Chapter 5

On Neutral Impulsive Functional Integro-Differential Equations With Nonlocal Condition In Banach Spaces

The purpose of this chapter is to study the existence, uniqueness and continuous dependence on initial data of mild solutions of an impulsive neutral integro-differential equations with nonlocal condition in Banach spaces. Our analysis is based on Banach contraction theorem, Krasnoselskii-Schaefer type fixed point theorem and the theory of fractional power of operators. 

1 Two research papers are accepted for the publication in journals, based on the text included in this chapter. Details are enclosed at the end.
5.1. Introduction

The theory of impulsive differential equations is much richer than the corresponding theory of differential equations without impulse effects. For example, initial value problems of such equations may not, in general, possess any solutions at all even when the corresponding differential equation is smooth enough, fundamental properties such as continuous dependence relative to initial data may be violated. Therefore the theory of impulsive differential equations has been studied extensively in past years, for example see [2], [3], [12], [13], [22], [52], [58], [63], [69], [71], [72], [92]-[94], [96]-[97], [108], [110] and the references cited therein. On the other hand, nonlocal condition has better effect on the solution and is more precise for physical measurements than the classical condition. That is the reason why initial value problems with nonlocal conditions have received much attention in recent years, for more information see [2]-[4], [7], [9], [17]-[19], [43], [63], [70]-[73], [84], [94],[101], [102], [110] and the references cited therein. Also neutral differential equations arise in many areas of applied mathematics and therefore these equations have studied by many researchers, for eg. see [3], [6], [10], [12], [21], [22], [25], [43], [57], [58], [74], [78], [80], [81], [108].

Recently, A. Anguraj and K. Karthikeyan [3], studied existence, uniqueness, and continuous dependence of mild solutions of the nonlocal neutral Cauchy problem of the form:

$$\frac{d}{dt}[u(t) + g(t, x_t)] = Ax(t) + f(t, u_t), \quad t \in (0, a], \quad t \neq \tau_k, k = 1, 2, \ldots, m$$
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\[ u(\tau_k + 0) = Q_k u(\tau_k) = u(\tau_k) + I_k u(\tau_k), \quad k = 1, 2, \ldots, m, \]

\[ u(t) + (g(u_{t_1}, \ldots, u_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0, \]

by using Banach contraction theorem. Hayder Akca, A. Bourcherif and V. Covachev [2], investigated existence, uniqueness, and continuous dependence of mild solutions of the nonlocal Cauchy problem of the form:

\[ u'(t) = Ax(t) + f(t, u_t), \quad t \in (0, a], \quad t \neq \tau_k, k = 1, 2, \ldots, m \]

\[ u(\tau_k + 0) = Q_k u(\tau_k) = u(\tau_k) + I_k u(\tau_k), \quad k = 1, 2, \ldots, m, \]

\[ u(t) + (g(u_{t_1}, \ldots, u_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0, \]

where \( A \) is \( C_0 \) semigroup of operators, using Banach contraction fixed point theorem. Also, J. P. Daur and N. I. Mahmudov [27], studied existence of mild solutions to the nonlinear neutral differential equation with nonlocal condition of the following form:

\[ \frac{d}{dt} [x(t) + g(t, x(t))] = Ax(t) + f(t, x(t)) \]

\[ x(0) = x_0 - h(x), \quad t \in [0, T], \]

using Krasnoselskii-Schaefer type fixed point theorem.

In the present chapter, we study the following nonlocal neutral integro-differential equations of the type:

\[ \frac{d}{dt} [x(t) - u(t, x_t)] = Ax(t) + f\left(t, x_t, \int_0^t k(t, s) h(s, x_s) ds\right), \]

\[ t \in (0, T], \quad t \neq \tau_k, k = 1, 2, \ldots, m \quad (5.1.1) \]
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\[ x(t) + (g(x_{t_1}, \ldots, x_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0, \]  
\[ \Delta x(\tau_k + 0) = I_k x(\tau_k), \quad k = 1, 2, \ldots, m, \]  
(5.1.2)  
(5.1.3)

where \(0 < t_1 < t_2 < \ldots < t_p \leq T, p \in \mathbb{N}, A \) and \(I_k(k = 1, 2, \ldots, k)\) are the linear operators acting in a Banach space \(X\). The functions \(f, h, g, k\) and \(\phi\) are given functions satisfying some assumptions. \(\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0)\) and the impulsive moments \(\tau_k\) are such that \(0 < \tau_1 < \tau_2 < \ldots < \tau_k < T, k \in \mathbb{N}\). For any continuous function \(x\) and for any \(t \in [0, T]\), we denote \(x_t\) the element of \(C([-r, 0], X)\) defined by \(x_t(\theta) = x(t + \theta)\) for \(\theta \in [-r, 0]\).

Equations of the form (5.1.1)-(5.1.3) or their special forms arise in some physical applications as a natural generalization of the classical initial value problems. The results for semilinear functional differential nonlocal problem are generalized for the case of impulsive effect.

As usual, in the theory of impulsive differential equations at the points of discontinuity \(\tau_i\) of the solution \(t \to x(t)\), we assume that \(x(\tau_i) = x(\tau_i - 0)\). It is clear that, in general the derivatives \(x'(\tau_i)\) do not exist. On the other hand, according to (5.1.1) there exist the limits \(x'(\tau_i \pm 0)\). With above convention, we assume \(x'(\tau_i) = x'(\tau_i - 0)\).

The aim of the present chapter is to study the existence, uniqueness and continuous dependence on initial data of mild solutions of nonlocal initial value problem for an impulsive neutral functional integro-differential equation. Our results generalize some of the results reported in [3], [58], [72]. We use the theory of fractional power of operators, Banach contraction theorem and Krasnoselskii-Schaefer type fixed point theorem to obtain our results.
5.2 Preliminaries and Hypotheses

Our work is motivated from [2], [3] and influenced by the work of J. P. Daur and N. I. Mahmudov [27].

This chapter is organized as follows: Section 5.2 presents the preliminaries and hypotheses. In Section 5.3, we prove existence and uniqueness of mild solution and section 5.4, deals with continuous dependence of mild solution on initial data.

5.2 Preliminaries and Hypotheses

Throughout this chapter $X$ will be a Banach space with norm $\|\cdot\|$. Let $PC([a,b],X)$ denote the set $\{x : [a,b] \to X : x(t) \text{ is continuous at } t \neq \tau_k, \text{ left continuous at } t = \tau_k, \text{ and the right limit } x(\tau_k + 0) \text{ exists for } k = 1,2,...,m\}$. Clearly, $PC([a,b],X)$ is a Banach space with the supremum norm $\|x\|_{PC([a,b],X)} = \sup\{\|x(t)\| : t \in [a,b] \setminus \{\tau_1, \tau_2, ..., \tau_m\}\}$ and $A : D(A) \to X$ be the infinitesimal generator of an analytic semigroup of linear operators $\{T(t)\}_{t \geq 0}$ on $X$. It is well known that there exists a constant $K$ such that $\|T(t)\| \leq K$, $t \geq 0$. We assume that $k : [0,T] \times [0,T] \to \mathbb{R}$ is continuous function and since the set $[0,T] \times [0,T]$, is compact, there exists a constant $L > 0$ such that $|k(t,s)| \leq L$, for $0 \leq s \leq t \leq T$. If $T$ is uniformly bounded analytic semigroup such that $0 \in \rho(A)$, then it is possible to define the fractional power $(-A)\alpha$, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)\alpha$. Furthermore, the subspace $D(-A)\alpha$ is dense in $X$ and the expression $\|x\|_\alpha = \|(-A)\alpha x\|$ defines a norm in $D(-A)\alpha$. If $X_\alpha$ represents the space $D(-A)\alpha$ endowed with the norm $\|\cdot\|$, then the following properties

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are well known.

Lemma 5.2.1 [90, p.74] : Let $0 < \alpha \leq \beta \leq 1$. Then the following properties hold.

(i) $X_\beta$ is a Banach space and $X_\beta \hookrightarrow X_\alpha$ is continuous.

(ii) The function $S \mapsto (A)^\alpha T(S)$ is continuous in the uniform operator topology on $(0, \infty)$ and $\exists$ a positive constant $C_\alpha$ such that $\|(-A)^\alpha T(t)\| \leq \frac{C_\alpha}{t^{\alpha}}$, for every $t > 0$.

Definition 5.2.2 A function $x \in PC([-r,T],X)$, $T > 0$ is called the mild solution of the problem (5.1.1)-(5.1.3) if $x(t) + (g(x_{t_1},...,x_{t_p}))(t) = \phi(t)$, $-r \leq t \leq 0$, the restriction of $x(.)$ to the interval $[0,T]$ is continuous and for each $0 \leq t < T$, the function $AT(t-s)u(s,x_s)$, $s \in [0,t)$, is integrable and the following integral equation

\[
x(t) = T(t)[\phi(0) - (g(x_{t_1},...,x_{t_p}))(0) - u(0,\phi(0) - (g(x_{t_1},...,x_{t_p}))(0))] + u(t,x_t)
+ \int_0^t T(t-s)f(s,x_s,\int_0^s k(s,\tau)h(\tau,x_{\tau})d\tau)ds + \int_0^t AT(t-s)u(s,x_s)ds
+ \sum_{0<\tau_k<t} T(t-\tau_k)I_k\phi(\tau_k), \quad t \in (0,T]
\]

is satisfied.

The following inequality will be useful while proving our result.

Lemma 5.2.3[93, p.12] Let a nonnegative piecewise continuous function $u(t)$ satisfy for $t \geq t_0$, the inequality

\[
u(t) \leq C + \int_{t_0}^t v(s)u(s)ds + \sum_{t_0<\tau_k<t} \beta_k u(\tau_k)
\]

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where $C \geq 0$, $\beta_i \geq 0$, $v(t) > 0$, $\tau_i$ are the first kind discontinuity points of the function $u(t)$. Then the following estimate holds for the function $u(t)$,

$$u(t) \leq C \prod_{t_0 < \tau_i < t} (1 + \beta_i) \exp(\int_{t_0}^{t} v(s)ds).$$

The following theorem is known as Krasnoselskii-Schaefer type fixed point theorem.

**Theorem 5.2.4**[16] Let $X$ be a Banach space and let $A, B : X \to X$ be two operators satisfying:

(i) $A$ is a contraction, and

(ii) $B$ is completely continuous.

Then, either,

(a) the operator equation $Ax + Bx = x$ has a solution, or

(b) the set $\Omega = \{u \in X : \lambda A(u) + \lambda Bu = u, 0 < \lambda < 1\}$ is unbounded.

Now we introduce the following hypotheses.

(\(H_1\)) There exists $\beta \in (0, 1)$.

(\(H_2\)) Let $f : [0, T] \times C([-r, 0], X_\beta) \times X_\beta \to X_\beta$. Then there exists a positive constant $F$ such that

$$\|f(t, x_t, \phi) - f(t, y_t, \psi)\| \leq F\left(\|x - y\|_{C([-r, t], X_\beta)} + \|\phi - \psi\|\right)$$

(\(H_3\)) Let $h : [0, T] \times C([-r, 0], X_\beta) \to X_\beta$ and there exists a positive constant $H$ such that

$$\|h(t, x_t) - h(t, y_t)\| \leq H\|x - y\|_{C([-r, t], X_\beta)}$$

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\((H_4)\) Let \(u : [0, T] \times C([-r, 0], X_\beta) \to X_\beta\) and there exists a positive constant \(U\) such that
\[
\|(-A)^\beta u(t, x_t) - (-A)^\beta u(t, y_t)\| \leq U \|x - y\|_{C([-r, t], X_\beta)}
\]

\((H_5)\) Let \(I_k : X_\beta \to X_\beta\) and there exists positive constants \(L_k\) such that
\[
\|I_k(v)\|_{X_\beta} \leq L_k \|v\|_{X_\beta}, \quad \phi, \psi, v \in X_\beta, \quad k = 1, 2, \ldots, m.
\]

\((H_6)\) For the function \(g : C([-r, 0], X_\beta)^P \to C([-r, 0], X_\beta)\), there exists a constant \(G > 0\) such that
\[
\|(g(x_{t_1}, \ldots, x_{t_p}))(t) - (g(y_{t_1}, \ldots, y_{t_p}))(t)\| \leq G \|x - y\|_{C([-r, T], X_\beta)}
\]

\((H_7)\) Assume \(\phi \in C([-r, 0], X_\beta)\)

\((H_8)\) for every \(w \in PC([-r, T], X_\beta), x \in X_\beta\) and \(t \in [0, T], f(., w_t, x) \in PC([-r, T], X_\beta)\) and there exists a nondecreasing continuous function \(p : [0, T] \to [0, \infty)\) such that
\[
\|f(t, x_t, \psi)\| \leq p(t) (\|x_t\|_{C([-r, 0], X_\beta)} + \|\psi\|)
\]

\((H_9)\) for every \(w \in PC([-r, T], X_\beta)\) and \(t \in [0, T], h(., w_t) \in PC([-r, T], X_\beta)\) and there exists a nondecreasing continuous function \(q : [0, T] \to [0, \infty)\) such that
\[
\|h(t, x_t)\| \leq q(t) \|x_t\|_{C([-r, 0], X_\beta)}
\]

\((H_{10})\) There exists positive constants \(C_1, C_2\) such that
\[
\|(-A)^\beta u(t, x_t)\| \leq C_1 \|x\|_{PC([-r, t], X_\beta)} + C_2
\]
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\( (H_{11}) \) There exists a positive constant \( G_1 \) satisfying

\[
\max \| g(x_{t_1}, x_{t_2}, ..., x_{t_p}) \| \leq G_1
\]

\( (H_{12}) \) For every positive integer \( m \), there exists \( \alpha_m \in L^1(0, T) \) such that

\[
\sup_{\|x\|_{C([-r,0], X_\beta)} \leq m} \| f(t, x_t, \psi) \| \leq \alpha_m(t), \quad a.e. \text{ for } t \in [0, T].
\]

\( (H_{13}) \) For each \( t \in [0, T] \), the function \( f(t, \cdot, \cdot) : C([-r,0], X_\beta) \times X_\beta \to X_\beta \)

is continuous and for each \((\psi, x) \in C([-r,0], X_\beta) \times X_\beta\) the function

\( f(\cdot, \psi, x) : [0, T] \to X_\beta \)

is strongly measurable.

\( (H_{14}) \) For each \( t \in [0, T] \), the function \( h(t, \cdot) : C([-r,0], X_\beta) \to X_\beta \)

is continuous and for each \( \psi \in C([-r,0], X_\beta) \) the function \( h(\cdot, \psi) : [0, T] \to X_\beta \)

is strongly measurable.

5.3 Existence and Uniqueness of Mild Solution

**Theorem 5.3.1** Suppose that hypotheses \((H_1)-(H_7)\) are satisfied and

\[
\Gamma_1 < 1 \quad \text{where} \quad \Gamma_1 = KG + K \|(-A)^{-\beta}\| UG + \|(-A)^{-\beta}\| U + C_{1-\beta} T^\beta U + K F[1 + L H T] T + K \sum_{0<\tau_k \leq t} L_k
\]

then the nonlocal impulsive Cauchy problem \((5.1.1)-(5.1.3)\) has a unique mild solution \( x \) on \([-r, T]\).

**Proof:** We introduce an operator \( \mathcal{F} \) on a Banach space \( PC([-r, T], X_\beta) \) as follows:

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\[
(Fx)(t) = \begin{cases} 
\phi(t) - (g(x_{t_1}, ..., x_{t_p}))(t) & \text{if } -r \leq t \leq 0 \\
T(t) \left[\phi(0) - (g(x_{t_1}, ..., x_{t_p}))(0) - u(0, \phi(0) - (g(x_{t_1}, ..., x_{t_p}))(0)\right] + u(t, x_t) \\
+ \int_0^t AT(t - s)u(s, x_s)ds + \int_0^t T(t - s)f(s, x_s, \int_0^s k(s, \tau)h(\tau, s_s)d\tau)ds \\
+ \sum_{0<\tau_k<t} T(t - \tau_k)I_kx(\tau_k) & \text{if } t \in (0, T]
\end{cases}
\tag{5.3.1}
\]

It is easy to see that \(F : C([-r, T], X_\beta) \to C([-r, T], X_\beta)\).

Now we will show that \(F\) is a contraction on \(PC([-r, T], X_\beta)\).

Let \(x, y \in C([-r, T], X_\beta)\). Then for \(t \in [-r, 0]\),

\[
\|(Fx)(t) - (Fy)(t)\| = \|(g(x_{t_1}, ..., x_{t_p}))(t) - (g(y_{t_1}, ..., y_{t_p}))(t)\|
\leq G\|x - y\|_{PC([-r, T], X_\beta)}
\tag{5.3.2}
\]

and using hypotheses \((H_2)\)\(-(H_6)\), for \(t \in (0, T]\), we get

\[
\|(Fx)(t) - (Fy)(t)\|
= \left\|T(t) \left[(g(x_{t_1}, ..., x_{t_p}))(0) - (g(y_{t_1}, ..., y_{t_p}))(0) + u(0, \phi(0) - (g(x_{t_1}, ..., x_{t_p}))(0)ight.\right.

- u(0, \phi(0) - (g(y_{t_1}, ..., y_{t_p}))(0))\right) + u(t, x_t) - u(t, y_t)

+ \int_0^t AT(t - s)[u(s, x_s) - u(s, y_s)]ds \\
+ \int_0^t T(t - s)f\left(s, x_s, \int_0^s k(s, \tau)h(\tau, s_s)d\tau\right) - f\left(s, y_s, \int_0^s k(s, \tau)h(\tau, y_s)d\tau\right)\right]\]ds

\[+ \sum_{0<\tau_k<t} T(t - \tau_k)[I_kx(\tau_k) - I_ky(\tau_k)]\right\|
\leq \|T(t)\|\|(g(x_{t_1}, ..., x_{t_p}))(0) - (g(y_{t_1}, ..., y_{t_p}))(0)\|

+ \|T(t)\|\|u(0, \phi(0) - (g(x_{t_1}, ..., x_{t_p}))(0)) - u(0, \phi(0) - (g(y_{t_1}, ..., y_{t_p}))(0)\|
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\[ + \| u(t, x_t) - u(t, y_t) \| + \int_0^t \| AT(t - s)\| \| u(s, x_s) - u(s, y_s) \| ds \]
\[ + \int_0^t \| T(t - s)\| \left\| f \left( s, x_s, \int_0^s k(s, \tau) h(\tau, x_\tau) d\tau \right) - f \left( s, y_s, \int_0^s k(s, \tau) h(\tau, y_\tau) d\tau \right) \right\| ds \]
\[ + \sum_{0 < \tau_k < t} \| T(t - \tau_k)\| \| I_k x(\tau_k) - I_k y(\tau_k) \| \]
\[ \leq KG \| x - y \|_{PC([-r,T],X_\alpha)} + KJ_1 + J_2 + J_3 + J_4 + J_5 \quad (5.3.3) \]

where

\[ J_1 = \| u(0, \phi(0) - (g(x_{t_1}, ..., x_{t_p}))(0)) - u(0, \phi(0) - (g(y_{t_1}, ..., y_{t_p}))(0)) \| \]
\[ = \| (-A)^{-\beta} \| (-A)^\beta \| (A) \| \| (A) \| ^\beta \| u(0, \phi(0) - (g(x_{t_1}, ..., x_{t_p}))(0)) \| \]
\[ \leq \| (-A)^{-\beta} \| U \| \phi(0) - (g(x_{t_1}, ..., x_{t_p}))(0) - \phi(0) + (g(y_{t_1}, ..., y_{t_p}))(0) \| \]
\[ \leq \| (-A)^{-\beta} \| UG \| x - y \|_{PC([-r,T],X_\alpha)} \quad (5.3.4) \]
\[ J_2 = \| u(t, x_t) - u(t, y_t) \| \leq \| (-A)^{-\beta} \| U \| \| x - y \|_{PC([-r,T],X_\alpha)} \quad (5.3.5) \]
\[ J_3 = \int_0^t \| AT(t - s)\| \| u(s, x_s) - u(s, y_s) \| ds \]
\[ = \int_0^t \| AT(t - s)\| \| (-A)^{-\beta} \| \| (A) \| ^\beta \| u(s, x_s) - u(s, y_s) \| ds \]
\[ \leq \int_0^t \| (-A)^{1 - \beta} T(t - s)\| \| U \| \| x - y \|_{PC([-r,s],X_\beta)} ds \]
\[ \leq \int_0^t \frac{C_{1 - \beta}}{(t - s)^{1 - \beta}} U \| x - y \|_{PC([-r,s],X_\beta)} ds \]
\[ \leq \frac{C_{1 - \beta}}{T^{1 - \beta}} U \| x - y \|_{PC([-r,T],X_\beta)} T \]
\[ = C_{1 - \beta} T^{\beta} U \| x - y \|_{PC([-r,T],X_\beta)} \quad (5.3.6) \]
\[ J_4 = \int_0^t \| T(t - s)\| \| f \left( s, x_s, \int_0^s k(s, \tau) h(\tau, x_\tau) d\tau \right) - \]

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\[ f \left( s, y, \int_0^s k(s, \tau) h(\tau, y) d\tau \right) ds \leq K \int_0^t F \left[ \| x - y \|_{PC([-r, s], X_\beta)} + \| k(s, \tau) \| h(\tau, y) \| d\tau \right] ds \]

\[ \leq K F \int_0^t \left[ \| x - y \|_{PC([-r, s], X_\beta)} + L \int_0^s H \| x - y \|_{C([-r, \tau], X_\beta)} d\tau \right] ds \]

\[ \leq K F \int_0^t \left[ \| x - y \|_{PC([-r, s], X_\beta)} + L H T \| x - y \|_{PC([-r, s], X_\beta)} \right] ds \]

\[ \leq K F \int_0^t \left[ 1 + L H T \right] \| x - y \|_{PC([-r, s], X_\beta)} ds \]

\[ J_5 = \sum_{0 < \tau_k < t} \| T(t - \tau_k) \| \| I_k x(\tau_k) - I_k y(\tau_k) \|_{X_\beta} \]

\[ \leq \sum_{0 < \tau_k < t} K \| I_k x(\tau_k) - I_k y(\tau_k) \|_{X_\beta} \]

\[ \leq K \sum_{0 < \tau_k < t} L_k \| x(\tau_k) - y(\tau_k) \|_{X_\beta} \]

\[ \leq K \sum_{0 < \tau_k < t} L_k \| x - y \|_{PC([-r, s], X_\beta)} \]  (5.3.8)

Using (5.3.4)-(5.3.8), inequality (5.3.3) becomes

\[ \| (F x)(t) - (F y)(t) \| \leq K G \| x - y \|_{PC([-r, T], X_\beta)} + \left( K \| (A)^{-\beta} \| U G + \| (A)^{-\beta} \| U + C_{1 - \beta} T^\beta U \right) \]

\[ + K F [1 + L H T] T + K \sum_{0 < \tau_k < t} L_k \| x - y \|_{PC([-r, T], X_\beta) \]  \ t \in [0, T]. \]  (5.3.9)

In view of inequality (5.3.2) and (5.3.9), we can say that inequality (5.3.9) holds good for \( t \in [-r, T] \). Therefore, for \( t \in [-r, T], \)

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\[
\|(F_x)(t) - (F_y)(t)\|
\leq KG \|x - y\|_{PC([-r,T],X_{\beta})} + \left( K \|(-A)^{-\beta}\|UG + \|(-A)^{-\beta}\|U + C_{1-\beta}T^3U + KF[1 + LHT]T + K \sum_{0<\tau_k<t} L_k \right) \|x - y\|_{PC([-r,T],X_{\beta})}
\]

which implies

\[
\|F_x - F_y\|_{PC([-r,T],X_{\beta})} \leq \Gamma_1 \|x - y\|_{PC([-r,T],X_{\beta})},
\]

where \(\Gamma_1 = KG + K \|(-A)^{-\beta}\|UG + \|(-A)^{-\beta}\|U + C_{1-\beta}T^3U + KF[1 + LHT]T + K \sum_{0<\tau_k<t} L_k\). Since \(\Gamma_1 < 1\), the operator \(F\) satisfies all the assumptions of Banach contraction theorem and therefore \(F\) has unique fixed point in the space \(PC([-r,T],X_{\beta})\) and which is the mild solution of nonlocal neutral initial value problem (5.1.1)-(5.1.3) with impulse effect. This completes the proof of the theorem.

**Theorem 5.3.2** Suppose that the hypotheses \((H_1), (H_4)-(H_{14})\) are satisfied and \(\Gamma_2 < 1\) where \(\Gamma_2 = KG + K \|(-A)^{-\beta}\|UG + \|(-A)^{-\beta}\|U + C_{1-\beta}T^3U + K \sum_{0<\tau_k<t} L_k\) then the nonlocal impulsive Cauchy problem (5.1.1)-(5.1.3) has a mild solution \(x\) on \([-r,T]\).

**Proof:** We introduce an operator \(F\) on a Banach space \(PC([-r,T],X_{\beta})\) as follows:
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\[
(Fx)(t) = \begin{cases} 
\phi(t) - (g(x_{t_1}, \ldots, x_{t_p}))(t) & \text{if } -r \leq t \leq 0 \\
T(t)[\phi(0) - (g(x_{t_1}, \ldots, x_{t_p}))(0) - u(0, \phi(0) - (g(x_{t_1}, \ldots, x_{t_p}))(0))] + u(t, x_t) & \\
+ \int_0^t AT(t - s)u(s, x_s)ds + \int_0^t T(t - s)f(s, x_s, \int_0^s k(s, \tau)h(\tau, s, \tau)d\tau)ds & \\
+ \sum_{0<\tau_k<t} T(t - \tau_k)I_k x(\tau_k) & \text{if } t \in (0, T]
\end{cases}
\] (5.3.10)

Let \( F = F_1 + F_2 \), where,

\[
(F_1x)(t) = \begin{cases} 
\phi(t) - (g(x_{t_1}, \ldots, x_{t_p}))(t) & \text{if } -r \leq t \leq 0 \\
T(t)[\phi(0) - (g(x_{t_1}, \ldots, x_{t_p}))(0) - u(0, \phi(0) - (g(x_{t_1}, \ldots, x_{t_p}))(0))] + u(t, x_t) & \\
+ \int_0^t AT(t - s)u(s, x_s)ds + \sum_{0<\tau_k<t} T(t - \tau_k)I_k x(\tau_k) & \text{if } t \in (0, T]
\end{cases}
\] (5.3.11)

and

\[
(F_2x)(t) = \begin{cases} 
0 & \text{if } -r \leq t \leq 0 \\
\int_0^t T(t - s)f(s, x_s, \int_0^s k(s, \tau)h(\tau, s, \tau)d\tau)ds & \text{if } t \in (0, T]
\end{cases}
\] (5.3.12)

It is easy to see that \( F_1, F_2 : PC([-r, T], X_\beta) \rightarrow PC([-r, T], X_\beta) \).

Now we will show that \( F_1 \) is a contraction on \( PC([-r, T], X_\beta) \). Let \( x, y \in PC([-r, T], X_\beta) \). Then for \( t \in [-r, 0] \),

\[
\|(F_1x)(t) - (F_1y)(t)\| = \|(g(x_{t_1}, \ldots, x_{t_p}))(t) - (g(y_{t_1}, \ldots, y_{t_p}))(t)\| \\
\leq G\|x - y\|_{PC([-r, T], X_\beta)}
\] (5.3.13)
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and for $t \in (0, T]$,

$$
\| (F_1x)(t) - (F_1y)(t) \|
= \left\| T(t) \left[ (g(x_{t_1}, \ldots, x_{t_p}))(0) - (g(y_{t_1}, \ldots, y_{t_p}))(0) + u(0, \phi(0) - (g(x_{t_1}, \ldots, x_{t_p}))(0) - u(0, \phi(0) - (g(y_{t_1}, \ldots, y_{t_p}))(0)) \right] + u(t, x_t) - u(t, y_t) \\
+ \int_0^t AT(t - s)[u(s, x_s) - u(s, y_s)]ds + \sum_{0<\tau_k<t} T(t - \tau_k)[I_kx(\tau_k) - I_ky(\tau_k)] \right\| \\
\leq \| T(t) \| \| (g(x_{t_1}, \ldots, x_{t_p}))(0) - (g(y_{t_1}, \ldots, y_{t_p}))(0) \| \\
+ \| T(t) \| \| u(0, \phi(0) - (g(x_{t_1}, \ldots, x_{t_p}))(0)) - u(0, \phi(0) - (g(y_{t_1}, \ldots, y_{t_p}))(0)) \| \\
+ \| u(t, x_t) - u(t, y_t) \| + \int_0^t \| AT(t - s) \| \| u(s, x_s) - u(s, y_s) \| ds \\
+ \sum_{0<\tau_k<t} \| T(t - \tau_k) \| \| I_kx(\tau_k) - I_ky(\tau_k) \| \\
\leq KG\| x - y \|_{PC([-r,T],X_\beta)} + KJ_1 + J_2 + J_3 + J_4 
$$

(5.3.14)

where

$$
J_1 = \| u(0, \phi(0) - (g(x_{t_1}, \ldots, x_{t_p}))(0)) - u(0, \phi(0) - (g(y_{t_1}, \ldots, y_{t_p}))(0)) \|
\leq \| (A)^{-\beta} \| \| (A)^{-\beta}u(0, \phi(0) - (g(x_{t_1}, \ldots, x_{t_p}))(0)) - (A)^{-\beta}u(0, \phi(0) - (g(y_{t_1}, \ldots, y_{t_p}))(0)) \| \\
\leq \| (A)^{-\beta} \| U \| \phi(0) - (g(x_{t_1}, \ldots, x_{t_p}))(0) - \phi(0) + (g(y_{t_1}, \ldots, y_{t_p}))(0) \| \\
\leq \| (A)^{-\beta} \| UG\| x - y \|_{PC([-r,T],X_\beta)} 
$$

(5.3.15)

$$
J_2 = \| u(t, x_t) - u(t, y_t) \| \leq \| (A)^{-\beta} \| U \| x - y \|_{PC([-r,T],X_\beta)} 
$$

(5.3.16)

$$
J_3 = \int_0^t \| AT(t - s) \| \| u(s, x_s) - u(s, y_s) \| ds \\
\leq \int_0^t \| - AT(t - s) \| \| (A)^{-\beta} \| \| (A)^{-\beta}[u(s, x_s) - u(s, y_s)] \| ds \\
\leq \int_0^t \| (A)^{-1-\beta}T(t - s) \| U \| x - y \|_{PC([-r,s],X_\beta)} ds 
$$
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\[ \leq \int_0^t \frac{C_1}{(t-s)^{1-\beta}} U \| x - y \|_{PC([-r,s],X_\beta)} ds \]

\[ = \int_0^t \frac{C_1}{T^{1-\beta}} U \| x - y \|_{PC([-r,s],X_\beta)} ds \]

\[ \leq \frac{C_1}{T^{1-\beta}} U \| x - y \|_{PC([-r,T],X_\beta)} T \]

\[ = C_1 T^\beta U \| x - y \|_{PC([-r,T],X_\beta)} \] (5.3.17)

\[ J_4 = \sum_{0 < \tau_k < t} \| T(t - \tau_k) \| \| I_k x(\tau_k) - I_k y(\tau_k) \|_{X_\beta} \]

\[ \leq \sum_{0 < \tau_k < t} K \| I_k x(\tau_k) - I_k y(\tau_k) \|_{X_\beta} \]

\[ \leq K \sum_{0 < \tau_k < t} L_k \| x(\tau_k) - y(\tau_k) \|_{X_\beta} \]

\[ \leq K \sum_{0 < \tau_k < t} L_k \| x - y \|_{PC([-r,T],X_\beta)} \] (5.3.18)

Using (5.3.15)-(5.3.18), inequality (5.3.14) becomes

\[ \| (\mathcal{F}_1 x)(t) - (\mathcal{F}_1 y)(t) \| \]

\[ \leq K G \| x - y \|_{PC([-r,T],X_\beta)} + \left( K \| (-A)^{-\beta} U G + \| (-A)^{-\beta} U \| + C_1 T^\beta U \right) \]

\[ + K \sum_{0 < \tau_k < t} L_k \| x - y \|_{PC([-r,T],X_\beta)} \quad t \in [0, T]. \] (5.3.19)

In view of inequality (5.3.13) and (5.3.19), we can say that inequality (5.3.19) holds good for \( t \in [-r, T] \). Therefore, for \( t \in [-r, T] \),

\[ \| (\mathcal{F}_1 x)(t) - (\mathcal{F}_1 y)(t) \| \leq K G \| x - y \|_{PC([-r,T],X_\beta)} + \left( K \| (-A)^{-\beta} U G + \| (-A)^{-\beta} U \| + C_1 T^\beta U \right) \]

\[ + K \sum_{0 < \tau_k < t} L_k \| x - y \|_{PC([-r,T],X_\beta)} \]

\[ \leq \left( K G + K \| (-A)^{-\beta} U G + \| (-A)^{-\beta} U \| + C_1 T^\beta u_\beta \right) \]

\[ + K \sum_{0 < \tau_k < t} L_k \| x - y \|_{PC([-r,T],X_\beta)} \]
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which implies
$$
\|F_1x - F_1y\|_{PC([−r,T],X_β)} \leq \Gamma_2 \|x - y\|_{PC([−r,T],X_β)}, \text{ where } \Gamma_2 = KG + K\|−A\|^{−\beta}\|UG\| + \|−A\|^{−\beta}\|U\| + C_1\sum_{0<\tau_n<T}I_k. \text{ Since } \\
\Gamma_2 < 1, \text{ the operator } F_1 \text{ is a contraction on } PC([−r,T],X_β).
$$

We denote $M(t) = \sup\{p(t),Lq(t)\}, t \in [0,T]$ and $M^* = \sup\{M(t) : t \in [0,T]\}$. Now we show that $F_2$ is completely continuous operator on $PC([−r,T],X_β)$. To prove this, first we prove that $F_2 : PC([−r,T],X_β) \to PC([−r,T],X_β)$ is continuous. Let $\{x_n\}$ be a sequence of elements of $PC([−r,T],X_β)$ converging to $x$ in $PC([−r,T],X_β)$. Then there exists an integer $N$ such that $\|x_n(t)\| \leq N$ for all $n$ and $t \in (0,T]$. So $x_n \in \{x \in PC([−r,T],X_β) : \|x\|_{PC([−r,T],X_β)} \leq N\}$. Then by using hypothesis $(H_{12})$-$(H_{14})$, we have
$$
f(t,x_n,\int_0^t k(t,s)h(s,x_n)ds) \to f(t,x,\int_0^t k(t,s)h(s,x)ds)
$$
for each $t \in (0,T]$. Since
$$
\left\|f(t,x_n,\int_0^t k(t,s)h(s,x_n)ds) - f(t,x,\int_0^t k(t,s)h(s,x)ds)\right\| \leq 2\alpha(N_1)\gamma(t)
$$
where $(N_1)' = \max\{N, NM^*T\}$.

Then by dominated convergence theorem, we have
$$
\left\|(F_2x_n)(t) - (F_2x)(t)\right\| \leq \int_0^t \left\|T(t-s)\left[f(s,x_n,\int_0^s k(s,\tau)h(\tau,x_n)d\tau) - f(s,x,\int_0^s k(s,\tau)h(\tau,x) d\tau)\right]\right\|ds
$$
$$
\to 0 \text{ as } n \to \infty, \forall \ t \in (0,T].
$$

Since, $\|F_2x_n - F_2x\|_{PC([−r,T],X_β)} = \sup_{t \in [−r,T]} \|(F_2x_n)(t) - (F_2x)(t)\|$,
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it follows that

\[ \|F_2 x_n - F_2 x\|_{PC([-r,T],X_\beta)} \to 0 \text{ as } n \to \infty \]

which implies \( F_2 x_n \to F_2 x \) in \( PC([-r,T],X_\beta) \) as \( x_n \to x \) in \( PC([-r,T],X_\beta) \). Therefore, \( F_2 \) is continuous.

Next we prove that \( F_2 \) maps a bounded set of \( PC([-r,T],X_\beta) \) into a relatively compact set of \( PC([-r,T],X_\beta) \). Let \( B_m = \{ x \in PC([-r,T],X_\beta) : \| x \|_{PC([-r,T],X_\beta)} \leq m \} \) for \( m \geq 1 \). We employ Arzela-Ascoli theorem to show that \( F_2(B_m) \) is relatively compact. For this we prove that \( F_2(B_m) \) is uniformly bounded. From equation (5.3.12) and using hypothesis \((H_{12})\) and the fact that \( \| x \|_{PC([-r,T],X_\beta)} \leq m, x \in B_m \) implies \( \| x_t \|_{C([-r,0],X_\beta)} \leq m, t \in (0,T] \). We obtain

\[
\| (F_2 x)(t) \| \leq \int_0^t \| T(t-s)f(s,x_s,\int_0^s k(s,\tau)h(\tau,x_\tau)d\tau) \| ds \\
\leq K \int_0^t \| f(s,x_s,\int_0^s k(s,\tau)h(\tau,x_\tau)d\tau) \| ds \\
\leq K \int_0^t N'(s) ds
\]

where \( N' = \max\{m, mM^*T\} \). This implies that the set

\( \{(F_2 x)(t) : \| x \|_{PC([-r,T],X_\beta)} \leq m, -r \leq t \leq T \} \) is uniformly bounded in \( X \) and hence \( F_2(B_m) \) is uniformly bounded.

Now we show that \( F_2 \) maps \( B_m \) into an equicontinuous family of functions with values in \( X \). Let \( x \in B_m \) and \( t_1, t_2 \in [-r,T] \). Then from the equation (5.3.20) and using the hypothesis \((H_{12})\), we have

Case 1: Suppose \( 0 \leq t_1 \leq t_2 \leq T \)

\[
(F_2 x)(t_2) - (F_2 x)(t_1) = \int_0^{t_2} T(t_2-s)f(s,x_s,\int_0^s k(s,\tau)h(\tau,x_\tau)d\tau) ds
\]

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\[- \int_0^{t_1} T(t_1 - s)f\left(s, x_s, \int_0^s k(s, \tau)h(\tau, x_\tau)d\tau\right)ds = \int_0^{t_1} [T(t_2 - s) - T(t_1 - s)]f\left(s, x_s, \int_0^s k(s, \tau)h(\tau, x_\tau)d\tau\right)ds + \int_{t_1}^{t_2} T(t_2 - s)f\left(s, x_s, \int_0^s k(s, \tau)h(\tau, x_\tau)d\tau\right)ds \]

\[\| (F_2x)(t_2) - (F_2x)(t_1) \|
\leq \int_0^{t_1} \| T(t_2 - s) - T(t_1 - s)\|\| f\left(s, x_s, \int_0^s k(s, \tau)h(\tau, x_\tau)d\tau\right)\| ds
+ \int_{t_1}^{t_2} \| T(t_2 - s)\|\| f\left(s, x_s, \int_0^s k(s, \tau)h(\tau, x_\tau)d\tau\right)\| ds
\leq \int_0^{t_1} \| T(t_2 - s) - T(t_1 - s)\|\alpha_N'(s)ds
+ \int_{t_1}^{t_2} \| T(t_2 - s)\|\alpha_N'(s)ds
\]

Case 2: Suppose \(-r \leq t_1 \leq 0 \leq t_2 \leq T\)

Then \((F_2x)(t_1) = 0\) and therefore proceeding as in Case 1, we get

\[\| (F_2x)(t_2) - (F_2x)(t_1) \|
\leq \int_0^{t_1} \| T(t_2 - s)\|\| f\left(s, x_s, \int_0^s k(s, \tau)h(\tau, x_\tau)d\tau\right)\| ds
\leq \int_0^{t_1} \| T(t_2 - s)\|\alpha_N'(s)ds
\]

As \(t_2 - t_1 \rightarrow 0\) implies \(t_2 \rightarrow t_1\) and \(t_1 \leq 0, \ t_2 \geq 0\) implies \(t_2 \rightarrow 0\).

Case 3: Suppose \(-r \leq t_1 \leq t_2 \leq 0\).Then

\[\| (F_2x)(t_2) - (F_2x)(t_1) \| = 0\]

Therefore, as \(t_2 - t_1 \rightarrow 0\), the R.H.S. in the cases 1-3, tends to zero. Thus \(F_2\) maps \(B_m\) into an equicontinuous family of functions with values in \(X\).

We have already shown that \(F_2(B_m)\) is an equicontinuous and uniformly bounded collection. To prove the set \(F_2(B_m)\) is relatively compact in
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$PC([−r,T],X_β)$, it is sufficient, by Arzela-Ascoli’s argument, to show that $F_2$ maps $B_m$ into a relatively compact set in $X$.

Let $0 < t ≤ T$ be fixed and $ε$ a real number satisfying $0 < ε < t$. Moreover for $x ∈ B_m$, we define

$$(F_2εx)(t) = \int_{t-ε}^{t} T(t-s)f(s,x_s,\int_{0}^{s} k(s,τ)h(τ,x_τ)dτ)ds = T(ε)\int_{0}^{t-ε} T(t-ε-s)f(s,x_s,\int_{0}^{s} k(s,τ)h(τ,x_τ)dτ)ds$$

Since $T(t)$ is the compact operator, the set $X_ε(t) = \{(F_2εx)(t) : x ∈ B_m\}$ is relatively compact in $X$ for every $ε$, $0 < ε < t$. Moreover for every $x ∈ B_m$, we have

$$\|(F_2x)(t) - (F_2εx)(t)\| ≤ \int_{t-ε}^{t} \|T(t-s)f(s,x_s,\int_{0}^{s} k(s,τ)h(τ,x_τ)dτ)\|ds ≤ K \int_{t-ε}^{t} \|f(s,x_s,\int_{0}^{s} k(s,τ)h(τ,x_τ)dτ)\|ds ≤ K \int_{t-ε}^{t} α_N(s)ds$$

This shows that there exist relatively compact sets arbitrarily close to the set $\{(F_2x)(t) : x ∈ B_m\}$. Hence the set $\{(F_2x)(t) : x ∈ B_m\}$ is relatively compact in $X$. This shows that $F_2$ is completely continuous operator.

To apply the Krasnoselskii-Schaefer theorem, it remains to show that the set $Ω(F) = \{x(.) : λF_1\left(\frac{x}{λ}\right) + λF_2(x) = x\}$ is bounded for $λ ∈ (0,1)$. We denote $L_k^* = \frac{L_k}{|A|}$, $Λ_1 = K(\|\phi\|_{C([-r,0],X_β)} + G_1 + K(1+ C_1 β T^β) + C_2(1+C_1 β T^β))$ and $Λ_2 = \frac{G_1}{|A|}(1+C_1 β T^β) + K(M^*)^2 T^2$.

To this end let $x(.) ∈ Ω(F)$. Then $λF_1\left(\frac{x}{λ}\right) + λF_2(x) = x$ for some
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\( \lambda \in (0, 1) \) and

\[
\|x(t)\| = |\lambda| \| (F_1 \left( \frac{x}{\lambda} \right))(t) + (F_2 x)(t) \|
\]

For \( t \in [-r, 0] \), using hypotheses (H_5) and (H_7)-(H_11), we get

\[
\|x(t)\| = \| \phi(t) - \left( g \left( \left( \frac{x}{\lambda} \right)_{t_1}, \ldots, \left( \frac{x}{\lambda} \right)_{t_p} \right) \right)(0) \| + \sum_{0 < \tau_k < t} \| T(t - \tau_k) \| \| I_k \left( \frac{x}{\lambda} \right)(\tau_k) \|
\]

\[
+ \| T(t) \| \| u(0, \phi(0) - \left( g \left( \left( \frac{x}{\lambda} \right)_{t_1}, \ldots, \left( \frac{x}{\lambda} \right)_{t_p} \right) \right)(0) \| + \| u(t, \left( \frac{x}{\lambda} \right)_t) \|
\]

\[
+ \int_0^t \| A T(t - s) \| \| u(s, \left( \frac{x}{\lambda} \right)_s) \| ds + \int_0^t \| T(t - s) f(s, x_s, \int_0^s k(s, \tau) h(\tau, x_\tau) d\tau) \| ds
\]

\[
\leq K \left( \| \phi \|_{C([-r,0],X_\beta)} + G_1 \right) + K \sum_{0 < \tau_k < t} L_k^* \| x(\tau_k) \|
\]

\[
+ \| T(t) \| \| (A)^{-\beta} \| \left( \frac{C_1}{\| x \|_{PC([-r,T],X_\beta)}} + C_2 \right) + C_1 \beta T^\beta \| (A)^{-\beta} \| \left( \frac{C_1}{\| x \|_{PC([-r,T],X_\beta)}} + C_2 \right)
\]

\[
+ K \int_0^t p(s) \left( \| x_s \|_{C([-r,0],X_\beta)} + L \int_0^s q(\tau) \| x_\tau \|_{C([-r,0],X_\beta)} d\tau \right) ds
\]

\[
\leq K \left( \| \phi \|_{C([-r,0],X_\beta)} + G_1 \right) + K \sum_{0 < \tau_k < t} L_k^* \| x(\tau_k) \|
\]

\[
+ K \| (A)^{-\beta} \| \left( \frac{C_1}{\| x \|_{PC([-r,T],X_\beta)}} + G_1 \right) + C_2 \beta T^\beta \| (A)^{-\beta} \| \left( \frac{C_1}{\| x \|_{PC([-r,T],X_\beta)}} + C_2 \right)
\]

\[
+ \left( \frac{C_1}{\| x \|_{PC([-r,T],X_\beta)}} + G_1 \right) + K \| x \|_{PC([-r,T],X_\beta)} ds
\]

\[
+ \left( \frac{C_1}{\| x \|_{PC([-r,T],X_\beta)}} + G_1 \right) + K \| (A)^{-\beta} \| \left( 1 + C_1 \beta T^\beta \right) \| (A)^{-\beta} \| \left( 1 + C_1 \beta T^\beta \right) \| \| x \|_{PC([-r,T],X_\beta)} ds
\]

\[
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\]
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\[ \leq [\Lambda_1 + \Lambda_2 \|x\|_{PC([-r,T],X_\beta)}] + \int_0^t KM(s)\|x_s\|_{C([-r,0],X_\beta)} ds \]

\[ + \sum_{0<\tau_k<t} KL_k^* \|x(\tau_k)\| \] \hspace{1cm} (5.3.21)

In view of inequalities (5.3.20) and (5.3.21), we have, for \( t \in [-r,T] \),

\[ \|x(t)\| \leq [\Lambda_1 + \Lambda_2 \|x\|_{PC([-r,T],X_\beta)}] + \int_0^t KM(s)\|x_s\|_{C([-r,0],X_\beta)} ds \]

\[ + \sum_{0<\tau_k<t} KL_k^* \|x(\tau_k)\| \]

Now applying Lemma 5.2.3, we get

\[ \|x(t)\| \leq [\Lambda_1 + \Lambda_2 \|x\|_{PC([-r,T],X_\beta)}] \prod_{0<\tau_k<t} (1 + KL_k^*) \exp(KM^*T) \]

By taking supremum over \( t \in [-r,T] \), we get,

\[ \|x\|_{PC([-r,T],X_\beta)} \leq \frac{\Lambda_1 \prod_{0<\tau_k<t} (1 + KL_k^*) \exp(KM^*T)}{[1 - \Lambda_2 \prod_{0<\tau_k<t} (1 + KL_k^*) \exp(KM^*T)]} = Q, \quad \text{constant} \]

This implies the set \( \Omega(F) = \{x(.) : \lambda F_1\left(\frac{x}{\lambda}\right) + \lambda F_2(x) = x\} \) is bounded for \( \lambda \in (0,1) \). hence by Krasnoselskii-Schaefier fixed point theorem, \( F \) has a fixed point and which is the required mild solution of equations (5.1.1)-(5.1.3).

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Theorem 5.4.1. Suppose that hypotheses \((H_1)-(H_7)\) are satisfied and \( \Gamma_1 < 1 \). Then for each \( \phi_1, \phi_2 \in C([-r,T],X_\beta) \) and for the corresponding mild
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solutions $x_1, x_2$ of the problems,

$$
\frac{d}{dt}[x(t) + u(t, x_t)] = Ax(t) + f(t, x_t, \int_0^t k(t, s)h(s, x_s)ds), \quad t \in (0, T],
$$

(5.4.1)

$$
\Delta x(\tau_k + 0) = I_k x(\tau_k), \quad k = 1, 2, \ldots, m,
$$

(5.4.2)

$$
x(t) + (g(x_{t_1}, \ldots, x_{t_p}))_{i}(t) = \phi_i(t), \quad i = 1, 2, \ldots, t \in [-r, 0]
$$

(5.4.3)

the following inequality holds

$$
\|x_1 - x_2\|_{PC([-r,T], X\beta)} \leq \prod_{0 < \tau_k < t} (1 + KL_k)\exp(KFT)[K + KU\|(-A)^{-\beta}\|

\times \|\phi_1 - \phi_2\|_{C([-r,0], X\beta)}
$$

(5.4.4)

where $\Lambda_1 = GK + K\|(-A)^{-\beta}\|UG + \|(-A)^{-\beta}\|U + C_{1-\beta}UT^\beta + KFLHT^2$.

Moreover, if $U = 0$ and $G = 0$, the above inequality reduces to classical inequality,

$$
\|x_1 - x_2\|_{PC([-r,T], X\beta)} \leq \prod_{0 < \tau_k < t} (1 + KL_k)\exp(KFT)

\leq \frac{K}{[1 - KFLHT^2 \prod_{0 < \tau_k < t} (1 + KL_k)\exp(KFT)]} \times \|\phi_1 - \phi_2\|_{C([-r,0], X\beta)}
$$

(5.4.5)

**Proof:** Let $\phi_1, \phi_2 \in C([-r,T], X\beta)$ be arbitrary functions and let $x_1, x_2$ be the mild solutions of the problem (5.4.1)-(5.4.3). Then we have

$$
x_1(t) - x_2(t) = T(t)[\phi_1(0) - \phi_2(0)] - T(t)[(g(x_{t_1}, \ldots, x_{t_p}))(0) - (g(x_{t_1}, \ldots, x_{t_p}))(0)]
$$
5.4. Continuous Dependence of a Mild Solution

\[ -T(t)[u(0, \phi_1(0) - (g(x_{1_1}, ..., x_{1_p}))(0)) - u(0, \phi_2(0) - (g(x_{2_1}, ..., x_{2_p}))(0))] + u(t, x_{1_1}) \]

\[ -u(t, x_{2_1}) + \int_0^t (-A)^{1-\beta}T(t-s)[(-A)^{-\beta}u(s, x_{1_1}) - (-A)^{-\beta}u(s, x_{2_1})]ds \]

\[ + \int_0^t T(t-s)\left[ f(s, x_{1_1}, \int_0^s k(s, \tau)h(s, x_{1_1})d\tau) - f(s, x_{2_1}, \int_0^s k(s, \tau)h(s, x_{2_1})d\tau) \right]ds \]

\[ + \sum_{0<\tau_k<t} T(t-\tau_k)(I_kx_1(\tau_k) - I_kx_2(\tau_k)), \quad t \in (0, T) \]  (5.4.6)

and for \( t \in [-r, 0] \),

\[ x_1(t) - x_2(t) = \phi_1(t) - \phi_2(t) - [g(x_{1_1}, ..., x_{1_p})(t) - g(x_{2_1}, ..., x_{2_p})(t)] \]

(5.4.7)

From (5.4.6) and using hypothesis (H2)-(H3), we get,

\[ \|x_1(t) - x_2(t)\| \]

\[ \leq K\|\phi_1 - \phi_2\|_{C([-r,0],X_\beta)} + GK\|x_1 - x_2\|_{PC([-r,T],X_\beta)} + Ku\|(-A)^{-\beta}\|\phi_1 - \phi_2\|_{C([-r,0],X_\beta)} \]

\[ + K\|(-A)^{-\beta}\|UG\|x_1 - x_2\|_{PC([-r,T],X_\beta)} + \|(-A)^{-\beta}\|U\|x_1 - x_2\|_{PC([-r,T],X_\beta)} \]

\[ + \int_0^t C_{1-\beta}UT^{\beta-1}\|x_1 - x_2\|_{PC([-r,s],X_\beta)}ds + K\int_0^t F\|x_1 - x_2\|_{PC([-r,s],X_\beta)}ds \]

\[ + K\int_0^t FLHT\|x_1 - x_2\|_{PC([-r,T],X_\beta)}ds + K\sum_{0<\tau_k<t} L_k\|x_1(\tau_k) - x_2(\tau_k)\|, \quad 0 \leq t \leq T \]

\[ \leq \left[ K + Ku\|(-A)^{-\beta}\|\right]\phi_1 - \phi_2\|_{C([-r,0],X_\beta)} + GK\|x_1 - x_2\|_{PC([-r,T],X_\beta)} \]

\[ + \|(-A)^{-\beta}\|UG\|x_1 - x_2\|_{PC([-r,T],X_\beta)} + \|(-A)^{-\beta}\|U\|x_1 - x_2\|_{PC([-r,T],X_\beta)} \]

\[ + C_{1-\beta}UT^{\beta}\|x_1 - x_2\|_{PC([-r,T],X_\beta)} + \int_0^t KF\|x_1 - x_2\|_{PC([-r,s],X_\beta)}ds \]

\[ + KFLHT^2\|x_1 - x_2\|_{PC([-r,T],X_\beta)} + K\sum_{0<\tau_k<t} L_k\|x_1(\tau_k) - x_2(\tau_k)\|, \quad 0 \leq t \leq T \]

\[ \leq \left[ K + Ku\|(-A)^{-\beta}\|\right]\phi_1 - \phi_2\|_{C([-r,0],X_\beta)} + \Lambda_1\|x_1 - x_2\|_{PC([-r,T],X_\beta)} \]

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\[
+ \int_{0}^{t} KF \| x_1 - x_2 \|_{PC([-r,s],X_\beta)} ds + K \sum_{0 < \tau_k < t} L_k \| x_1(\tau_k) - x_2(\tau_k) \|, \quad 0 < \tau_k \leq t
\]

Simultaneously, by (5.4.7) and hypothesis \((H_6)\), we get,

\[
\| x_1(t) - x_2(t) \| \leq \| \phi_1 - \phi_2 \|_{C([-r,0],X_\beta)} + G \| x_1 - x_2 \|_{PC([-r,T],X_\beta)}, \quad t \in [-r,0].
\]

Since \( K \geq 1 \), the inequalities (5.4.8) and (5.4.9) imply

\[
\| x_1 - x_2 \|_{PC([-r,T],X_\beta)}
\leq \left[ (K + KU \| (A)^{-\beta} \| \phi_1 - \phi_2 \|_{C([-r,0],X_\beta)} + \Lambda_1 \| x_1 - x_2 \|_{PC([-r,T],X_\beta)} \right]
+ \int_{0}^{t} KF \| x_1 - x_2 \|_{PC([-r,s],X_\beta)} ds + K \sum_{0 < \tau_k < t} L_k \| x_1(\tau_k) - x_2(\tau_k) \|,
\]

\[0 < \tau_k \leq t, \quad t \in [0,T].\] (5.4.10)

Now applying Lemma 5.2.3 to the inequality (5.4.10), we get,

\[
\| x_1 - x_2 \|_{PC([-r,T],X_\beta)}
\leq \prod_{0 < \tau_k < t} (1 + KL_k) \exp(KFT) \left( [K + KU \| (A)^{-\beta} \| \phi_1 - \phi_2 \|_{C([-r,0],X_\beta)} + \Lambda_1 \| x_1 - x_2 \|_{PC([-r,T],X_\beta)} \right)
\]

hence, we get,

\[
\| x_1 - x_2 \|_{PC([-r,T],X_\beta)}
\leq \prod_{0 < \tau_k < t} (1 + KL_k) \exp(KFT) \left( [K + KU \| (A)^{-\beta} \| \phi_1 - \phi_2 \|_{C([-r,0],X_\beta)} + \Lambda_1 \| x_1 - x_2 \|_{PC([-r,T],X_\beta)} \right)
\]

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The inequalities given by (5.4.4) and (5.4.5) are easy consequence of above inequality. This completes the proof.