily surveyable models for the anisotropic distribution, based on the above solution.

4. PARTICULAR CASE K=2.

We shall now discuss in detail an anisotropic fluid sphere model based on the above class of solutions for the particular choice $K=2$. The purpose of setting $K=2$, is twofold

(i) In chapter-2, we have obtained a physically plausible metric describing the pseudo spheroidal space-time of a spherical distribution of perfect fluid at rest with $K=2$. Accordingly this case will provide an anisotropic counterpart of those solutions.

(ii) The expressions for radial and tangential pressures become relatively simple when $K=2$ rendering the corresponding model as easily surveyable.

The expressions for $\rho, P_r, S, P_\perp, C$ and D for $K=2$, are respectively given below:

$$8\pi\rho = \frac{3+2\dfrac{r^2}{R^2}}{R^2\left(1+2\dfrac{r^2}{R^2}\right)^2}, \tag{7.28}$$

$$8\pi P_r = \frac{\sqrt{1+\frac{a^2}{R^2}}\left(3+4\frac{a^2}{R^2}\right)-\sqrt{1+\frac{r^2}{R^2}}\left(3+4\frac{r^2}{R^2}\right)}{R^2\left(1+2\frac{r^2}{R^2}\right)^2\left[\sqrt{1+\frac{a^2}{R^2}}\left(3+4\frac{a^2}{R^2}\right)-\sqrt{1+\frac{r^2}{R^2}}\right]}, \tag{7.29}$$

$$8\pi\sqrt{3}S = \frac{\frac{r^2}{R^2}\left(2-\frac{r^2}{R^2}\right)}{R^2\left(1+2\frac{r^2}{R^2}\right)^3}, \tag{7.30}$$

$$8\pi P_\perp = \frac{\sqrt{1+\frac{a^2}{R^2}}\left(3+4\frac{a^2}{R^2}\right)-\sqrt{1+\frac{r^2}{R^2}}\left(3+4\frac{r^2}{R^2}\right)}{R^2\left(1+2\frac{r^2}{R^2}\right)^2\left[\sqrt{1+\frac{a^2}{R^2}}\left(3+4\frac{a^2}{R^2}\right)-\sqrt{1+\frac{r^2}{R^2}}\right]} - \frac{\frac{r^2}{R^2}\left(2-\frac{r^2}{R^2}\right)}{R^2\left(1+2\frac{r^2}{R^2}\right)^3}, \tag{7.31}$$

$$C = -\frac{1}{2\left(1+2\frac{a^2}{R^2}\right)^{7/4}}, \tag{7.32}$$

$$D = \frac{\sqrt{1+\frac{a^2}{R^2}}\left(3+4\frac{a^2}{R^2}\right)}{2\left(1+2\frac{a^2}{R^2}\right)^{7/4}} \quad . \tag{7.33}$$

It is observed that the radial pressure P_r is positive throughout the distribution. The tangential pressure $P_\perp$ also remains non-negative throughout if $a \geq \sqrt{2}R$. This can be seen from the following discussion.

Let $x = \frac{r^2}{R^2}$ and $2+b = \frac{a^2}{R^2}$ where $b \geq 0$. Then expression (7.31) for $P_\perp$

becomes

$$8\pi P_\perp = \frac{\sqrt{3+b}(11+4b)-\sqrt{1+x}(3+4x)}{R^2(1+2x)^2\left[\sqrt{3+b}(11+4b)-\sqrt{1+x}\right]} - \frac{x(2-x)}{R^2(1+2x)^3}$$

$$= \frac{\sqrt{3+b}(11+4b)\left(1+x^2\right)-\sqrt{1+x}\left(3+8x+9x^2\right)}{R^2(1+2x)^3\left[\sqrt{3+b}(11+4b)-\sqrt{1+x}\right]}.$$

The non negativity of $P_\perp$ is ensured if

$$\sqrt{3+b}(11+4b)\left(1+x^2\right)-\sqrt{1+x}\left(3+8x+9x^2\right)\geq 0.$$

We note that $\sqrt{3+b}\geq\sqrt{1+x}$ and $b\geq 0$ and find that

$$11\left(1+x^2\right)-\left(3+8x+9x^2\right)\geq 0,$$

implying $P_\perp \geq 0$.

After a lengthy but straightforward computation one finds that

$$8\pi\frac{dP_r}{dr} = -\frac{\frac{r}{R^2}\left\{\left(1+2\frac{r^2}{R^2}\right)\left(11+12\frac{r^2}{R^2}\right)F(r)+G(r).H(r)\right\}}{R^2\sqrt{1+\frac{r^2}{R^2}}\left(1+2\frac{r^2}{R^2}\right)^3[F(r)]^2}, \tag{7.34}$$

where

$$F(r) = \sqrt{1+\frac{a^2}{R^2}}\left(3+4\frac{a^2}{R^2}\right)-\sqrt{1+\frac{r^2}{R^2}}, \tag{7.35}$$

$$G(r) = \sqrt{1+\frac{a^2}{R^2}}\left(3+4\frac{a^2}{R^2}\right)-\sqrt{1+\frac{r^2}{R^2}}\left(3+4\frac{r^2}{R^2}\right), \tag{7.36}$$

and

$$H(r) = \sqrt{1+\frac{r^2}{R^2}}\sqrt{1+\frac{a^2}{R^2}}\left(24+32\frac{a^2}{R^2}\right)-\left(9+10\frac{r^2}{R^2}\right). \tag{7.37}$$

In a perfect fluid distribution, $\sqrt{\frac{dP}{d\rho}}$ represents the speed of sound. Any physically plausible distribution of matter should comply with the requirement $\frac{dP}{d\rho} < 1$ which ensures the fulfilment of causality condition. So in anisotropic fluid distribution also we expect that causality condition should not be violated. Therefore we expect that $\frac{dP_r}{d\rho} < 1$ and $\frac{dP_\perp}{d\rho} < 1$, throughout the distribution. Owing to the complexity of expressions $\frac{dP_r}{d\rho}$ and $\frac{dP_\perp}{d\rho}$ it is difficult to verify the fulfilment of these conditions throughout the distribution. However we found that these conditions are satisfied at the centre and at the boundary of the distribution.

From the expression

$$\rho - P_r - 2P_\perp = \frac{2\frac{r^2}{R^2}\left(3+4\frac{a^2}{R^2}\right)\left(3+\frac{r^2}{R^2}\right)\sqrt{1+\frac{a^2}{R^2}} + 2\sqrt{1+\frac{r^2}{R^2}}\left[3+9\frac{r^2}{R^2}+11\left(\frac{r^2}{R^2}\right)^2\right]}{8\pi R^2\left(1+2\frac{r^2}{R^2}\right)^3\left[\left(3+4\frac{a^2}{R^2}\right)\sqrt{1+\frac{a^2}{R^2}} - \sqrt{1+\frac{r^2}{R^2}}\right]},$$

(7.38)

it follows that $\rho - P_r - 2P_\perp > 0$ implying that strong energy condition is satisfied throughout the distribution.

The scheme for the computation of mass and size of the fluid distribution discussed in Chapter-2, can be used in the case of anisotropic distribution also. Following this scheme we adopt $\rho(a) = 2\times 10^{14}\, gm/cm^3$ and introduce density variation parameter

$$\frac{\rho(a)}{\rho(0)} = \lambda. \qquad (7.39)$$

Then

$$\frac{a^2}{R^2} = \frac{1-6\lambda+\sqrt{24\lambda+1}}{6\lambda K}. \tag{7.40}$$

The condition $\frac{a^2}{R^2} \geq 2$ which ensures that $P_\perp \geq 0$ subsequently restricts

$$\lambda \leq 0.093. \tag{7.41}$$

This indicates that the introduction of anisotropy results in a high degree of density variation as one moves from centre to boundary. The perfect fluid models with pseudo spheroidal geometry for its physical space do not admit high density variation. We have given in Table - 7.1 the results of computation of mass, size and other relevant quantities for the anisotropic fluid models with $K=2$. These estimates indicate that the models with $\lambda \leq 0.093$ may be suitable to describe the interiors of superdense spherical distributions of matter like neutron stars.

For fluid spheres with $K=2$ and $a = \sqrt{2}R$, both P_r and $P_\perp$ are decreasing functions of r and $P_r \geq P_\perp$ throughout, the equality holds at the centre. A noteworthy feature of this model is that the tangentical pressure $P_\perp$ also vanishes at the boundary of the distribution. The anisotropic fluid sphere of this model will contain matter with total mass $m=1.663M_0$ at a boundary radius of *12.263* kms when the matter density on its boundary is 2×10^{14} gms/cm^3 and $\frac{\rho(a)}{\rho(0)} = 0.093$.

Table - 7.1

Masses and radii of superdense star models corresponding to K=2, $\rho(a) = 2 \times 10^{14} gm / cm^3$ and $\lambda \leq 0.093$.

λ	*R(km)*	*a(km)*	*m(km)*	*C*	*D*
0.093	8.670	12.263	2.453	-0.120	2.285
0.090	8.516	12.257	2.469	-0.119	2.359
0.080	8.028	12.233	2.516	-0.110	2.413
0.070	7.510	12.201	2.564	-0.100	2.591
0.060	6.953	12.159	2.613	-0.089	2.755
0.050	6.347	12.107	2.661	-0.078	2.976