2.1. INTRODUCTION

Basu (1971) has proposed a generalized difference estimator for estimating a population mean. Some well-known design unbiased estimators can be obtained as particular cases of this estimator. Admissibility of this estimator is established by CSW (1977). In this chapter we suggest a generalized difference type estimator for the population variance, which is design unbiased. Following CSW (1977) admissibility of the estimator in the class of all unbiased estimators $\mathcal{M}_u$ is proved with an optimal property of uniform admissibility, that is, the admissibility of the estimator for every FES(n) design having nonzero inclusion probabilities, in $\mathcal{M}_u(n)$. Following Tripathi and Chaubey (1992) we propose an improved estimator over $q_{HT}$, which is a linear combination of HT type estimators. We also consider estimation of the mean square error of the proposed estimator.
2.2. PROPOSED ESTIMATOR

We first state the lemma due to Hanurav (1966), which gives necessary and sufficient conditions for the existence of an unbiased estimator of $\frac{s^2}{y}$.

Lemma 2.1 For any given design with $\pi_i$ and $\pi_{ij}$ ($1 \leq i \neq j \leq N$), there is at least one unbiased estimator of $\frac{s^2}{y}$ if and only if $\pi_i > 0$ and $\pi_{ij} > 0$ ($1 \leq i \neq j \leq N$).

For any given design with $\pi_i > 0$ and $\pi_{ij} > 0$ ($1 \leq i \neq j \leq N$), the proposed generalized difference type estimator of $\frac{s^2}{y}$ denoted by $q_{OD}$, is

$$q_{OD} = \frac{Y^2_i - (\bar{e})^2}{\pi_i} - c_2 \sum_{s} \frac{Y_{ij} - \bar{e} e_j}{\pi_{ij}}$$

where $\bar{e} = (e_1, e_2, \ldots, e_N)$ is an arbitrary but known real vector in $\mathbb{R}^N$, $s^2 = c_1 \sum_{i=1}^{N} e_i^2 - c_2 \sum_{j=1}^{N} e_j e_i$ and $\pi_i = \frac{1}{2} \sum_{s} p(s)$ and $\pi_{ij} = \sum_{s} p(s)$ are the inclusion probabilities of unit i and units i and j respectively. Note that the conditions given in (2.3) of Chapter 1 are satisfied by $q_{OD}$ if we take

$$b_s = \frac{Y^2 - (\bar{e})^2/\pi_i}{\pi_i} + c_2 \sum_{s} \frac{e_j/\pi_{ij}}{\pi_{ij}}$$

$$b(s, ii) = \frac{c_1}{\pi_i} \quad (i=1, \ldots, N)$$

$$b(s, ij) = - \frac{c_2}{\pi_{ij}} \quad (i \neq j=1, \ldots, N).$$
2.3. ADMISSIBILITY

The admissibility of the generalized difference type estimator is established in the following Theorem 2.1, from which we obtain, as corollaries, certain admissibility results for the Horvitz-Thompson type estimator. The proof is based on the following lemma.

For a given vector $e \in \mathbb{R}^N$, define $\Omega_m (m=0,1,\ldots,N)$ as

$$\Omega_m = \{y \in \mathbb{R}^N : y_k \neq e_k \text{ for exactly } m \text{ components of } y\}$$

Lemma 2.2. For any given design $p$, if there exists $q \in \mathcal{A}_u$ satisfying

(a) $V(p,q) \leq V(p,q_{GD})$ \hspace{1cm} $y \in \mathbb{R}^N$

(b) $q(s,y) = q_{GD}(s,y)$ \hspace{1cm} $s \in S^+$

when $y \in \Omega_m$ then $q(s,y) = q_{GD}(s,y)$ for all $s \in S^+$, when $y \in \Omega_{m+1}$.

Proof. Let $y_o$ be an arbitrary vector in $\Omega_{m+1}$. Let $S_k (k = 0,1,\ldots,m+1)$ be defined as

$$S_k = \{s \in S : y_{oi} \neq e_i \text{ for exactly } k \text{ labels } i \in s\}$$

Any outcome, $\{(i,y_i) : i \in s\}$, that can occur when $s \in \bigcup_{k=0}^{m} S_k$ and $y = y_o$ can also occur when $y \in \Omega_m$. Hence by condition...
If $s \notin S$ and $y = y$, that is, if every $s \in S$ contains those $m+1$ labels, $i$, for which $y_i - e_i \neq 0$, then

$$q_{GD}(s, y) = \sum_{i=1}^{m+1} (y_i - e_i)^2 - \sum_{i=1}^{m+1} (y_i - e_i) + e_i^2$$

which is constant. Hence, for $y = y_0$,

$$q_{GD}(s, y) = c \quad s \in S \text{ with } p(s) > 0. \quad (3.2)$$

Since $q$ is unbiased for $S^2$, we have from (3.1) that, for $Y = Y_0$

$$E(q - q_{GD}) = \sum_{s \in S} p(s)(q - q_{GD}) = 0 \quad (3.3)$$

Further, we have from (3.1) and condition (a) of the lemma, for $y = y_0$

$$\nabla(p, q) - \nabla(p, q_{GD}) = \sum_{s \in S} p(s)\{q^2 - q_{GD}^2\} < 0 \quad (3.4)$$

Now, from (3.1) to (3.4) it follows that for $y = y_0$

$$\sum_s p(s) [q - q_{GD}]^2$$

$$= \sum_{s \in S} p(s)(q^2 - q_{GD}^2) + 2 \sum_{s \in S} p(s)q_{GD}(q_{GD} - q)$$
This means that \( q(s, y) = q_{\text{GD}}^*(s, y) \) for all \( s \in S^+ \), when \( y = y_0 \). Since \( y_0 \) was arbitrarily chosen from \( \Omega_{m+1} \) the lemma is proved.

Theorem 2.1. For any given design with \( a_{ij} > 0 \) (\( i, j = 1, \ldots, N \)) and for any given vector \( \mathbf{e} \in \mathbb{R}^N \), \( q_{\text{GD}}(s, y) \) is admissible in \( \mathcal{A} \).

Proof. If the theorem is not true, then there exists an estimator \( q \in \mathcal{A} \) such that
\[
V(p, q) \leq V(p, q_{\text{GD}}) \quad \forall y \in \mathbb{R}^N
\]
with strict inequality for at least one \( y \in \mathbb{R}^N \). When \( y \in \Omega_{\text{lo}} \), we have that \( q_{\text{GD}}(s, y) = S_e^2 \) for all \( s \in S \), which implies
\[
V(p, q_{\text{GD}}) = 0 \quad \text{for} \ y \in \Omega_{\text{lo}}
\]
with strict inequality for at least one \( y \in \mathbb{R}^N \). As \( E(q) = S_e^2 \) for \( y \in \Omega_{\text{lo}} \), \( q(s, y) = S_e^2 = q_{\text{GD}}(s, y) \) for all \( s \in S^+ \), when \( y \in \Omega_{\text{lo}} \). By repeated use of Lemma 2.2 we get \( q(s, y) = q_{\text{GD}}(s, y) \) for all \( s \in S^+ \), when \( y \in \Omega_{\text{lo}}, \Omega_1, \ldots, \Omega_N \), that is, \( q(s, y) = q_{\text{GD}}(s, y) \) for all \( s \in S^+ \) and for all \( y \in \mathbb{R}^N \). Thus for any estimator \( q \in \mathcal{A} \) such that \( V(p, q) \leq V(p, q_{\text{GD}}) \) for all
y \in \mathbb{R}_N, \text{ it is impossible to have } V(p,q) < V(p,q_{GD}) \text{ for all } y \in \mathbb{R}_N. \text{ Therefore } q_{GD} \text{ is admissible in } \mathcal{A}_u.

By considering special values of the constant vector e, we obtain various corollaries.

When e = 0, q_{GD} is equal to q_{HT}, hence the following statement due to Liu(1974b):

Corollary 2.1. For any given design with \( \pi_i > 0, \pi_{ij} > 0 \) \((i,j = 1,2,\ldots,N)\) the Horvitz-Thompson type estimator

\[
q_{HT} = \frac{\sum_s y_i^2}{(N\pi_i)} - \frac{\sum_s y_i y_j}{(N(N-1)\pi_{ij})}
\]

is admissible in \( \mathcal{A}_u \).

Taking \( e = cx \), where \( x \) is a known vector of auxiliary variable values, and \( c \) is a predetermined constant, we obtain second conclusion:

Corollary 2.2. For any given design with \( \pi_i > 0, \pi_{ij} > 0 \) \((i,j = 1,\ldots,N)\), and for any real constant \( c \), the estimator

\[
q_{GD} = c \sum_i \frac{y_i^2}{\pi_i} - c \sum_i \sum_j \frac{y_i y_j}{\pi_{ij}}
+ c \left\{ \frac{s_x^2}{s^2} - \left( c \sum_i \frac{x_i^2}{\pi_i} - c \sum_i \sum_j \frac{x_i x_j}{\pi_{ij}} \right) \right\}
\]

is admissible in \( \mathcal{A}_u \).
2.4. UNIFORM ADMISSIBILITY

In this section we show that the generalized
difference type estimator and a sampling design of fixed size
and fixed inclusion probabilities are together uniformly
admissible in the class of design-estimator pairs called
strategies when the sampling design belongs to a class of
designs of fixed size and fixed inclusion probabilities and
the estimator is restricted to the class of all unbiased
estimators.

Lemma 2.3. If \((p_{0}, q_{GD})\) and \((p_{1}, q_{1})\) are strategies in \(X_{u}(n)\)
satisfying

\[
V(p_{1}, q_{1}) \leq V(p_{0}, q_{GD}) \quad \forall y \in R_{N} \tag{4.1}
\]

then \(\pi_{ijkl}(0) = \pi_{ijkl}(1)\) for \(i, j, k, l = 1, \ldots, N\), that is,
first four order inclusion probabilities under designs \(p_{0}\) and
\(p_{1}\) are equal.

Proof. Let \((p_{0}, q_{GD})\) and \((p_{1}, q_{1})\) be two arbitrary strategies
in \(X_{u}(n)\) satisfying condition (4.1). Then it implies that,
for all \(y \in R_{N}\),

\[
\sum_{s} p_{1}(s)(q_{1} - S_{e}^{2})^{2} \leq \sum_{s} p_{0}(s)(q_{GD} - S_{e}^{2})^{2} \tag{4.2}
\]

First, take \(y = e\), then \(q_{GD} = S_{e}^{2}\) \(s \in S\), and remembering
that \( q \) is an estimator, we have for \( y = e \).

\[
q(s, y) = s^2 \quad \forall s \text{ with } p_1(s) > 0 \quad (4.3)
\]

Next consider a point \( y = y' \) such that \( y' - e \) has only one non-vanishing coordinate, say \( y'_i - e_i \neq 0 \). Unbiasedness of \( q \) at \( y' \) together with (4.3) gives

\[
\sum_{s \in \mathcal{S}} p_1(s) (q - S^2) = c_1 (y'_i^2 - e_i^2). \quad (4.4)
\]

So that by the Cauchy-Schwartz inequality

\[
\left\{ \sum_{s \in \mathcal{S}} p_1(s) (q - S^2) \right\}^2 \leq \pi_i(1) \sum_{s \in \mathcal{S}} p_1(s) (q - S^2)^2 \quad (4.5)
\]

Since for \( y = y' \),

\[
\sum_{s \in \mathcal{S}} p_1(s) (q - S^2)^2 = \frac{c_1 (y'_i^2 - e_i^2)}{\pi_i(0)} \quad (4.6)
\]

we obtain from (4.2), (4.4), (4.5) and (4.6)

\[
\frac{(y'_i^2 - e_i^2)^2}{\pi_i(1)} \leq \frac{(y'_i^2 - e_i^2)^2}{\pi_i(0)}
\]

which implies

\[
\pi_i(0) \leq \pi_i(1).
\]

Since \( i \) is arbitrary and as both strategies belong to \( \mathcal{H}_u(n) \), we must have
\( \pi_i(0) = \pi_i(1) = \pi_i \) say \( i = 1, \ldots, N \)

It now follows, for \( y = y' \), that

\[
V(p_1, q_1) = V(p_0, q_{\text{GP}}) = \frac{c_i(y_i^2 - e_i^2)^2}{\pi_i}
\]

and that equality holds in (4.5), so that

\[
q_i - s_i^2 = \frac{c_i(y_i^2 - e_i^2)}{\pi_i} \quad \forall \ s \ni i \text{ with } p_i(s) > 0. \quad (4.7)
\]

Second, consider a point \( y = y'' \) such that \( y'' - e \) has only two non-vanishing coordinates, say \( y_i'' - e_i^2 \neq 0 \) and \( y_j'' - e_j^2 \neq 0 \). When \( y = y'' \), it follows from (4.3) and (4.7), provided \( p_i(s) > 0 \), that

\[
q_i - s_i^2 = 0 \quad \forall \ s \ni ij
\]

\[
= \frac{c_i(y_i'' - e_i^2)}{\pi_i} \quad \forall \ s \ni i \text{ and } s \ni j
\]

\[
= \frac{c_j(y_j'' - e_j^2)}{\pi_j} \quad \forall \ s \ni j \text{ and } s \ni j
\]

Hence, for \( y = y'' \)

\[
- \sum_{s \ni j} p_i(s)(q_i - s_i^2) = \sum_{s \ni i} p_i(s)\left\{ \frac{c_i(y_i'' - e_i^2)}{\pi_i} + \frac{c_j(y_j'' - e_j^2)}{\pi_j} \right\}
\]

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\[ + \sum_{s \in i} \frac{c_i (y_i^n - e_i^2)}{n_i} + \sum_{s \in j} \frac{c_j (y_j^n - e_j^2)}{n_j} \]

\[ = \sum_{s \in i} p_i(s) (q_i - s_i^2) - \pi_{ij}(1)c_i \left\{ \frac{y_i^n - e_i^2}{n_i} - \frac{y_j^n - e_j^2}{n_j} \right\} \]

\[ + c_i \left\{ (y_i^n - e_i^2) + (y_j^n - e_j^2) \right\} \]

and as \( q_i \) is unbiased for \( s_i^2 \), we have

\[ \sum_{s \in i} p_i(s) (q_i - s_i^2) \]

\[ = \pi_{ij}(1)c_i \left\{ \frac{y_i^n - e_i^2}{n_i} + \frac{y_j^n - e_j^2}{n_j} \right\} - c_i (y_i^n - e_i) (y_j^n - e_j) \cdot \quad (4.8) \]

From Cauchy-Schwartz inequality,

\[ \left\{ \sum_{s \in i} p_i(s) (q_i - s_i^2) \right\}^2 \leq \pi_{ij}(1) \sum_{s \in j} p_i(s) (q_i - s_i^2)^2 \quad (4.9) \]

Hence for \( y = y^n \),

\[ \sum_s p_i(s) (q_i - s_i^2)^2 \]

\[ = \sum_{s \in i, j} p_i(s) (q_i - s_i^2)^2 - \pi_{ij}(1) \left\{ \frac{c_i^2 (y_i^n - e_i^2)^2}{n_i^2} + \frac{c_j^2 (y_j^n - e_j^2)^2}{n_j^2} \right\} \]

\[ + c_i^2 \left\{ \frac{(y_i^n - e_i^2)^2}{n_i^2} + \frac{(y_j^n - e_j^2)^2}{n_j^2} \right\} \]
From this it follows, by use of (4.9), that

\[ \sum_{i} p_{i}(s) \left( q_{i} - s_{e}^{2} \right)^{2} \]

\[ \geq \frac{1}{\pi_{i}^{2}(1)} \left\{ \sum_{s \geq i} p_{i}(s) \left( q_{i} - s_{e}^{2} \right)^{2} \right\} \]

\[ - \pi_{i}(1) \left\{ \left( \frac{c_{i}^{2}(y_{i}^{2} - e_{i}^{2})^{2}}{\pi_{i}} \right) + \frac{c_{i}^{2}(y_{j}^{2} - e_{j}^{2})^{2}}{\pi_{j}} \right\} \]

\[ + c_{i}^{2} \left\{ \frac{(y_{i}^{2} - e_{i}^{2})^{2}}{\pi_{i}} + \frac{(y_{j}^{2} - e_{j}^{2})^{2}}{\pi_{j}} \right\} \]

which reduces, with the help of (4.8) to

\[ \sum_{i} p_{i}(s) \left( q_{i} - s_{e}^{2} \right)^{2} \]

\[ \geq 2c_{i}^{2} \pi_{i}^{2}(1) \left\{ \left( \frac{y_{i}^{2} - e_{i}^{2}}{\pi_{i}} \right) \left( \frac{y_{j}^{2} - e_{j}^{2}}{\pi_{j}} \right) + \frac{c_{i}^{2}(y_{i}^{2} - e_{i}^{2})^{2}}{\pi_{i}} \right\} \]

\[ + c_{i}^{2} \frac{(y_{j}^{2} - e_{j}^{2})^{2}}{\pi_{j}} + \frac{c_{j}^{2}(y_{j}^{2} - e_{j}^{2})^{2}}{\pi_{j}} \]

\[ + \frac{c_{i}^{2}}{\pi_{i}(1)} \left\{ \frac{y_{i}^{2} - e_{i}^{2}}{\pi_{i}} + \frac{y_{j}^{2} - e_{j}^{2}}{\pi_{j}} \right\} \left( y_{i}^{2} - e_{i}^{2} \right) \]

\[ - 2c_{i}^{2} c_{j}^{2} \frac{(y_{i}^{2} - e_{i}^{2})}{\pi_{i}} + \frac{(y_{j}^{2} - e_{j}^{2})}{\pi_{j}} \left( y_{i}^{2} - e_{i}^{2} \right) \]

\[ = 2c_{i}^{2} \pi_{i}(0) \left\{ \left( \frac{y_{i}^{2} - e_{i}^{2}}{\pi_{i}} \right) \left( \frac{y_{j}^{2} - e_{j}^{2}}{\pi_{j}} \right) + \frac{c_{i}^{2}(y_{i}^{2} - e_{i}^{2})}{\pi_{i}} \right\} \]

Next since

\[ \sum_{i} p_{i}(s) \left( q_{i} - s_{e}^{2} \right)^{2} \]

\[ = 2c_{i}^{2} \pi_{i}(0) \left\{ \left( \frac{y_{i}^{2} - e_{i}^{2}}{\pi_{i}} \right) \left( \frac{y_{j}^{2} - e_{j}^{2}}{\pi_{j}} \right) + \frac{c_{i}^{2}(y_{i}^{2} - e_{i}^{2})}{\pi_{i}} \right\} \]
comparing (4.10) with (4.11), we find, using (4.2), that

\[
2c^2 \pi_{i,j}(1) \left( \frac{y^2_i - e^2_i}{\pi_i} \right) \left( \frac{y^2_j - e^2_j}{\pi_j} \right) + c^2 \pi_{i,j}(0) \leq 2c^2 \pi_{i,j}(0) \left( \frac{y^2_i - e^2_i}{\pi_i} \right) \left( \frac{y^2_j - e^2_j}{\pi_j} \right) + c^2 \pi_{i,j}(0)
\]

or equivalently

\[
2c^2 \left( \frac{y^2_i - e^2_i}{\pi_i} \right) \left( \frac{y^2_j - e^2_j}{\pi_j} \right) \pi_{ij}(0) \pi_{ij}(1) \pi_{ij}(1) \pi_{ij}(0) \leq c^2 (y^2_i y^2_j - e^2_i e^2_j) \pi_{ij}(1) \pi_{ij}(0).
\]

This inequality must hold for both positive and negative values of the product \((y^2_i - e^2_i)(y^2_j - e^2_j)\). It follows that \(\pi_{ij}(0) = \pi_{ij}(1)\). Since \(i\) and \(j\) were arbitrarily chosen, we conclude that

\[
\pi_{ij}(0) = \pi_{ij}(1) = \pi_{ij} \quad \forall \quad i \neq j = 1, \ldots, N
\]

Third, consider a vector \(y''\) such that \(y'' - e\) has
exactly three nonzero components, say, \( y'''_i - e_i \neq 0 \), \( y'''_j - e_j \neq 0 \) and \( y'''_k - e_k \neq 0 \). When \( y = y''' \), it follows from (4.3) and (4.7), provided \( p_1(s) > 0 \), that

\[
q_1 - s^2_0 = 0 \quad \forall \ s \in c_1 \cup c_j \cup c_k
\]

\[
= z_i \quad \forall \ s \in c_i - c_j - c_k + c_{ijk}
\]

\[
= z_j \quad \forall \ s \in c_j - c_i - c_k + c_{ijk}
\]

\[
= z_k \quad \forall \ s \in c_k - c_i - c_j + c_{ijk}
\]

\[
= -zz_{ij} \quad \forall \ s \in c_{ij} - c_{ijk}
\]

\[
= -zz_{ik} \quad \forall \ s \in c_{ik} - c_{ijk}
\]

\[
= -zz_{jk} \quad \forall \ s \in c_{jk} - c_{ijk}
\]

where \( z_i = c_i (y'''_i - e_i^2) \), \( z_j = c_j (y'''_j - e_j e_j) \), \( z_k = c_k (y'''_k - e_k) \) and \( c_{ijk} = \{ s : i, j, k \in s \} \) etc. Unbiasedness of \( q_1 \) at \( y''' \) together with (4.3) and (4.13) gives

\[
\sum_{s \in c_{ijk}} p_1(s) (q_1 - s^2_0) = \pi_{ijk} \left( \frac{z_i}{\pi_i} + \frac{z_j}{\pi_j} + \frac{z_k}{\pi_k} - 2 \left( \frac{z_{ij}}{\pi_{ij}} + \frac{z_{ik}}{\pi_{ik}} + \frac{z_{jk}}{\pi_{jk}} \right) \right)
\]

\[
+ \pi_{ij} \left( \frac{z_i}{\pi_i} + \frac{z_j}{\pi_j} \right) + \pi_{ik} \left( \frac{z_i}{\pi_i} + \frac{z_k}{\pi_k} \right) + \pi_{jk} \left( \frac{z_j}{\pi_j} + \frac{z_k}{\pi_k} \right)
\]

\[
= \phi_{ijk} \quad \text{say}
\]

(4.13)

Using the Cauchy-Schwartz inequality, we have

\[
\left\{ \sum_{s \in c_{ijk}} p_1(s) (q_1 - s^2_0) \right\}^2 \leq \pi_{ijk} \left( \sum_{s \in c_{ijk}} p_1(s) (q_1 - s^2_0)^2 \right)
\]

(4.14)
Noting (4.13) and (4.14), the second moments of \((q_i - s_i^2)\),
for \(y = y''\) can be found as

\[
\sum_{i,j,k} \pi_i(s)(q_i - s_i^2)^2 \geq \frac{1}{\pi_{ijk}(1)} \left\{ \frac{1}{\pi_{ijk}} \right\}^2
\]

\[
- \pi_{ijk}(1) \left\{ \frac{z_i^2}{\pi_i} + \frac{z_j^2}{\pi_j} + \frac{z_k^2}{\pi_k} + 4 \left( \frac{z_i^2}{\pi_{ij}} + \frac{z_k^2}{\pi_{ik}} + \frac{z_{jk}^2}{\pi_{jk}} \right) \right\}
\]

Using \(\pi_{ij} / \pi_{ijk} \geq \pi_{ij} / \pi_{ij} = 1/\pi_{ij}\) and \(\pi_{ijk} / \{\pi_{i}, \pi_{ij} \} \leq 1/\pi_{ij}\), we have

\[
\sum_{i,j,k} \pi_i(s)(q_i - s_i^2)^2 \geq \frac{z_i^2}{\pi_i} + \frac{z_j^2}{\pi_j} + \frac{z_k^2}{\pi_k} + 4 \left( \frac{z_i^2}{\pi_{ij}} + \frac{z_k^2}{\pi_{ik}} + \frac{z_{jk}^2}{\pi_{jk}} \right)
\]

\[
+ 2 \left[ \frac{\pi_{ij}}{\pi_i} z_i z_j + \frac{\pi_{ik}}{\pi_i} z_i z_k + \frac{\pi_{jk}}{\pi_i} z_j z_k - 2 \left( \frac{z_i^2}{\pi_i} + \frac{z_{ik}}{\pi_i} + \frac{z_{jk}}{\pi_i} + \frac{z_{ij}}{\pi_j} + \frac{z_{ik}}{\pi_j} + \frac{z_{jk}}{\pi_j} + \frac{z_{ij}}{\pi_k} + \frac{z_{ik}}{\pi_k} + \frac{z_{jk}}{\pi_k} \right) \right]
\]

\[
- 4\pi_{ijk}(1) \left[ \frac{z_{ij} z_{jk}}{\pi_{ij} \pi_{jk}} + \frac{z_{ij} z_{ik}}{\pi_{ij} \pi_{ik}} + \frac{z_{ik} z_{jk}}{\pi_{ik} \pi_{jk}} - 2 \left\{ \frac{z_{ij} z_{ik}}{\pi_{ij} \pi_{ik}} + \frac{z_{ij} z_{jk}}{\pi_{ij} \pi_{jk}} + \frac{z_{ik} z_{jk}}{\pi_{ik} \pi_{jk}} \right\} \right]
\]

(c4.15.)
on simplification.

Since for \( y = y'' \),

\[
\sum_{\text{sec} i,j,k} p_{0}(s)\{q_{0d} - S_{e}^{2}\}^{2} = \frac{z_{i}}{n_{i}} + \frac{z_{j}}{n_{j}} + \frac{z_{k}}{n_{k}} + 4\left(\frac{z_{i}^{2}}{n_{i}} + \frac{z_{j}^{2}}{n_{j}} + \frac{z_{k}^{2}}{n_{k}}\right)
\]

\[+ 2\left[\frac{n_{i,j}}{n_{i}} \frac{n_{i,j}}{n_{j}} z_{i} z_{j} + \frac{n_{i,k}}{n_{i}} \frac{n_{i,k}}{n_{k}} z_{i} z_{k} + \frac{n_{j,k}}{n_{j}} \frac{n_{j,k}}{n_{k}} z_{j} z_{k}\right]
\]

\[- 2\left(\frac{n_{i,j}}{n_{i}} + \frac{n_{i,k}}{n_{i}} + \frac{n_{j,k}}{n_{j}} + \frac{n_{k}}{n_{k}}\right)\]

\[- 4\pi \left(0\right)\left[\frac{z_{i} z_{j}}{n_{i,j}} + \frac{z_{i} z_{k}}{n_{i,k}} + \frac{z_{j} z_{k}}{n_{j,k}}\right].\]

(4.16)

We obtain from (4.2), (4.14), (4.15) and (4.16)

\[
\pi_{ijk}^{(1)} \left[\frac{z_{i} z_{j} z_{k}}{n_{i,j,k}} \frac{z_{i} z_{j} z_{k}}{n_{i,j,k}} + \frac{z_{i} z_{j} z_{k}}{n_{i,j,k}} + \frac{z_{i} z_{j} z_{k}}{n_{i,j,k}}\right] - 2\left(\frac{z_{i} z_{j} z_{k}}{n_{i,j,k}} + \frac{z_{i} z_{j} z_{k}}{n_{i,j,k}} + \frac{z_{i} z_{j} z_{k}}{n_{i,j,k}}\right)
\]

\[
\geq \pi_{ijk}^{(0)} \left[\frac{z_{i} z_{j} z_{k}}{n_{i,j,k}} + \frac{z_{i} z_{j} z_{k}}{n_{i,j,k}} + \frac{z_{i} z_{j} z_{k}}{n_{i,j,k}}\right] - 2\left(\frac{z_{i} z_{j} z_{k}}{n_{i,j,k}} + \frac{z_{i} z_{j} z_{k}}{n_{i,j,k}} + \frac{z_{i} z_{j} z_{k}}{n_{i,j,k}}\right)
\]

This inequality must hold for both positive and negative values of the product \( z_{ij} = c_{c}(y'' - e^{2})(y'' - e^{2}) \). It follows that \( \pi_{ijk}^{(0)} = \pi_{ijk}^{(1)} \). Since \( i, j \) and \( k \) were arbitrarily chosen we conclude that

\[
\pi_{ijk}^{(0)} = \pi_{ijk}^{(1)} = \pi_{ijk} \quad \text{say} \quad \forall \ i \neq j \neq k = 1, \ldots, N.
\]
With quite a cumbersome algebra it can also be shown that

$$\pi_{ijkl}(0) = \pi_{ijkl}(1)$$

which proves the lemma.

Theorem 2.2. Any strategy \((p_0, q_{GD}) \in \mathcal{X}_u(n)\) is admissible in \(\mathcal{X}_u(n)\).

Proof. Given any strategy \((p_1, q_1) \in \mathcal{X}_u(n)\) with

$$V(p_1, q_1) \leq V(p_1, q_{GD}) \quad y \in R_N$$

(4.12)

We shall show that the strict inequality in (4.12) cannot hold at any point \(y\).

From Lemma 2.3, \(\pi_{ijkl}(0) = \pi_{ijkl}(1)\) for \(i,j,k,\ell = 1, \ldots, N\),

$$V(p_0, q_{GD}) = V(p_1, q_{GD}) \quad y \in R_N$$

Since \(q_{GD}\) is admissible in \(\mathcal{X}_u\) under the design \(p_1\), strict inequality cannot hold in (4.12) for any \(y \in R_N\). Hence the strategy \((p_1, q_1)\) cannot be better than \((p_0, q_{GD})\). We conclude that no strategy in \(\mathcal{X}_u(n)\) can be better than \((p_0, q_{GD})\) which means that \((p_0, q_{GD})\) is admissible.

Again for the special values \(e = 0\) we obtain
Corollary 2.3. Any strategy \((p, q_{HT}) \in \mathcal{R}_u(n)\) is admissible in \(\mathcal{R}_u(n)\).

And for \(e = cx\)

Corollary 2.4. Any strategy \((p, q_{OD}) \in \mathcal{R}_u(n)\), where \(q_{OD}\) defined is in (3.7), is admissible in \(\mathcal{R}_u(n)\).

Remark 2.1. Result of Sengupta (1988) is deducible from Corollary 2.3 by taking the design SRSWOR.

2.5. IMPROVED ESTIMATOR

In addition to the unknown \(\{y_1, \ldots, y_N\}\), suppose there is a set of numbers \(\{x_1, \ldots, x_N\}\), associated with this population, which are known and positive. Let \(S_x^2 = c_1 \sum U_i x_i^2 - c_2 \sum U_i x_i\) be the population variance of \(x\). Note that when \(S_x^2\) is known several estimators of \(S_y^2\), including ratio-type and regression-type estimators (Isaki, 1983), are available (see Table 1.1 in Chapter 1) which improve upon the estimator based only upon the principal characteristic. In this section our objective is to investigate possible improvements over the Horvitz-Thompson type estimator, \(q_{HTy}\), using the paired sample \((y_i, x_i), i \in s\), when \(S_x^2\) is unknown.

We propose the estimator \(q_\omega\) as
\[ q_\omega = (1 - \omega)q_{HTY} + \omega q_{HTX} \]  \hspace{1cm} (5.1)

where \( \omega \) is a constant to be determined.

The following proposition exhibits the range of values of \( \omega \) such that \( q_\omega \) is an improvement over \( q_{HTY} \) in the sense of having smaller MSE.

**Proposition 2.1.**

(i) The minimum MSE of \( q_\omega \) is attained for \( \omega = \omega_o \) where,

\[
\omega_o = \frac{V(q_{HTY}) - \text{Cov}(q_{HTY}, q_{HTX})}{V(q_{HTY}) - 2\text{Cov}(q_{HTY}, q_{HTX}) + V(q_{HTX}) + (S_{y}^{2} - S_{x}^{2})^{2}}
\]  \hspace{1cm} (5.2)

(ii) \( \text{MSE}(q_{\omega}) < \text{V}(q_{HTY}) \) for \( \omega \in (0, 2\omega_o) \) if \( \omega_o > 0 \) and for \( \omega \in (2\omega_o, 0) \) for \( \omega_o < 0 \).

**Proof.** By minimizing the MSE(\( q_{\omega} \)) given by

\[
\text{MSE}(q_{\omega}) = V(q_{HTY}) + \omega^{2}[V(q_{HTY} - q_{HTX}) + (S_{y}^{2} - S_{x}^{2})^{2}] - 2\omega \text{Cov}(q_{HTY}, q_{HTY} - q_{HTX})
\]  \hspace{1cm} (5.3)

We easily obtain the optimum value of \( \omega \) given in (5.2). Now (5.3) may be written as
\[ \text{MSE}(q) = \nu(q_{HTY}) + \omega(\omega - 2\nu)\{\nu(q_{HTY} - q_{HTX}) + (s^2 - s^2)^2\} \]

and the second part of the proposition follows.

Since the optimum value \( \omega_0 \) depends on unknown parameters, an estimator of \( \omega_0 \) may be obtained from the sample. Then, of course, the resulting estimator of \( s^2 \) is no longer optimal. Thus Proposition 2.1 is not of direct practical use except that it allows one to explore the general conditions under which a known range of \( \omega \) may be used to improve upon \( q_{HTY} \) using \( q_{HTX} \). Some such conditions are given in the following proposition.

**Proposition 2.2.** Let \( \rho \) denote the correlation coefficient between \( q_{HTY} \) and \( q_{HTX} \), \( K = \sigma(q_{HTX})/\sigma(q_{HTY}) \), \( D = (s^2 - s^2)/\sigma(q_{HTY}) \) and \( \beta = \rho K \) denote the regression coefficient of \( q_{HTX} \) on \( q_{HTY} \).

(i) For \( K = \frac{1}{2} \rho \left[ 1 \pm \left\{ 1 - \frac{4D^2}{\rho^2} \right\}^{1/2} \right] \), the optimum value of \( \omega \) is 1 and for \( \beta = 1 \) one obtains \( \omega_0 = 0 \).

(ii) If \( \rho/K = \beta \geq 1 + (D/K)^2 \) then for any value of \( \omega \) in the interval \((0, 2)\), \( q_\omega \) is better than \( q_{HTY} \).

(iii) If \( K^2 + D^2 = \frac{\nu(q_{HTX})}{\nu(q_{HTY})} + \frac{(s^2 - s^2)^2}{\nu(q_{HTX})} < 1 \) then for any \( \omega \) in the interval \((0, 1)\), \( q_\omega \) is better than \( q_{HTY} \).
Proof. Note that (5.2) can be written as

\[ \omega_0 = \frac{1 - \rho K}{A} \]  

(5.4)

where

\[ A = 1 - 2\rho K + K^2 + D^2 > 0 \]

Equating \( \omega_0 \) to 1 and solving for \( K^* \) gives the first part of the condition (i) and second part is obvious.

To prove part (ii) of the proposition write \( \omega_0 \) as

\[ \omega_0 = 1 + K^2\left(\frac{\rho}{K} - 1 - \frac{D}{K}\right) \]

which implies that \( \omega_0 > 1 \) if the condition in (ii) holds.

Further we have from (5.4)

\[ \omega_0 = \frac{1}{2} + \frac{1 - K^2 - D^2}{2A} \]

which is positive if \( K^2 + D^2 < 1 \). Thus under this condition the interval \((0, 1)\) is a subset of \((0, 2\omega_0)\) which gives part (iii) of the above proposition using (ii) of Proposition 2.1.

Now we consider the estimation of \( \text{MSE}(\omega) \), assuming that \( \omega \) is a known constant. An unbiased estimator of \( \text{MSE}(\omega) \) is given by

\[ \text{mse}_{\omega}(\omega) = v(\omega_{HTY}) + \omega^2(\omega_{HTe})^2 - 2\omega \text{ cov}(\omega_{HTY}, \omega_{HTe}) \]
where

\[ q_{HTe} = c_1 \sum_i e_i \frac{1}{\pi_i} - c_2 \sum_{i,j} e_{i,j} \]

\[ e_i^2 = y_i^2 - x_i^2 \]

\[ e_{i,j} = y_{i,j} - x_{i,j} \]

\[ \text{cov}(q_{HTy}, q_{HTe}) = c_1^2 \left[ \sum_i \left( \frac{1}{\pi_i} - 1 \right) \frac{y_i^2 e_i^2}{\pi_i} + \sum_{i,j} \left( \frac{\pi_{i,j}}{\pi_i \pi_j} - 1 \right) \frac{y_{i,j} e_{i,j}}{\pi_{i,j}} \right] \]

\[ + c_2^2 \left[ 2 \sum_i \left( \frac{1}{\pi_{i,j}} - 1 \right) \frac{y_{i,j} e_{i,j}}{\pi_{i,j}} + 4 \sum_{i,j,k} \left( \frac{\pi_{i,j,k}}{\pi_{i,j} \pi_{i,k}} - 1 \right) \frac{y_{i,j,k} e_{i,j,k}}{\pi_{i,j,k}} \right] \]

\[ + \sum_{i,j,k,l} \left( \frac{\pi_{i,j,k,l}}{\pi_{i,j} \pi_{i,k} \pi_{i,l}} - 1 \right) \frac{y_{i,j,k,l} e_{i,j,k,l}}{\pi_{i,j,k,l}} \]

\[ - 2c_1 c_2 \left[ 2 \sum_i \left( \frac{1}{\pi_i} - 1 \right) \frac{y_i^2 e_i}{\pi_i} + \sum_{i,j,k} \left( \frac{\pi_{i,j,k}}{\pi_i \pi_{i,k}} - 1 \right) \frac{y_{i,j,k} e_{i,j,k}}{\pi_{i,j,k}} \right] \]