# CHAPTER 6

# EINSTEIN CLUSTERS ON PSEUDO SPHEROIDAL SPACE-TIMES

## 1. INTRODUCTION

According to Newtonian theory of gravitation, a spherically symmetric distribution of incoherent matter (dust) at rest can not maintain itself in equilibrium. The system would collapse under its own gravitation due to the absence of radial force to counter the collapse. Einstein investigated the gravitational field inside the system of particles moving in randomly oriented, non-intersecting, circular orbits under their own gravitation. Such systems of particles are called Einstein clusters. In such systems equilibrium is maintained even though radial outward force is absent. Einstein showed that the space-time of an Einstein cluster is static and spherically symmetric.

Gilbert (1954) and Hogan (1973) derived the expression for energy-momentum tensor in curvature coordinates for such systems. Einstein clusters is a special type of anisotropic distribution of fluid for which the radial stress is zero and non-vanishing tangential stress maintain the equilibrium. Florides (1974) gave a general scheme for constructing solutions of Einstein's field equations for Einstein clusters. Following Florides scheme Patel (1984) showed that the static spheroidal space-time of Vaidya-Tikekar type is appropriate to describe the space-time of Einstein clusters. Subsequently the pseudo spheroidal space-times which are useful in describing the equilibrium distributions of perfect fluid, charged perfect fluid, charged dust, can be expected to represent Einstein clusters also. We have investigated this possibility in this chapter.

## 2. FLORIDES METHOD

We discuss Florides method in brief and then use it to describe Einstein clusters on pseudo spheroidal space-time. Since a spherically symmetric distribution of incoherent matter can not keep equilibrium by itself, Florides assumed that Einstein's Field equations

$$\Re_i^j - \frac{1}{2}\Re\delta_i^j = -8\pi T_i^j, \tag{6.1}$$

with $T_0^0$ the only non-vanishing component of the energy momentum tensor, can not admit a spherically symmetric solution. He obtained the solution for Einstein clusters while attempting to answer the question why equation (6.1) cannot admit a solution under the above assumption.

Assuming that the space-time inside Einstein cluster has the explicit form

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2. \tag{6.2}$$

Florides began the search for a solution of Einstein field equations such that

(i) $T_0^0 = \rho(r)$, the matter density and the metric coefficients

(ii) $e^{\lambda}$, $e^{\lambda}$ and $\nu'$ are continuous across the boundary $r = a$ of the distribution. The space-time outside the sphere is described by Schwarzschild exterior metric.

Einsteins field equations for the metric (6.2) lead to the following system of three equations.

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + 8\pi T_1^1 = 0, \tag{6.3}$$

$$\frac{1}{2}e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\lambda'\nu'}{2}\right) + 8\pi T_2^2 = 0, \tag{6.4}$$

$$e^{-\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)-\frac{1}{r^2}+8\pi T_0^0=0, \tag{6.5}$$

with $T_1^1 = T_2^2 = T_3^3 = 0$ and $T_0^0 = \rho$.

Equation (6.5) gives $e^{-\lambda}$ as

$$e^{-\lambda}=1-\frac{2M}{r}\ , \tag{6.6}$$

where

$$M=\int_0^r 4\pi\rho r^2 dr\ . \tag{6.7}$$

The continuity of $g_{11}$ across the boundary $r=a$ implies

$$\left(1-\frac{2m}{r}\right)_{r=a}=\left(1-\frac{2M}{r}\right)_{r=a},$$

where $m=M(a)$, the gravitational mass of the spherical distribution of radius $a$.

Setting $T_1^1=0$ in (6.3) and integrating for $\nu$ we get

$$\nu=\int_a^r \frac{2M}{r^2}\left[1-\frac{2M}{r}\right]^{-1} dr+B, \tag{6.8}$$

where $B$ is the arbitrary constant of integration.

The continuity of $g_{00}$ across $r=a$ gives

$$e^{\nu(a)}=1-\frac{2m}{a}. \tag{6.9}$$

Subsequently (6.8) and (6.9) determine the arbitrary constant $B$ as

$$B=\ln\left(1-\frac{2m}{a}\right)\ . \tag{6.10}$$

Accordingly $\nu(r)$ has the explicit expression

$$\nu(r) = \int_a^r \frac{2M}{r^2}\left[1 - \frac{2M}{r}\right]^{-1} dr + \ln\left(1 - \frac{2m}{a}\right). \tag{6.11}$$

Substitution for $\lambda'$, $\nu'$ and $\nu''$ in (6.4) leads to

$$\frac{8\pi\rho M}{2r}\left(1 - \frac{2M}{r}\right)^{-1} + 8\pi T_2^2 = 0 \;. \tag{6.12}$$

Thus the relativistic field equation (6.4) is not satisfied for the above values of $\nu$ and $\lambda$. So the field equations do not admit any solution for Einstein clusters complying with stress structure

$$T_0^0 = \rho,\;\; T_1^1 = T_2^2 = T_3^3 = 0 \quad \text{and} \quad T_i^j = 0 \;\; \text{For } i \neq j. \tag{6.13}$$

Newtonian theory is known to permit the existence of Einstein clusters in which equilibrium is maintained even in the absence of radial repulsive force. Accordingly such configurations should be permissible in the formalism of general relativity. This is possible only if the energy-momentum tensor for their matter content has the form

$$T_0^0 = \rho, \tag{6.14}$$

$$T_1^1 = 0\;,\;\; T_2^2 = T_3^3 = -\frac{\rho M}{2r}\left(1 - \frac{2M}{r}\right)^{-1}. \tag{6.15}$$

With this choice for the components of energy-momentum tensor the field equations (6.3) through (6.5) are satisfied.

Thus the metric

$$ds^2 = \left(1 - \frac{2m}{a}\right)\exp\left\{\int_a^r \frac{2M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1}\right\}dt^2 - \frac{dr^2}{\left(1 - \dfrac{2M}{r}\right)} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 ,$$

(6.16)

is a solution of Einsteins field equations with non-vanishing components of energy-momentum tensor given by (6.14) and (6.15).

Gilbert (1954) and Hogan (1973) have shown that if

$$ds^2 = e^{\beta(r)} dt^2 - e^{\alpha(r)}dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 , \tag{6.17}$$

represents the metric inside Einstein cluster, then the only non-vanishing components of the energy-momentum tensor are

$$\tilde{T}_0^0 = \tilde{\rho}\left(1 - \frac{r\beta'}{2}\right)^{-1}, \tag{6.18}$$

$$\tilde{T}_2^2 = \tilde{T}_3^3 = -\frac{1}{4}\tilde{\rho} r\beta'\left(1 - \frac{r\beta'}{2}\right)^{-1}, \tag{6.19}$$

where $\tilde{\rho}$ represents the proper matter density of Einstein cluster and the quantities with tilde refer to Einstein cluster.

Identifying $\alpha = \lambda$ and $\beta = \nu$ and compairing metrics (6.2) and (6.17) , we obtain the exact relationship between the energy tensors given by equations (6.14) , (6.15) and (6.18), (6.19) as

$$T_o^o = \frac{\tilde{\rho}\left(1 - \frac{2M}{r}\right)}{1 - \frac{3M}{r}}, \tag{6.20}$$

$$\tilde{T}_2^2 = \tilde{T}_3^3 = -\frac{\tilde{\rho}}{2r}\frac{M}{\left(1 - \frac{3M}{r}\right)}. \tag{6.21}$$

Comparing (6.14), (6.15) and (6.20), (6.21), we see that the energy-momentum tensors $\tilde{T}_i^j$ and $T_i^j$ are identical provided

$$\tilde{\rho} = \frac{\rho\left(1 - \frac{3M}{r}\right)}{1 - \frac{2M}{r}}. \tag{6.22}$$

In the next section we investigate the appropriateness of pseudo spheroidal space time for describing space-time of Einstein clusters.

## 3. A SOLUTION FOR EINSTEIN CLUSTERS.

We shall use Florides' method to examine the suitability of the pseudo spheroidal space-time metric

$$ds^2 = e^{\nu(r)}dt^2 - \frac{1 + K\frac{r^2}{R^2}}{1 + \frac{r^2}{R^2}}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta^2 d\phi^2, \tag{6.23}$$

for describing space-time of Einstein clusters.

We reproduce the component $T_0^0$ of the energy-momentum tensor for the space-time metric (6.23)

$$8\pi T_0^0 = 8\pi\rho = \frac{3(K-1)\left(1 + \frac{K}{3}\frac{r^2}{R^2}\right)}{R^2\left(1 + K\frac{r^2}{R^2}\right)^2}. \tag{6.24}$$

Identifying the space-time metric (6.23) with the metric (6.16) suggested by Florides as the metric of an Einstein cluster, we find

$$\frac{2M}{r}=\frac{(K-1)\frac{r^2}{R^2}}{1+K\frac{r^2}{R^2}}, \tag{6.25}$$

which demonstrate the dependence of the mass function $M(r)$ of the Einstein cluster with the parameter $K$ of the pseudo spheroidal space-time.

Consequently

$$\int_a^r \frac{2M}{r^2}\left(1-\frac{2M}{r}\right)^{-1} dr=\frac{K-1}{2}\ln\left[\frac{1+\frac{r^2}{R^2}}{1+\frac{a^2}{R^2}}\right]. \tag{6.26}$$

The continuity of the metric coefficient $e^{\lambda}$ of (6.23) across the boundary $r=a$ of the Einstein cluster implies

$$1-\frac{2m}{a}=\frac{1+\frac{a^2}{R^2}}{1+K\frac{a^2}{R^2}}. \tag{6.27}$$

Then equation (6.11) determines

$$e^{\nu(r)}=\left(1-\frac{2m}{a}\right)\exp\left\{\int_a^r \frac{2M}{r^2}\left(1-\frac{2M}{r}\right)^{-1} dr\right\},$$

$$=\frac{\left(1+\frac{a^2}{R^2}\right)^{\frac{3-K}{2}}\left(1+\frac{r^2}{R^2}\right)^{\frac{K-1}{2}}}{\left(1+K\frac{a^2}{R^2}\right)}. \tag{6.28}$$

The explicit form of the space-time metric of the Einstein's cluster in this set up is

$$ds^2 = \frac{\left(1+\frac{a^2}{R^2}\right)^{\frac{3-K}{2}}\left(1+\frac{r^2}{R^2}\right)^{\frac{K-1}{2}}}{\left(1+K\frac{a^2}{R^2}\right)}dt^2 - \frac{1+K\frac{r^2}{R^2}}{1+\frac{r^2}{R^2}}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2. \tag{6.29}$$

The non-vanishing components of energy momentum tensor corresponding to (6.29) are obtained using (6.20) and (6.21)

$$8\pi\tilde{T}_0^0 = \tilde{\rho}\frac{1+\frac{r^2}{R^2}}{1+\left(\frac{3-K}{2}\right)\frac{r^2}{R^2}}, \tag{6.30}$$

$$8\pi\tilde{T}_2^2 = 8\pi\tilde{T}_3^3 = -\frac{\tilde{\rho}(K-1)r^2}{4R^2\left(1+\frac{r^2}{R^2}\right)}. \tag{6.31}$$

The proper density of Einstein cluster follows from (6.22) and (6.24)

$$\tilde{\rho} = \frac{(K-1)\left(3+K\frac{r^2}{R^2}\right)\left[1-\left(\frac{K-3}{2}\right)\frac{r^2}{R^2}\right]}{8\pi R^2\left(1+K\frac{r^2}{R^2}\right)^2\left(1+\frac{r^2}{R^2}\right)}. \tag{6.32}$$

The condition $\tilde{\rho} > 0$ imposes the restriction

$$\frac{K-3}{2}\cdot\frac{r^2}{R^2} < 1.$$

When $K > 3$ the size of the configuration $a$ is restricted by

$$a < \sqrt{\frac{2}{K-3}}R \; . \tag{6.33}$$

The total gravitational mass of the configuration follows from (6.27) as

$$m = \frac{(K-1)a^3}{2R^2\left(1+K\dfrac{a^2}{R^2}\right)} \tag{6.34}$$

The condition (6.33), when used in (6.34) implies

$$3m<a \tag{6.35}$$

The orbiting period $\Delta\tau$ measured by an observer travelling with the particle in circular orbits in Schwarzschild exterior field is given by

$$\Delta\tau = 2\pi\left[\frac{r^3}{mc^2}\left(1-\frac{3m}{r}\right)\right]^{\frac{1}{2}} \tag{6.36}$$

where $m = \dfrac{GM}{c^2}$, where $M$ is the mass of the body about which the particle orbits.

$\Delta\tau > o$ imposes the restriction $r > 3m$ on the size of the orbit of the particle. The restriction (6.35) is in agreement with this condition. Thus the pseudo spheroidal space-times are useful to describe the gravitational field of Einstein clusters.