Chapter 5

Stability of Nonlinear Fractional Partial Differential Equations

“The tool which serve as intermediary between theory and practice, between thought and observation, is mathematics, it is mathematics which builds the linking bridges and gives the ever more reliable forms. From this it has come about that our entire contemporary culture, in as much as it is based the intellectual penetration and the exploitation of nature, has its foundations in mathematics.”

DAVID HILBERT.

The content of this chapter is accepted in the following paper.
i) Stability of Nonlinear Fractional Diffusion Equation, Int.J.Appl.Pure Math.(Accepted)

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5.1 Introduction

The weighted average finite difference scheme is more general scheme in the study of nonlinear partial differential equations. Explicit, implicit and Crank-Nicolson finite difference schemes are the particular cases of weighted average finite difference scheme. The highly nonlinear partial differential equations are difficult to solve not only analytically but also numerically. In such cases generally linearization method is used. Ritchymer [88] developed the Ritchymer Linearization method for particular type of nonlinear diffusion equation (5.2.1) (for $\alpha = 1$). Also it is shown that the method is conditionally stable. Using finite difference schemes several researchers have obtained the solution of fractional diffusion equations. Time fractional diffusion equation studied numerically by number of authors such as Zhung and Liu [123], developed implicit finite difference approximation method for Dirichlet initial boundary value problem, while Liu et al. [67] obtained the solution of time fractional diffusion equation by using explicit finite difference method and discussed the stability and convergence of the scheme. However numerical methods for nonlinear fractional partial differential equations are limited to date. Yang et al. [113] solved the time space fractional Fokker-Plank equation with nonlinear source term as well as Zhang and Liu [121] considered Riesz space fractional diffusion equation with nonlinear source term.
We organize the chapter as follows. In section 5.2 we study numerically the first initial boundary value problem for nonlinear time fractional diffusion equation and develop the more general weighted average finite difference scheme. In section 5.3 we prove that the method is conditionally stable. In the last section a suitable numerical example is illustrated. The graphical representation of solution is also given using MATLAB software.

5.2 Nonlinear Fractional Diffusion Equation

In this section, we study numerically the first initial boundary value problem (IBVP) for nonlinear time fractional diffusion equation. Consider the nonlinear time fractional diffusion equation of the form

\[
D_t^\alpha u(x, t) = \frac{\partial^2 u^m}{\partial x^2}, \quad m \geq 2, \quad x \in (a, b), \quad 0 < t \leq T
\]  

(5.2.1)

with initial condition \( u(x, 0) = g(x), \quad x \in (a, b) \)  

(5.2.2)

boundary conditions \( u(a, t) = u(b, t) = 0, \quad t > 0 \)  

(5.2.3)

It is called the first IBVP for nonlinear time fractional diffusion equation. For \( m = 1 \) the above equation (5.2.1) reduces to linear time fractional diffusion equation which is studied by number of authors [82, 99, 115] with Dirichlet boundary condition and Yang et al.[114] with Neumann boundary condition. Now, we present numerical method to simulate the nonlinear behavior of the nonlinear first IBVP (5.2.1)-(5.2.3). Let \( x_i = ih (i = 0, 1, ..., M) \) and
\( t_k = k\tau (k = 0, 1, ..., N) \), where \( h = \frac{b-a}{M} \) and \( \tau = \frac{T}{N} \) be the spatial and temporal approximation to \( u(x_i, t_k) \). We discretize the equation (5.2.1) as follows:

\[
\frac{\partial^\alpha u(x_i, t_{k+1})}{\partial t^\alpha} \quad \text{where} \quad \theta \text{ is weight factor (} 0 \leq 1 - \theta \leq 1) \text{ which control the degree of implicitness. For} \theta = 0, \frac{1}{2} \text{ and } 1 \text{ gives the explicit, Crank-Nicolson and fully implicit finite difference method respectively. Therefore the weighted average finite difference method is more general finite difference method. The time fractional derivative term in the equation (5.2.1), is approximated by the following scheme: Therefore, we can write it as } u_{i,k} = u(x_i, t_k), \text{ we have}
\]

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[ u_{i,k+1} - u_{i,k} \right] + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} \left[ u_{i,k+j-1} - u_{i,k-j} \right] b_j
\]

(5.2.5)

where \( b_j = (j + 1)^{1-\alpha} - j^{1-\alpha}, j = 0, 1, 2, ..., k \). Using the time fractional approximation for equation (5.2.1), we get

\[
\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[ u_{i,k+1} - u_{i,k} \right] + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} b_j \times
\]

\[
\left[ u_{i,k+1-j} - u_{i,k-j} \right] = \frac{\theta \delta^2_x(u_{i,k+1}) + (1 - \theta) \delta^2_x(u_{i,k})}{h^2}
\]
Rearrange the above equation, we get

\[ u_{i,k+1} - u_{i,k} = r\theta \delta^2_x(u_{i,k+1}^m) + r(1 - \theta)\delta^2_x(u_{i,k}^m) - \sum_{j=1}^{k} \left[ u_{i,k+1-j} - u_{i,k-j} \right] b_j \]  

(5.2.6)

where \( r = \frac{\tau^n \Gamma(2-\alpha)}{h^2} \). In the above equation the nonlinear term \( (\delta^2_x(u_{i,k+1}^m)) \) create the difficulty to obtain the solution of equation (5.2.6). To avoid this difficulty, the nonlinear term \( (\delta^2_x(u_{i,k+1}^m)) \) is linearized by applying the procedure discussed in Richtmyers and Morton [88], Crank[29], Morton et al.[81] , Ames [10] and Smith [100] for integer order nonlinear partial differential equation. We linearize the nonlinear term \( \delta^2_x(u_{i,k+1}^m) \) by using Taylor’s series expansion about the point \( (i,k) \), we have

\[ u_{i,k+1}^m = u_{i,k}^m + k \frac{\partial u_{i,k}^m}{\partial t} + \frac{k^2}{2!} \frac{\partial^2 u_{i,k}^m}{\partial t^2} + \ldots \]

\[ = u_{i,k}^m + k \frac{\partial u_{i,k}^m}{\partial t} + \ldots \]

which is truncated up to order \( k \), we get

\[ u_{i,k+1}^m = u_{i,k}^m + m u_{i,k}^{m-1} (u_{i,k+1} - u_{i,k}) \]  

(5.2.7)

Using equation (5.2.7) in equation (5.2.6), we have

\[ u_{i,k+1} - u_{i,k} = r\theta \delta^2_x \left( u_{i,k+1}^m + m u_{i,k}^{m-1} (u_{i,k+1} - u_{i,k}) \right) + \]

\[ r(1 - \theta)\delta^2_x(u_{i,k}^m) - \sum_{j=1}^{k} \left[ u_{i,k+1-j} - u_{i,k-j} \right] b_j \]  

(5.2.8)
Putting \( w_i = u_{i,k+1} - u_{i,k} \) in equation (5.2.8), we obtain

\[
w_i = r\theta \delta_x^2 (mu_{i,k}^{m-1} w_i) + r\delta_x^2 (u_{i,k}^m) - \sum_{j=1}^{k} [u_{i,k+1-j} - u_{i,k-j}] b_j \quad (5.2.9)
\]

Using central difference, we have

\[
-mr\theta w_{i-1} u_{i-1,k}^m + (1 + 2mr\theta u_{i,k}^{m-1}) w_i - mr\theta w_{i+1} u_{i+1,k}^{m-1} = ru_{i-1,k}^m - 2ru^m_{i,k} + ru^m_{i+1,k} - \sum_{j=1}^{k} [u_{i,k+1-j} - u_{i,k-j}] b_j \quad (5.2.10)
\]

Substitute \( m = 2 \) in equation (5.2.10), we have

\[
-2r\theta w_{i-1} u_{i-1,k}^1 + (1 + 4r\theta u_{i,k}) w_i - 2r\theta w_{i+1} u_{i+1,k} = ru_{i-1,k}^2 - 2ru^2_{i,k} + ru^2_{i+1,k} - \sum_{j=1}^{k} [u_{i,k+1-j} - u_{i,k-j}] b_j \quad (5.2.11)
\]

For \( k = 0, \ i = 1, 2, ..., M - 1 \) the equation (5.2.11) can be written in the system of \( (M - 1) \) equations. The matrix equation of the system is as follows.

\[
AW = BU_0 + b 
\quad (5.2.12)
\]

where

\[
A = \begin{pmatrix}
1 + 4r\theta u_{1,0} & 2r\theta u_{2,0} \\
-2r\theta u_{1,0} & 1 + 4r\theta u_{2,0} & 2r\theta u_{3,0} \\
& & \\
& & \\
-2r\theta u_{M-3,0} & 1 + 4r\theta u_{M-2,0} & 2r\theta u_{M-1,0} \\
& & \\
-2r\theta u_{M-2,0} & 1 + 4r\theta u_{M-1,0}
\end{pmatrix}
\]
Nonlinear Fractional Diffusion Equation

\[
B = \begin{pmatrix}
-2ru_{1,0} & ru_{2,0} \\
ru_{1,0} & -2ru_{2,0} & ru_{3,0} \\
& & \ddots \\
r_{u_{M-3,0}} & -2ru_{M-2,0} & ru_{M-1,0} \\
ru_{M-2,0} & -2ru_{M-1,0}
\end{pmatrix}
\]

\[
W = [w_1, w_1, ..., w_{M-1}]^T, \quad U_0 = [u_{1,0}, u_{2,0}, ..., u_{M-1,0}]^T,
\]

\[
b = [ru_{0,0}^2 + 2r\theta u_{0,0}(u_{0,1} - u_{0,0}), 0, ..., 0, ru_{2,0}^2 + 2r\theta u_{M,0}(u_{M,1} - u_{M,0})]^T
\]

Before writing the system of \((M - 1)\) equations for \(k \geq 1\), we rearrange the last term of the equation (5.2.11) as follows.

\[
\sum_{j=1}^{k} \left[u_{i,k+1-j} - u_{i,k-j}\right] b_j = b_1 u_{i,k} + \sum_{j=1}^{k-1} (b_{j+1} - b_j) u_{i,k-j} - b_k u_{i,0}
\]

(5.2.13)

Substitute equation (5.2.13) in equation (5.2.11), we get

\[
-2r\theta w_{i-1,k} - 2r\theta w_{i+1,k} = ru_{i-1,0}^2 - 2ru_{i,k}^2 + ru_{i+1,k} - b_1 u_{i,k} - \sum_{j=1}^{k-1} (b_{j+1} - b_j) u_{i,k-j} + b_k u_{i,0}
\]

(5.2.14)

Putting \(k = 1, i = 1, 2, ..., M - 1\) in equation (5.2.14), we obtain

\[
-2r\theta w_{0,1} + (1 + 4r\theta) w_{1} - 2r\theta w_{2,1} = ru_{0,1}^2 - 2ru_{1,1}^2 + ru_{2,1}^2 - b_1 u_{1,1} + b_1 u_{1,0} - 2ru_{2,1}^2 + ru_{3,1}^2 - b_1 u_{2,1} + b_1 u_{2,0}
\]
\[-2r\theta w_{M-2} u_{M-2,1} + (1 + 4r\theta u_{M-1,1}) w_{M-1} - 2r\theta w_{M} u_{M,1} = ru_{M-2,1}^2 - 2ru_{M-1,1}^2 + ru_{M,1}^2 - b_{1} u_{M-1,1} + b_{1} u_{M-1,0}\]

The system of \((M - 1)\) equations can be represented by matrix equation

\[AW = CU_1 + b_1 U_0\]

Similarly, for \(k = 2\), the matrix equation is

\[AW = CU_2 + c_1 U_1 + d_2 U_0\]

In general, for each \(k \geq 1\), \((k = 1, 2, ..., N)\) the matrix equation can be written as

\[AW = CU_k + c_1 U_{k-1} + c_2 U_{k-2} + c_3 U_{k-3} + ... + c_k U_1 + b_k U_0\quad (5.2.15)\]

where

\[C = \begin{pmatrix}
-2ru_{1,1} - b_1 & ru_{2,1} \\
ru_{1,1} & -2ru_{2,1} - b_1 & ru_{3,1} \\
 & & \ddots & \ddots & \ddots \\
 & & & ru_{M-3,1} & -2ru_{M-2,1} - b_1 & ru_{M-1,1} \\
 & & & & ru_{M-2,1} & -2ru_{M-1,1} - b_1
\end{pmatrix}\]

\[W = [w_1, w_1, ..., w_{M-1}]^T, \quad c_k = b_k - b_{k+1}, \quad U_k = [u_{1,k}, u_{2,k}, ..., u_{M-1,k}]^T\]
5.3 Stability

In this section, we prove that the weighted average finite difference scheme developed in last section is conditionally stable.

**Theorem 5.1.** The weighted average finite difference scheme (5.2.11) is conditionally stable.

**Proof.** To prove the stability we consider the following two cases.

**Case-I** For \( k = 0 \), the matrix equation (5.2.12) is

\[
AW = BU_0 + b.
\]

where A and B are tridiagonal matrices and b is column matrix. It can written as

\[
(I - 2rc\theta T_{M-1})(u_{k+1} - u_k) = rcu_k T_{M-1}
\]

where

\[
T_{M-1} = \begin{pmatrix}
-2 & 1 & & & \\
1 & -2 & 1 & & \\
& 1 & -2 & 1 & \\
& & 1 & -2 & 1
\end{pmatrix},
\]

\[
c = \max_{i,k}\{u_{i,k}\} \text{ and } I \text{ is identity matrix.}
\]
\[(I - 2rc\theta T_{M-1})(u_{k+1} - u_k) = \left(rcT_{M-1} + (I - 2rc\theta T_{M-1})\right)u_k\]

\[u_{k+1} = \frac{I + (1 - 2\theta)rcT_{M-1}}{I - 2rc\theta T_{M-1}}u_k\]

The eigenvalues of the above matrix are

\[\lambda_s = \frac{1 - 4rc(1 - 2\theta)\sin^2 \left(\frac{s\pi}{2M}\right)}{1 + 8rc\theta \sin^2 \left(\frac{s\pi}{2M}\right)}, \quad s = 1(1)(M - 1) \quad (5.3.1)\]

For the stability condition the eigenvalue lie in \(-1 \leq \lambda \leq 1\) (see \[44\]). The one side of inequality is trivial, so we have to prove only that \(-1 \leq \lambda\).

From the equation (5.3.1), we have

\[-1 \leq \frac{1 - 4rc(1 - 2\theta)\sin^2 \left(\frac{s\pi}{2M}\right)}{1 + 8rc\theta \sin^2 \left(\frac{s\pi}{2M}\right)}\]

\[-1 - 8rc\theta \sin^2 \left(\frac{s\pi}{2M}\right) \leq 1 - 4rc(1 - 2\theta)\sin^2 \left(\frac{s\pi}{2M}\right)\]

\[-1 - 16rc\theta \sin^2 \left(\frac{s\pi}{2M}\right) \leq 1 - 4rc\sin^2 \left(\frac{s\pi}{2M}\right)\]

\[-1 - 4rc(4\theta - 1)\sin^2 \left(\frac{s\pi}{2M}\right) \leq 1\]

\[4rc(1 - 4\theta)\sin^2 \left(\frac{s\pi}{2M}\right) \leq 2\]

\[r \leq \frac{1}{2c(1 - 4\theta)\sin^2 \left(\frac{s\pi}{2M}\right)} \quad (0 \leq \theta < \frac{1}{4})\]

**Case-II** For \(k \geq 1\), the matrix equation (5.2.15) is

\[AW = CU_k + c_1U_{k-1} + c_2U_{k-2} + c_3U_{k-3} + \ldots + c_kU_1 + b_kU_0, \quad k \geq 1\]

where
\[
C = \begin{pmatrix}
-2ru_{1,1} - b_1 & ru_{2,1} \\
ru_{1,1} & -2ru_{2,1} - b_1 & ru_{3,1} \\
& \ddots & \ddots & \ddots \\
& & ru_{M-3,1} & -2ru_{M-2,1} - b_1 & ru_{M-1,1} \\
& & & ru_{M-2,1} & -2ru_{M-1,1} - b_1
\end{pmatrix}
\]

It can be written as

\[
(I - 2rc\theta T_{M-1})(u_{k+1} - u_k) = (rcT_{M-1} + (I - 2rc\theta T_{M-1}) - b_1)u_k
\]

\[
u_{k+1} = \frac{(I - b_1) + (1 - 2\theta)rcT_{M-1}}{I - 2rc\theta T_{M-1}}u_k
\]

The eigenvalues of the matrix C are

\[
\lambda_s = \frac{(1 - b_1) - 4rc(1 - 2\theta)sin^2\left(\frac{s\pi}{2M}\right)}{1 + 8rc\theta sin^2\left(\frac{s\pi}{2M}\right)}, \quad s = 1(1)(M - 1) \quad (5.3.2)
\]

For the stability condition the eigenvalues lie in \(-1 \leq \lambda \leq 1\) (see [44]).

The one side of inequality is trivial. We prove only \(-1 \leq \lambda\). From the equation (5.3.2), we have

\[
-1 \leq \frac{(1 - b_1) - 4rc(1 - 2\theta)sin^2\left(\frac{s\pi}{2M}\right)}{1 + 8rc\theta sin^2\left(\frac{s\pi}{2M}\right)}
\]

\[
-1 - 8rc\theta sin^2\left(\frac{s\pi}{2M}\right) \leq (1 - b_1) - 4rc(1 - 2\theta)sin^2\left(\frac{s\pi}{2M}\right)
\]

\[
-1 - 16rc\theta sin^2\left(\frac{s\pi}{2M}\right) \leq (1 - b_1) - 4rcsin^2\left(\frac{s\pi}{2M}\right)
\]

\[
-1 - 4rc(4\theta - 1)sin^2\left(\frac{s\pi}{2M}\right) \leq (1 - b_1)
\]
Therefore, we conclude that the proposed weighted average finite difference scheme is stable only when
\[ r \leq \frac{(1 - b_1)}{4c(1 - 4\theta)sin^2(\frac{s\pi}{2M})} \quad (0 \leq \theta < \frac{1}{4}) \]

for \(0 < \alpha < 1\).

### 5.4 Numerical Example

**Example 5.1.** Consider the nonlinear time fractional diffusion equation

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} \quad 0 < x < 1, \ t > 0 \tag{5.4.1}
\]

with initial condition

\[ u(x, 0) = 4x(1 - x) \tag{5.4.2} \]

the boundary conditions

\[ u(0, t) = 0 = u(1, t) \tag{5.4.3} \]

The discrete form of nonlinear IBVP (5.4.1)-(5.4.3) is

\[
-2r\theta w_{i-1}u_{i-1,k} + (1 + 4r\theta u_{i,k})w_i - 2r\theta w_{i+1}u_{i+1,k} = ru_{i-1,k}^2 -
2ru_{i,k}^2 + ru_{i+1,k}^2 - \sum_{j=1}^{k}[u_{i,k+1-j} - u_{i,k-j}]b_j
\]

initial condition \(u_{i,0} = 4ih(1 - ih), i = 0, 1, ..., 10. (\cdot h = 0.1)\)

boundary conditions \(u_{0,N} = 0 = u_{M,N}\)
The following table records the numerical solution for several time-steps by applying weighted average finite difference method when $\alpha = 1$ and its graphical representation is given in Fig.5.1(a).

<table>
<thead>
<tr>
<th>t</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.000</td>
<td>0.3600</td>
<td>0.6400</td>
<td>0.8400</td>
<td>0.9600</td>
<td>1.0000</td>
<td>0.9600</td>
<td>0.8400</td>
<td>0.6400</td>
</tr>
<tr>
<td></td>
<td>0.005</td>
<td>0.4156</td>
<td>0.6414</td>
<td>0.8027</td>
<td>0.8995</td>
<td>0.9318</td>
<td>0.8995</td>
<td>0.8027</td>
<td>0.6414</td>
</tr>
<tr>
<td></td>
<td>0.010</td>
<td>0.4374</td>
<td>0.6346</td>
<td>0.7707</td>
<td>0.8520</td>
<td>0.8791</td>
<td>0.8520</td>
<td>0.7707</td>
<td>0.6346</td>
</tr>
<tr>
<td></td>
<td>0.015</td>
<td>0.4417</td>
<td>0.6223</td>
<td>0.7424</td>
<td>0.8130</td>
<td>0.8365</td>
<td>0.8130</td>
<td>0.7424</td>
<td>0.6223</td>
</tr>
</tbody>
</table>

Fig.5.1(a): Numerical solution at different values of $t$ when $\alpha = 1$

The following table records the numerical solution for several time-steps by applying weighted finite difference method when $\alpha = 0.9$, whose graphical representation at different time levels is given in Fig.5.1(b).

<table>
<thead>
<tr>
<th>t</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.000</td>
<td>0.3600</td>
<td>0.6400</td>
<td>0.8400</td>
<td>0.9600</td>
<td>1.0000</td>
<td>0.9600</td>
<td>0.8400</td>
<td>0.6400</td>
</tr>
<tr>
<td></td>
<td>0.005</td>
<td>0.4028</td>
<td>0.6421</td>
<td>0.8131</td>
<td>0.9156</td>
<td>0.9498</td>
<td>0.9156</td>
<td>0.8131</td>
<td>0.6421</td>
</tr>
<tr>
<td></td>
<td>0.010</td>
<td>0.3977</td>
<td>0.6397</td>
<td>0.8111</td>
<td>0.9139</td>
<td>0.9481</td>
<td>0.9139</td>
<td>0.8111</td>
<td>0.6397</td>
</tr>
<tr>
<td></td>
<td>0.015</td>
<td>0.3993</td>
<td>0.6392</td>
<td>0.8097</td>
<td>0.9118</td>
<td>0.9459</td>
<td>0.9118</td>
<td>0.8097</td>
<td>0.6392</td>
</tr>
</tbody>
</table>
Fig.5.1(b): Numerical solution at different values of $t$ and $\alpha = 0.9$

5.5 Concluding Remarks

*Remark* 5.5.1. A new weighted average finite difference scheme for nonlinear first initial value problem is developed and demonstrated with suitable example. The conditional stability of this scheme is proved.

*Remark* 5.5.2. It is shown that the weighted average finite difference scheme is conditionally stable by using Greshgorin’s circle theorem. Numerical solution of the test problem is obtained and it is simulated by using MATLAB software.