2.1. Introduction

Let \( x_1, x_2, \ldots, x_n \) be a random sample of size \( n \) from an exponential distribution with p.d.f.

\[
 f(x/\theta) = \frac{1}{\theta} \exp\left(-\frac{x}{\theta}\right), \quad x > 0, \quad \theta > 0.
\] (2.1.1)

The problem is to estimate \( \theta \) when some prior information about \( \theta \) is available in the form of a guess or initial estimate, say \( \theta_0 \). If there would have been no apriori information about \( \theta \), then it is known that the sample mean \( \bar{x} = \frac{1}{n} \sum x_i \) is the best (UMVU) estimator of \( \theta \). However, in the present situation when an initial estimate of \( \theta \) is available, we would like to use the available information in constructing an estimator of \( \theta \). Thompson (1968) proposed an estimator \( T \) for \( \theta \), \( T = K\bar{x} + (1-K)\theta_0 \), in which \( K \) is a
constant lying between 0 and 1 and is to be specified by the experimenter according to his belief in $x$ and $\theta_0$. A value of $K$ near zero implies a strong belief in $\theta_0$, whereas a value of $K$ near one implies a strong belief in $\overline{x}$ (sample values). Attempts have been made to get a value of $K$ for which mean square error (MSE) of $T$ is minimum. The value of $K$ for which the mean square error of $T$ is minimum, unfortunately, comes out to be a function of $\theta$ and hence, remains unknown. Replacing $\theta$ by appropriate estimators (consistent, unbiased etc) one gets a number of adhoc estimators of $K$ and hence for $\theta$. Comparing these estimators in terms of mean square errors, we find that these are preferrable when the sample size is small, and $\theta_0$ is in the vicinity of $\theta$ i.e., $|\theta - \theta_0|$ is small.

Another way of estimating $\theta$ would be to use a pre-test for the hypotheses $H_0: \theta = \theta_0$ against the two sided alternative $H_1: \theta \neq \theta_0$ and depending on the result of the pre-test, we either use $K \overline{x} + (1-K)\theta_0$ or $\overline{x}$ as a pre-test estimator of $\theta$. This has been done by several workers including Bhattacharya and Srivastava (1974), who take $K = 1$.

In Bayesian approach we take prior belief in the form of a distribution called the prior distribution before
the data have actually been obtained or the result of the experiment is known and obtain posterior distribution of \( \theta \), given the observations. With the help of this posterior distribution we obtain Bayes estimator of \( \theta \) having some optimal property. Which prior distribution should one take is always a problem in Bayesian analysis save for asymptotic results. When information about \( \theta \) is 'vague' or practically there is no information about \( \theta \), Jefferey (1961) proposed the prior density \( g(\theta) \propto \sqrt{I(\theta)} \), where \( I(\theta) \) is the Fisher's information. This is known as Jefferey's invariance principle and the use of this principle has resulted in the derivation of a number of interesting and useful estimators. Jefferey's principle includes a wide class of priors including improper or quasi priors. Savage (1962) principle of stable estimation leads one to believe that in a state of ignorance an uniform prior would be a valid choice. In this chapter we have proposed some estimators of \( \theta \) and studied their properties.

2.2. Shrunken estimators

As already remarked in Section 2.1, Thompson (1968) has proposed a shrunken estimator of \( \theta \) which is of the form \( T = \theta_0 + K(\bar{x} - \theta_0), (0 \leq K \leq 1) \). Since \( \theta \) is a scale parameter in the exponential density, we consider another class of estimators
\[ T_1 = \theta_0 (1 + K_1 \mu), \quad (2.2.1) \]

where
\[ \mu = \frac{\theta_0}{n\bar{x}}. \]

The mean square of \( T_1 \) is given by
\[
\text{MSE}(T_1) = E(T_1 - \theta)^2
\]

\[
= \int_0^\infty \left( \theta_0 + K_1 \frac{\theta_0}{n\bar{x}} - \theta \right)^2 f(\bar{x}) \, d\bar{x}
\]

\[
= \int_0^\infty \left( \frac{n}{\theta} \right)^2 \frac{n-1}{\Gamma_n} e^{-n\bar{x}/\theta} \, d\bar{x} + 2(\theta_0 - \theta) \int_0^\infty \theta_0 K_1
\]

\[
= \frac{(n-1)^2}{\theta} e^{-n\bar{x}/\theta} \, d\bar{x} + \frac{K_1^2 \theta_0^2}{\theta^2 \Gamma_n} \int_0^\infty \frac{n}{\theta} \, d\bar{x}
\]

\[
= (\theta_0 - \theta)^2 + 2K_1(\theta_0 - \theta) \frac{\theta_0}{(n-1)} + \frac{K_1^2 \theta_0^2}{\theta^2(n-1)(n-2)}
\]

\[ (2.2.2) \]

\[
= \theta^2 \left[ r^2 + \frac{2K_1r}{(n-1)} (1+r) + \frac{K_1^2(1+r)^2}{(n-1)(n-2)} \right], \quad (2.2.3)
\]
where
\[ r = (\frac{\theta}{\hat{\theta}} - 1). \]

Expression (2.2.3) becomes minimum for \( K_1 = -\frac{(n-2)x}{(1+r)^2} \). Since \( \theta \) is unknown, we cannot know \( r \). If we replace \( \theta \) in \( r \) by MLE of \( \theta, \hat{\theta} \), then \( r \) is estimated by
\[ \hat{r} = (\frac{\theta}{\hat{\theta}} - 1) \] \hspace{1cm} (2.2.4)

Thus, the estimated value of \( K_1 \), say \( K_1^* \) is
\[ K_1^* = -\left[ (n-2) \left( \frac{\theta}{\hat{\theta}} - 1 \right) \right] \left/ \left( \frac{\theta}{\hat{\theta}} \right) \right)^2 \]
\[ = \frac{(n-2)(\bar{x} - \theta_0)\bar{x}}{\theta_0^2} \] \hspace{1cm} (2.2.5)

Substituting this value in \( T_1 \) we get
\[ \hat{T}_1 = \theta_0 + (1 - \frac{2}{n})(\bar{x} - \theta_0) \] \hspace{1cm} (2.2.6)

which has
\[ \text{MSE}(T_1) = \frac{\sigma^2}{n} \left[ \left( \frac{r-2}{n} \right)^2 + \frac{4}{n} r^2 \right]. \] \hspace{1cm} (2.2.7)

The traditional shrunken estimator of \( \theta, Y = \frac{n}{n+1} \bar{x} \) has
mean square error

$$\text{MSE}(Y) = \frac{\sigma^2}{n+1} \quad (2.2.8)$$

Therefore, $T_1$ has smaller mean square than $Y$ if

$$r^2 \leq \frac{n-1}{n} - \frac{n}{4(n+1)} \quad (2.2.9)$$

Thus $T_1$ performs better than $Y$ when $\theta_o$ is in the vicinity of $\theta$ and also when $\theta_o$ is moderately away from $\theta$. From (2.2.9) we observe that as $n \to \infty$, $r^2 \leq .75$. Hence, in large sample $T_1$ is better than $Y$, if $-\frac{\sqrt{3}}{2} \leq r \leq +\frac{\sqrt{3}}{2}$. The estimator proposed by Pandey (1983) is

$$\hat{\theta}_1 = \theta_o + \frac{(\bar{x} - \theta_o)^3}{(\bar{x} - \theta_o)^2 + \bar{x}^2/n} \quad (2.2.10)$$

which performs better than $Y$, if $-.30 \leq r \leq .30$. Thus, the estimator $T_1$ besides being simple to calculate has more applicability. It should be noted that whilst $T_1$ has an improvement over $\hat{\theta}_1$, in this region there may be other estimators of the form $\theta_o + K_1(\bar{x} - \theta_o)$ which are better at least over the part of the region. We can extend $T_1$ by introducing further polynomial terms in $\theta_o$ and $\bar{x}$ to find...
better estimators. Let such an estimator be

\[ T_2 = e_0 (1 + K_1 \mu + K_2 \mu^2) \]  
(2.2.11)

The mean square error of \( T_2 \) can be calculated easily. Its final expression is

\[
\text{MSE}(T_2) = \sigma^2 \left[ r^2 + \frac{K_1^2 (1+r)^4}{(n-1)(n-2)} + \frac{2K_1 r(1+r)^2}{(n-1)} \right. \\
+ \frac{K_2^2 (1+r)^6}{(n-1)(n-2)(n-3)(n-4)} + \frac{2K_2 r(1+r)^3}{(n-1)(n-2)} + \frac{2K_1 K_2 (1+r)^5}{(n-1)(n-2)(n-3)} \left. \right]
\]
(2.2.12)

The values of \( K_1, K_2 \) for which \( \text{MSE}(T_2) \) is minimum can be obtained by solving the equations \( \frac{\text{MSE}}{\partial K_1} = 0, \frac{\text{MSE}}{\partial K_2} = 0 \). These values are

\[
K_1^* = \frac{-2r(n-3)}{(1+r)^2}, \quad K_2^* = \frac{r(n+3)(n+4)}{(1+r)^3}
\]
(2.2.13)

As \( K_1^*, K_2^* \) depend on \( r \)(so on \( \theta \)), they remain unknown unless we replace \( r \) by its estimate. If we substitute
from (2.2.4) in (2.2.11) we get the estimated estimator

\[ T^*_2 = \theta_0 + \left(1 + \frac{1}{n} - \frac{12}{n^2}\right) (\bar{x} - \theta_0) \]  \hspace{1cm} (2.2.14)

which has

\[ \text{MSE}(T^*_2) = \frac{\theta^2}{n} \left[ \left(1 + \frac{1}{n} - \frac{12}{n^2}\right)^2 + \frac{1}{n} \left(1 - \frac{12}{n}\right)^2 \right] \]  \hspace{1cm} (2.2.15)

The estimator \( T^*_2 \) has smaller mean square error than \( Y \) or \( \hat{\theta}_0 \), in a limited range of the parameter space of \( \theta \). Thus a large number of adhoc estimators of the form \( T_1 = \theta_0 + K_1 (\bar{x} - \theta_0) \) could be proposed which are better over part of the region and the fact that the experimenter cannot know the optimum range for each, these are of limited value. It is well known that much improvement can be done if we use Bayesian analysis.

2.3. Bayes estimators

We assume that the prior information on the mean life \( \theta \) given to us in the form of an initial estimate \( \theta_0 \) can be used for the selection of a prior distribution. We take inverted gamma distribution as the prior distribution for \( \theta \) having the mean at \( \theta = \theta_0 \). A random variable
\( \theta \) is said to have inverted gamma distribution with parameters \( \alpha \) and \( \beta \) if

\[
g(\theta) = \frac{1}{\Gamma(\alpha)} \cdot \frac{1}{\theta^{\alpha+1}} \exp\left(-\frac{\theta}{\beta}\right), \quad \theta > 0 \quad (2.3.1)
\]

It has mean \( E(\theta) = \frac{\beta}{\beta - 1} \), variance \( \text{Var}(\theta) = \frac{\theta}{\beta (\alpha - 1)^2 (\alpha - 2)} \), and coefficient of variation \( C.V. = \frac{1}{\sqrt{\alpha - 2}} \). If we have some idea about the coefficient of variation of the prior distribution, it will help us in deciding the appropriate choice of \( \alpha \). Also, we want \( E(\theta) = \theta_0 \) i.e., \( \frac{1}{\beta (\alpha - 1)} = \theta_0 \) and \( \alpha > 2 \) so as to have the variance of prior distribution finite. Then, we should have

\[
g(\theta) = \frac{\alpha - 1}{\alpha \theta^{\alpha}} \exp\left\{-\frac{\alpha - 1}{\alpha} \frac{\theta_0^\alpha}{\theta}\right\}, \quad \theta > 0, \quad \theta_0 > 0, \quad \alpha > 2 \quad (2.3.2)
\]

For the squared error loss function \( L(\theta, d) = (\theta - d)^2 \), the Bayes estimator is the mean of the posterior distribution. Let \( x_1, x_2, \ldots, x_n \) be a random sample of size \( n \) from (2.1.1). Then the posterior density of \( \theta \) given the sample observations will be

\[
f(\theta|x_1, \ldots, x_n) = \frac{\left[\sum x_i + (\alpha - 1)\theta_0\right]}{\Gamma(n+\alpha) \theta^{n+\alpha+1}} \exp\left\{-\frac{\left[\sum x_i + (\alpha - 1)\theta_0\right]}{\theta}\right\}. \quad (2.3.3)
\]
From (2.3.3) we find the Bayes estimator of \( \theta \) (posterior mean) is

\[
B = \theta_0 + (1 - \frac{\alpha - 1}{n + \alpha - 1}) (\bar{x} - \theta_0) \tag{2.3.4}
\]

This is a shrinkage estimator with shrinkage factor

\[
1 - \frac{\alpha - 1}{n + \alpha - 1} = \frac{n}{n + \alpha - 1} < 1,
\]

and will approach to \( \bar{x} \) for large \( n \).

The risk and Bayes risk of \( B \) are respectively

\[
R(\theta, B) = \frac{\theta^2}{n} \left[ \frac{n^2}{(n + \alpha - 1)^2} + \frac{n(\alpha - 1)^2 r^2}{(n + \alpha - 1)^2} \right], \tag{2.3.5}
\]

\[
r(\theta, B) = \frac{(\alpha - 1)\theta_0^2}{(n + \alpha - 1)(\alpha - 3)}, \quad \alpha > 3. \tag{2.3.6}
\]

The Bayes estimator \( B \) will have smaller risk than \( Y \) if

\[
\left( \frac{\theta_0}{\theta} - 1 \right)^2 \leq \left[ \frac{1}{n+1} + \frac{2n}{(n+1)(\alpha - 1)} - \frac{n}{(n+1)(\alpha - 1)^2} \right]. \tag{2.3.7}
\]

As \( n \to \infty \), this reduces to

\[
\left( \frac{\theta_0}{\theta} - 1 \right)^2 \leq \frac{2}{(\alpha - 1)} - \frac{1}{(\alpha - 1)^2} \tag{2.3.8}
\]
The largest gain is obtained for larger values of $\alpha$.

For a given prior $g(\theta)$ the probability that $r$ lies in the given range determined by (2.3.8) is very high.

2.4. Justification for the use of Bayes estimator

The probability that $\theta$ will lie in the interval determined by (2.3.8) is

$$P = \int_{L}^{U} f(\theta/\alpha) \, d\theta, \quad (2.4.1)$$

where

$$L = \theta_0/\left[1+\left\{\frac{1}{n+1} + \frac{2n}{(n+1)(\alpha-1)} - \frac{n}{(n+1)(\alpha-1)^2}\right\}^{1/2}\right],$$

$$U = \theta_0/\left[1-\left\{\frac{1}{n+1} + \frac{2n}{(r+1)(\alpha-1)} - \frac{n}{(n+1)(\alpha-1)^2}\right\}^{1/2}\right].$$

Substituting these values in (2.4.1) and for $f(\theta/\alpha)$, we get

$$P = \int_{(\alpha-1)\left[1+\left\{\frac{1}{n+1} + \frac{2n}{(n+1)(\alpha-1)} - \frac{n}{(n+1)(\alpha-1)^2}\right\}^{1/2}}^{(\alpha-1)\left[1-\left\{\frac{1}{n+1} + \frac{2n}{(n+1)(\alpha-1)} - \frac{n}{(n+1)(\alpha-1)^2}\right\}^{1/2}} \frac{e^{-y}}{\Gamma} \frac{y^{\alpha-1}}{\Gamma} \, dy \quad (2.4.2)$$

Values of (2.4.2) or $P$ for different values of $n$ and $\alpha$ can
be calculated using tables of incomplete gamma distribution. In Table 2.1, we have calculated the values of $P$ for $n = 5, 10, \infty$, $\alpha = 3, 4, 9, 15, 19, 25$.

Table 2.1
Values of probability

<table>
<thead>
<tr>
<th>$n$</th>
<th>3</th>
<th>4</th>
<th>9</th>
<th>15</th>
<th>19</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>.7227</td>
<td>.7780</td>
<td>.8830</td>
<td>.9347</td>
<td>.9534</td>
<td>.9731</td>
</tr>
<tr>
<td>10</td>
<td>.7189</td>
<td>.7682</td>
<td>.8218</td>
<td>.9006</td>
<td>.9189</td>
<td>.9399</td>
</tr>
<tr>
<td>$\infty$</td>
<td>.7145</td>
<td>.7542</td>
<td>.8102</td>
<td>.8169</td>
<td>.8243</td>
<td>.8356</td>
</tr>
</tbody>
</table>

From Table 2.1 we find that the probability of $\theta$ belonging to interval determined by (2.3.8) is very high for smaller values of $n$ and large values of $\alpha$. The last row in Table 2.1 shows the lower bounds for the probability that $B$ will be a better estimator of $\theta$ than $Y$. We, therefore, conclude that $B$ with larger values of $\alpha$ can be used when $\theta_0$ is far away from $\theta$ and its performance then is better than both $Y$ as well as $\widehat{\theta}_1$.

Pandey (1979) considered an estimator of the form

$$T' = c \left[ \frac{n \bar{x}}{n + \alpha - 1} + \frac{\alpha - 1 \theta_0}{n + \alpha - 1} \right], \quad 0 \leq c \leq 1 \quad (2.4.3)$$
and studied its properties. He investigated whether there is a \( c \) for which \( T \) has higher probability of being a better estimator of \( \theta \) than \( Y \). For a set of values of \( n \), and \( c \) he has calculated the probability that \( \theta \) will lie in the interval

\[
\frac{1-cK}{(1-K)c} \pm \left[ \left\{ \frac{1}{(n+1)} - \frac{c^2K^2}{n} \right\} \frac{1}{(1-K)^2} \right]^{1/2} \tag{2.4.4}
\]

and found that the best choice of \( c \) is 1. We have, therefore, limited our study of \( B \) by taking \( c = 1 \) only.

2.5. Two parameter exponential distribution

We shall now consider estimation of parameters from a two parameter exponential distribution with probability density function

\[
f(x/\mu, \theta) = \frac{1}{\theta} \exp - \left( \frac{x-\mu}{\theta} \right), \quad -\infty < x < \infty, \quad \theta > 0 \tag{2.5.1}
\]

Let \( X_1 \leq X_2 \leq \cdots \leq X_r \), \( r \leq n \) denote the first \( r \) ordered statistics in a random sample of size \( n \) from (2.5.1).

Alam and Kirmani (1981) assume that the parameters \( \mu \) and \( \theta \) in (2.5.1) are the realized values of independent random variables having the respective quasi densities
Using these prior quasi densities they have obtained Bayes estimators of \( \mu, \theta, \mu + \theta, \) and \( \theta^2 \) of a two parameter exponential distribution from censored samples. The parametric functions \( \mu + \theta \) and \( \theta^2 \) are also of importance as they represent respectively the mean and variance of the r.v. \( X \) in (2.5.1). The estimators of \( \mu, \theta, \mu + \theta, \) and \( \theta^2 \) given by Alam and Kirmani (1981) are

\[
\mu^* = x(1) - \frac{S_r - x(1)}{a+r-3},
\]

(2.5.4)

\[
\theta^* = \frac{n(S_r - x(1))}{a+r-3},
\]

(2.5.5)

\[
(\mu + \theta)^* = \frac{(n-1)(S_r - x(1))}{a+r-3} + x(1),
\]

(2.5.6)

\[
\theta^{2*} = \frac{\{n(S_r - x(1))\}^2}{(a+r-3)(a+r-4)},
\]

(2.5.7)

where

\[
S_r = \frac{1}{n} \left[ \sum_{i=1}^{r} x(i) + (n-r)x(r) \right], \quad r + a > 4.
\]
These estimators have either upward or downward biases. Hence in order to reduce the biases in (2.5.4) to (2.5.7) we have taken the quasi-prior densities of $\mu$ and $\theta$ as follows:

$$
\begin{align*}
    g_1(\mu) &= 1, \quad -\infty \leq \mu \leq \infty, \\
    g_2(\theta) &= \frac{1}{\theta^a} e^{-b/\theta}, \quad \theta > 0, \quad a > 0, \quad b \geq 0
\end{align*}
$$

(2.5.8)

and have obtained Bayes estimates of $\mu$, $\theta$, $\mu + \theta$ and $\theta^2$. We have derived expressions for biases and mean square errors of the Bayes estimators and have compared them with the estimators proposed by Alam and Kirmani (1981).

2.5.1. Bayes estimators

The likelihood function in censored sampling from (2.5.1) is given by

$$
\begin{align*}
    f(x^*/\mu, \theta) = L(x^*/\mu, \theta) &= \frac{n!}{(n-r)! \theta^r} \\
    &\times \exp \left\{ - \sum_{i=1}^{n} \frac{(x(i) - \mu)^2 + (n-r)(x(r) - \mu)}{\theta} \right\} \\
    &= \frac{K}{\theta^r} \exp \left\{ - \frac{r (s_r + (x(1) - \mu))}{\theta} \right\}
\end{align*}
$$

(2.5.1.1)
where $K$ is a normalizing constant and $x^* = (x_{(1)} \leq x_{(2)} \ldots x_{(r)})$. If we take the prior densities (2.5.8), we get

the posterior distribution of $(\mu, \theta)$ given by

$$
\Pi(\mu, \theta | x^*) = \frac{n \{n(S_r - x_{(1)}) + b\}^{a+r-2}}{\theta^{r+a}} \exp \left\{ - \frac{n(S_r - x_{(1)}) + b}{\theta} \right\}
$$

$$
x_{(1)} \geq \mu, \quad \theta > 0, \quad r+a \geq 2 \quad (2.5.1.2)
$$

The marginal posterior of $\mu$ is

$$
\Pi(\mu | x^*) = \int_0^\infty \Pi(\mu, \theta | x^*) d\theta
$$

$$
\varphi_\mu = \frac{(r+a-2)(S_r - x_{(1)})^{r+a-2}}{(S_r - \mu)_+^{r+a-1}}, \quad -\infty < \mu < x_{(1)}, \quad (2.5.1.3)
$$

Similarly, the marginal posterior of $\theta$ is

$$
\Pi(\theta | x^*) = \int_{-\infty}^{x_{(1)}} \Pi(\mu, \theta | x^*) d\mu
$$

$$
= \frac{n(S_r - x_{(1)})}{\theta^{r+a-1}} \exp \left\{ - \frac{n(S_r - x_{(1)}) + b}{\theta} \right\}, \quad \theta > 0
$$

(2.5.1.4)
The Bayes estimators of $\mu$, $\theta$, $\mu+\theta$ and $\theta^2$ can be worked out easily and these are as follows:

$$\mu^* = x(1) - \frac{S_r - x(1)}{a+r-3} - \frac{b}{a+r-3}, \quad a+r > 3 \quad (2.5.1.5)$$

$$\theta^* = \frac{n(S_r - x(1))}{a+r-3} + \frac{b}{a+r-3}, \quad a+r > 3 \quad (2.5.1.6)$$

$$(\mu + \theta)^* = \frac{(n-1)(S_r - x(1))}{a+r-3} + x(1), \quad a+r > 3, \quad (2.5.1.7)$$

and

$$\theta^2 = \frac{[n(S_r - x(1)) + b]^2}{(a+r-3)(a+r-4)}, \quad a+r > 4 \quad (2.5.1.8)$$

If we work out the expected values of estimators (2.5.1.5) to (2.5.1.8), we get

$$E(\mu^*) = \mu + \frac{(a-2)\theta}{n(a+r-3)} - \frac{b}{r+a-3}, \quad (2.5.1.9)$$

$$E(\theta^*) = \frac{(r-1)\theta}{(r+a-3)} + \frac{b}{(r+a-3)}, \quad (2.5.10)$$
From these expressions we can find out expressions for
biased easily. For example, from (2.5.1.9) and (2.5.1.10)
we observe that the biases in \( \mu^* \) and \( \theta^* \) are respectively

\[
\text{Bias (} \mu^* \text{)} = \frac{\theta(a-2)}{n(a+r-3)} - \frac{b}{(a+r-3)} ,
\]

\[
\text{Bias (} \theta^* \text{)} = \frac{\theta(2-a)}{r+a-3} + \frac{b}{(a+r-3)}
\]

They have smaller biases than \( \mu \) and \( \theta \).

2.5.2. Mean square errors of the estimators

If we define

\[
U = \frac{2n(x(1) - \mu)}{\theta}, \quad V = \frac{2n(\sum_r - x(1))}{\theta}
\]
then $U$ is distributed $\chi^2$ with 2 degrees of freedom and $V$ is distributed $\chi^2$-with $2(r-1)$ degrees of freedom. Further $U$ and $V$ are independent. We may then derive mean square errors of the estimators using the relation

$$\text{Mean square Error (t) = Var(t) + \left[ \text{Bias(t)} \right]^2}$$

If we work out mean square errors of the estimators, we get the final expressions as follows

$$\text{MSE( } \mu \text{**) = } \frac{(r+a-3)^2 + (r-1) + (a-2) \frac{\theta^2}{n^2} + b^2 + 2(2-a)b}{(r + a - 3)^2}, \quad (2.5.2.2)$$

$$\text{MSE( } \theta \text{**) = } \frac{1}{(r+a-3)^2} \left[ (r-1) + ((2-a)\theta+b)^2 \right], \quad (2.5.2.3)$$

$$\text{MSE( } \mu + \theta \text{**) = } \frac{(n-1)^2 (r-1) + (a-2)^2 + (r+a-3)^2 \theta^2}{n^2 (r+a-3)^2}, \quad (2.5.2.4)$$

$$\text{MSE( } \theta^2 \text{**) = } \left[ 2r(r-1)(2r+1) + \left\{ r(r-1)-(r+a-3)(r+a-4) \right\}^2 \right] \theta^4$$
2.5.3. Comparison with Alam and Kirmani's estimators

If we put $b = 0$, we get the results of Alam and Kirmani (1981). From (2.5.1.12) and (2.5.1.13) we see that $\mu^*$ and $\theta^*$ have smaller biases than $\mu$ and $\theta$. Also, we have

$$\text{MSE}(\mu^*) \leq \text{MSE}(\mu)\;\text{if}\;0 \leq b \leq \frac{2(a-2)\theta}{n}, \quad (2.5.3.1)$$

$$\text{MSE}(\theta^*) \leq \text{MSE}(\theta^*), \text{if}\;0 \leq b \leq 2(a-2)\theta, \quad (2.5.3.2)$$

$$\text{MSE}(\theta^{**}) \leq \text{MSE}(\theta^{2*}), \text{if}\;b^3 + 4(r-1)b^2\theta + b\theta^2\{6r(r-1)-2(r+a-3)
(r+a-4)\} + 4(r-1)\theta^3 \{r(r+1)
- (r+a-3)(r+a-4)\} \leq 0 \quad (2.5.3.3)$$

Further,
Bias (\(\mu^*\)) = 0, if \(b = (a-2)\theta/n\),

Bias (\(\theta^*\)) = 0, if \(b = (2-a)\theta\), \hspace{1cm} (2.5.3.4)

Bias (\(\mu + \theta\)) = 0, if \(b = (2-a)\).}

For \(a = 2, b = 0\), the Bayes estimators \(\mu^*, \theta^*, (\mu + \theta)^*\) reduce to MVU estimators respectively.

For \(a = 3, b = 0\), the Bayes estimator \(\theta^{2**}\) reduces to MVUE.

For \(a = 5, b = 0, r = n\), the Bayes estimator \(\theta^{2**}\) reduces to minimum mean square error (MMSE) estimator.

\[\theta^2 = \frac{n^2(x - x_1)^2}{(n+1)(n+2)}\] \hspace{1cm} (2.5.3.5)

which is the same as in Pandey and Singh (1977). For other choices of \(a\) we can also find \(b\) for which the proposed estimators will be either minimum variance unbiased or minimum mean square error estimators. For example, if \(a = 3\), we can select \(b = \frac{\hat{\theta}}{n}\) (\(\hat{\theta}\) being MVUE) to get minimum variance unbiased estimator of \(\mu\). This is
Thus we conclude that with proper choices of $a$ and $b$, Bayes estimators of parameters or parametric functions superior to the estimators proposed by Alam and Kirmani (1981) can be obtained.