Introduction :-

As we are interested to study the axially symmetric space-time therefore it is essential to know the general nature of an axisymmetric metric. Here in this chapter, let us present some important results about the axisymmetric metric given by Chandrasekhar[1]. After that a brief information about geometry of the axisymmetric space-time metric is presented. In the first article of this chapter, deduction of non-static axially symmetric space-time metric is presented. The second article is about the tetrad formalism. In this article, brief discussion about curvature one-form, curvature two-form and calculations of Riemann curvature tensor and Ricci-tensor for non-static axisymmetric space-time have been given. In the third article, the equations of null geodesics for the same metric are presented and then discussion which is necessary for study of black-hole and Petrov classification [2] of the space-time is given. Later the Kerr [3] metric and its different forms, which are useful to understand the symmetry and rational nature of the space-time are presented. In the last article of the introductory chapter, a brief discussion about the gravitational collapse of spherically symmetric and an
axially symmetric space-time is given. Geometry of the Kerr black-hole is also presented.

1.1 Tetrad Formalism:

In the Riemannian geometry we have $n$ linearly independent vector fields $e^{(a)}$, $a = 1, 2, 3, \ldots, n$, which form the basis for the space. Covariant derivative of $e^{(a)}$ expressed in terms of rotation coefficients $\Gamma^a_{bc}$ are as follows

$$e^{(a)}_{\alpha;\beta} = -\Gamma^a_{bc} e^{(b)}_\alpha e^{(c)}_\beta.$$ ...

The Ricci rotation coefficients $\Gamma^a_{bc}$ associated with the basis vector $e^{(a)}$ are a set of $n^3$ functions. A set of 1-forms is defined as

$$\theta = e^{(a)}_\alpha dx^\alpha.$$ ...

Associate to each basis vector $e^{(a)}$, we have 1-form

$$\theta^a = e^{(a)}_\alpha dx^\alpha.$$ ...

Here and what follows the Latin letters $a, b, c, \ldots$ represent tetrad components and the Greek letters $\alpha, \beta, \ldots$ represent tensor components. Connection 1-forms are defined as

$$\omega^a_b = \Gamma^a_{bc} \theta^c.$$ ...

Similarly, 2-forms associated with the frame components

$$R^{(a)}_{(b)(c)(d)}$$

of the curvature tensor are defined as

$$\Omega^a_b = R^{(a)}_{(b)(c)(d)} \theta^c \wedge \theta^d.$$ ...
The relation between tetrad and tensor components is

\[ T_{(a)(b)\cdots} = T_{\alpha\beta\cdots} e^{(a)}_{\alpha} e^{(b)}_{\beta} \cdots \quad (1.1.6) \]

Let us introduce the matrix of scalar products of the basis vectors as

\[ g^{(a)(b)} = e_{(a)} e^{(b)} \quad \cdots \quad (1.1.7) \]

Then by definition (1.1.6)

\[ g^{(a)(b)} = g_{\alpha\beta} e^{(a)}_{\alpha} e^{(b)}_{\beta} \quad \cdots \quad (1.1.8) \]

Let us define the dual basis

\[ e^{(a)} = g^{(a)(b)} e^{(b)} \quad \cdots \quad (1.1.9) \]

It's clear that

\[ e_{(a)} e^{(b)} = \delta^{a}_{b} \quad \cdots \quad (1.1.10) \]

Therefore, \( e^{(a)}_{\alpha} \) and \( e_{(b)}^{\beta} \) are inverse matrices to each other.

Inverse of one form defined in (1.1.3) is

\[ dx^{\alpha} = e_{(a)}^{\alpha} \theta^{a} \quad \cdots \quad (1.1.11) \]

Hence

\[ ds^{2} = g_{\alpha\beta} dx^{\alpha} dx^{\beta} \]

\[ = g_{\alpha\beta} e^{(a)}_{\alpha} e^{(b)}_{\beta} \theta^{a} \theta^{b} \]

\[ = g^{(a)(b)} \theta^{a} \theta^{b} \quad \cdots \quad (1.1.12) \]
Now from the equation (1.1.1), it is easy to obtain the result

$$\Gamma^a_{bc} = e^{(a)}_{\beta;\alpha} e^{(\beta)}_{(b)} e^{(\alpha)}_{(c)} \cdots \cdots \cdots (1.1.13)$$

where semi-colon indicates covariant derivative. Use of equation (1.1.8) in equation (1.1.13) gives

$$W^a_b = -e^{(a)}_{\beta;\alpha} e^{(\beta)}_{(b)} e^{(\alpha)}_{(c)} \cdots \cdots \cdots (1.1.14)$$

where De^{(a)} represents the covariant differentiation of e^{(a)}, but

$$e^{(a)} e^{(b)} = g^a_b = \delta^a_b$$

$$\Rightarrow D(e^{(a)} e^{(b)}) = 0.$$  

$$\Rightarrow De^{(a)} e^{(b)} + e^{(a)} De^{(b)} = 0.$$  

$$\Rightarrow De^{(a)} e^{(b)} = -De^{(b)} e^{(a)} \cdots \cdots \cdots (1.1.15)$$

Therefore

$$W^a_b = -De^{(a)} e^{(b)} = De^{(b)} e^{(a)} \cdots \cdots \cdots (1.1.16)$$

and

$$De^{(b)} = W^a_b e^{(a)} \cdots \cdots \cdots (1.1.16)$$
Since \( g_{(a)(b)} \) are scalars and hence

\[
dg_{(a)(b)} = D g_{(a)(b)}
= D \left( e_{(a)} e_{(b)} \right)
= e_{(a)} D e_{(b)} + e_{(b)} D e_{(a)}
= e_{(a)} W^c_{\ b} e_{(c)} + e_{(b)} W^c_{\ a} e_{(c)}
= g_{(a)(c)} W^c_{\ b} + g_{(b)(c)} W^c_{\ a}
= W^a_{\ ab} + W^b_{\ ba}.
\]

(1.1.17)

Usually we choose \( g_{(a)(b)} \) as constants. Therefore

\[
W^a_{\ ab} = - W^a_{\ ba}.
\]

(1.1.18)

Hence, there are only six non-trivial connection 1-forms \( W^a_{\ b} \), instead of forty Christoffel symbols in traditional tensor approach. By definition the exterior derivative of one form \( \Theta^a \) is

\[
d\Theta^a = 2! e_{(\ a} \partial_{\beta}) \ dx^\beta \ dy^\alpha,
\]

where square brackets indicates anti-symmetrization, i.e.

\[
dx^\beta \ dy^\alpha = (1/2!) \left\{ dx^\beta \ dy^\alpha - dx^\alpha \ dy^\beta \right\}.
\]

Then

\[
d\Theta^a = 2! \Gamma^a_{\ bc} e_{(b} e_{(c)} \ dx^\beta \ dy^\alpha
= - \Gamma^a_{\ bc} \Theta^c \wedge \Theta^b.
\]

\( \ldots \ldots \ldots (1.1.19) \)
Therefore
\[ d\theta^a = -\left( W^a_b \wedge \theta^b \right). \] ... ... ... (1.1.20)

Equation (1.1.20) is known as Cartan's first equation of structure.

Let us derive the simple relation (second equation of structure) which connects the curvature tensor to the exterior differentials of the connection forms \( W^a_b \). By the definition
\[ W^a_b = \Gamma^a_{bc} \theta^c = \Gamma^a_{bc} e^{(c)} \alpha dx^\alpha. \]

The exterior differentiation of this equation gives
\[ dW^a_b = 2! \left[ \Gamma^a_{bc} e^{(c)} \alpha \right]_{;\beta} dx^{[\beta} dy^{\alpha]} \] ... ... ... (1.1.21)

But we know that
\[ e^{(a)}_{\alpha;\beta;\delta} = e^{(a)}_{\alpha;\delta;\beta} = e^{(a)} e^\epsilon R^\epsilon_{\alpha\beta\delta}. \]

Therefore
\[ 2e^{(a)}_{\alpha;[\beta;\delta]} = e^{(a)} e^\epsilon R^\epsilon_{\alpha\beta\delta}. \] ... ... ... (1.1.22)

Now covariant differentiation of
\[ e^{(a)}_{\alpha;\beta} = -\Gamma^a_{bc} e^{(c)}_{\beta} e^{(b)} \alpha \]
gives
\[ e^{(a)}_{\alpha;\beta;\delta} = -\left( \Gamma^a_{bc} e^{(c)}_{\beta} \right)_{;\delta} e^{(b)}_{\alpha} - e^{(b)}_{\alpha} \left( \Gamma^a_{bc} e^{(c)}_{\beta} \right)_{;\delta}. \] ... ... ... (1.1.23)
By virtue of equation (1.1.1) the last term of equation (1.1.23) can be written as (with a change of dummy indices)

$$- \Gamma^a_{hc} e^{(c)}_e \delta = - \left( \Gamma^a_{hc} e^{(c)}_e \right) \left( - \Gamma^h_{bd} e^{(b)}_e \delta \right)$$

Put this in equation (1.1.23) and multiply both sides by $dx^\beta dy^\delta$, we get

$$e^{(a)}_{\alpha;\beta;\delta} dx^\beta dy^\delta = \left\{ - \left( \Gamma^a_{hc} e^{(c)}_e \right) \delta dx^\beta dy^\delta \right\}$$

$$+ \left( \Gamma^a_{hc} e^{(c)}_e \right) \left( \Gamma^h_{bd} e^{(d)}_e \delta \right) dx^\beta dy^\delta \right\} e^{(b)}_\alpha$$

Using equations (1.1.21) and (1.1.22) we write

$$\frac{1}{2} e^{(a)}_\xi R^{\xi}_{\alpha\beta\delta} dx^\beta dy^\delta = \left\{ dW^a_b + W^a_c \Lambda W^c_b \right\} e^{(b)}_\alpha \quad \ldots \ldots \ldots (1.1.24)$$

Now

$$\hat{\Omega}^a_{\beta} = \frac{1}{2} R^{(a)}_{(b)(c)(d)} \delta^\beta \Lambda \delta^d.$$

$$= \frac{1}{2} R^{(a)}_{(b)(c)(d)} \left\{ 2! e^{(c)}_\delta e^{(d)}_\delta dx^\beta dy^\delta \right\}.$$

$$= e^{(a)}_\xi R^{\xi}_{\alpha\beta\delta} e^{(b)}_\alpha dx^\beta dy^\delta \quad \ldots \ldots \ldots (1.1.25)$$

From equations (1.1.24) and (1.1.25) we can write

$$\hat{\Omega}^a_{\beta} = dW^a_b + W^a_c \Lambda W^c_b \quad \ldots \ldots \ldots (1.1.26)$$
Equation (1.1.26) is known as second equation of Cartan. Now the exterior derivative of both sides of equation (1.1.26) gives

\[ d\Omega^a_b = d\omega^a_b + d(\omega^a_c \Lambda \omega^c_b) \]

\[ = d\omega^a_c \Lambda \omega^c_b - \omega^a_c \Lambda d\omega^c_b \]

\[ = \left( \Omega^a_c - \omega^a_d \Lambda \omega^d_c \right) \Lambda \omega^c_b - \omega^a_c \Lambda \left( \Omega^c_b - \omega^c_d \Lambda \omega^d_b \right) \]

\[ = \Omega^a_c \Lambda \omega^c_b - \omega^a_c \Lambda \Omega^c_b. \]

Thus

\[ d\Omega^a_b = \left( \Omega^a_c \Lambda \omega^c_b - \omega^a_c \Lambda \Omega^c_b \right) \cdots \cdots \cdots (1.1.27) \]

These are the Bianchi-identities. For another identity, consider the first equation of structure,

\[ d\theta^a_b = -\omega^a_b \Lambda \theta^b. \]

Take exterior differentiation

\[ d^2\theta^a_b = - d\omega^a_b \Lambda \theta^b + \omega^a_b \Lambda d\theta^b \]

\[ = - \left( \Omega^a_b \Lambda \omega^c_b \right) \Lambda \theta^b - \omega^a_b \Lambda \omega^b_c \Lambda \theta^c. \]

(Because \( d^2\theta^a_b = 0 \) and by using equations (1.1.26) and (1.1.20)).

Therefore

\[ \Omega^a_b \Lambda \theta^b = 0. \cdots \cdots \cdots (1.1.28) \]
This can be written as

\[ R(a)(b)(c)(d) \epsilon^c \Lambda \epsilon^d \Lambda \epsilon^b = 0. \]

This implies a cyclic identity

\[ R_{[abcd]} = 0 \]

and all symmetries of Riemann tensor

\[ R(a)(b)(c)(d) = - R(b)(a)(c)(d) \]
\[ R(a)(b)(c)(d) = - R(a)(b)(d)(c) \]
\[ R(a)(b)(c)(d) = R(c)(d)(a)(b). \]

1.2 Non-static axially symmetric space-time metric :-

For the deduction of non-static axially symmetric space-time metric Hartle and Sharp [4], Bardeen [5], Lewis [6], Papetrou [7], Cohen and Brill [8], Leavy [9] have given some calculations.

We are restricting ourselves to system which is axially symmetric at all times, hence the metric tensor is independent of one of the coordinate \( \phi \). The coordinate \( \phi \) is cyclic. (in the sense that we regain the same event if it is increased by \( 2\pi \) keeping other three coordinates fixed).

Let the space like coordinates besides \( \phi \), be \( x^2 \) and \( x^3 \) and \( t (=x^0) \) the time like coordinate. Now contravariant form of the
The metric tensor is
\[ d s^2 = g^{ij} d x_i d x_j \] .......................... (1.2.1)

Since all the components of the metric \( g^{ij} \) are independent of \( \phi \), it is clear that by a transformation involving only three coordinates \((t, x^2, x^3)\), the \(3 \times 3\) matrix \( g^{ij} \) where \( i, j = 0, 2, 3 \) reduces to its diagonal form, hence
\[ g^{02} = g^{03} = g^{23} = 0. \] .......................... (1.2.2)

Let us take
\[ g^{00} = e^{-2\nu}, \]
\[ g^{22} = e^{-2\mu_2}, \]
\[ g^{33} = e^{-2\mu_3}. \] .......................... (1.2.3)

where \( \nu, \mu_2 \) and \( \mu_3 \) are functions of \( t, x^2 \) and \( x^3 \). Now write the remaining coefficients of \( g^{ij} \) in the forms
\[ g^{01} = -\omega e^{-2\nu}, \]
\[ g^{12} = q_2 e^{-2\mu_2}, \]
\[ g^{13} = q_3 e^{-2\mu_3}, \]
\[ g^{11} = e^{-2\nu} - \omega^2 e^{-2\nu} + q_2^2 e^{-2\mu_2} + q_3^2 e^{-2\mu_3}. \] .......................... (1.2.4)

We have not restricted the gauge unduly in considering equations (1.2.2), (1.2.3) and (1.2.4). We have only used the possibility of reducing \( g^{02}, g^{03} \) and \( g^{23} \) simultaneously to zero by coordinate
transformation involving only $t, x^2$ and $x^3$. The covariant form of
the metric (1.2.1) becomes

$$ds^2 = - e^{2\nu} dt^2 + e^{2\psi} (d\phi - q_2 dx^2 - q_3 dx^3 - \omega dt)^2$$

$$+ e^{2\mu_2} (dx^2)^2 + e^{2\mu_3} (dx^3)^2.$$ ........... (1.2.5)

Here in this metric there are seven functions namely $\nu, \psi, q_2, q_3,$
$\omega, \mu_2$ and $\mu_3$ and on the other hand Einstein field equations
provide only six linearly independent equations, so it is obvious
that these seven functions can't be independent functions. In fact
there are only six independent quantities which can be seen as
under. Consider the coordinate transformation

$$\phi = \phi' + f(t', r', \theta')$$

$$t = t', \ r = r', \ \theta = \theta'$$ ........... (1.2.6)

then

$$g_{\phi'\phi'} = g_{\phi\phi'},$$

$$g_{i'j'} = g_{ij} \quad (i, j = t, r, \theta)$$

and

$$g_{i'1'} = g_{i1} + \left( \frac{\partial f}{\partial x^i} \right) g_{11},$$ ........... (1.2.7)

where $i = t, r, \theta$. From above relations we can say that the
functions $q_2, q_3$ and $\omega$ as they appear in equation (1.2.5), can
occur in the field equations only in the combinations

\[
  h_t = \frac{\partial q_2}{\partial \theta} - \frac{\partial q_3}{\partial r},
\]

\[
  h_r = \frac{\partial \omega}{\partial r} - \frac{\partial q_2}{\partial t},
\]

\[
  h_\theta = \frac{\partial \omega}{\partial \theta} - \frac{\partial q_3}{\partial t}.
\]

(1.2.8)

From these three equations we can find an identity

\[
  \frac{\partial h_r}{\partial \theta} - \frac{\partial h_\theta}{\partial r} = - \frac{\partial h_t}{\partial t}.
\]

(1.2.9)

Accordingly, the seven functions \( \nu, \psi, \mu_2, \mu_3, \omega, q_2 \) and \( q_3 \) will appear in the field equations only as six independent quantities.

It will be observed that the metric (1.2.5) includes the form

\[
  ds^2 = - e^{2\nu} dt^2 + e^{2\psi} (d\phi - \omega dt)^2 + e^{2\mu_2} (dx^2)^2
\]

\[+ e^{2\mu_3} (dx^3)^2, \quad \ldots \quad \ldots \quad (1.2.10)\]

which is appropriate metric for a stationary axisymmetric system.

In the stationary case, one has additional freedom of gauge to restrict the functions \( \mu_2 \) and \( \mu_3 \) by a coordinate condition

\[
  e^{\mu_3} = x^2 e^{\mu_2}, \quad \ldots \quad \ldots \quad (1.2.11)
\]

Now taking \( \mu_2 = \beta \) and \( e^{\mu_3} = r e^{\beta} \) the metric reduces to the form

\[
  ds^2 = - e^{2\nu} dt^2 + e^{2\sigma} (d\phi - \omega dt)^2 + e^{2\beta} (dr^2 + r^2 d\theta^2)
\]

\[\ldots \quad \ldots \quad (1.2.12)\]

where \( \psi, \sigma, \omega \) and \( \beta \) are functions of \( r, \theta \) and \( t \).
Let us obtain Riemann and Ricci-tensor by using tetrad method introduced in section (1.1), by defining the basis 1-form for the given metric (1.2.12) as

\[ \theta^1 = e^\sigma (d\phi - \omega dt), \]
\[ \theta^2 = e^\beta \, dr, \]
\[ \theta^3 = r \, e^\beta \, d\theta, \]
\[ \theta^0 = e^\psi \, dt. \] \hspace{1cm} \ldots \ldots \ldots (1.2.13)

The inverse relations are

\[ dt = e^{-\psi} \, \theta^0, \]
\[ dr = e^{-\beta} \, \theta^2, \]
\[ d\theta = r^{-1} \, e^{-\beta} \, \theta^3, \]
\[ d\phi = \omega \, e^{-\psi} \, \theta^0 + e^{-\sigma} \, \theta^1. \] \hspace{1cm} \ldots \ldots \ldots (1.2.14)

The metric (1.2.12) transforms to the form

\[ ds^2 = -(\theta^0)^2 + (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2. \] \hspace{1cm} \ldots \ldots \ldots (1.2.15)

We will obtain the exterior derivatives of Cartan's 1-form \( \theta^a \) as follows

\[ d\theta^0 = - \left[ e^{-\beta} \psi_2 \, \theta^0 \wedge \theta^2 + \frac{1}{r} \, e^{-\beta} \psi_3 \, \theta^0 \wedge \theta^3 \right]. \] \hspace{1cm} \ldots \ldots (1.2.16.a)
\[
d\theta^1 = e^\sigma - \beta - \psi \left( \omega_2 \theta^0 \Lambda \theta^2 + \omega_3 \frac{1}{r} \theta^0 \Lambda \theta^3 \right) + \sigma_0' e^{-\psi} \theta^0 \Lambda \theta^1 \\
- \sigma_2' e^{-\beta} \theta^1 \Lambda \theta^2 - \sigma_3' \frac{1}{r} e^{-\beta} \theta^1 \Lambda \theta^3, \ldots \ldots \ (1.2.16.b)
\]
\[
d\theta^2 = \beta_3' \frac{1}{r} e^{-\beta} \theta^3 \Lambda \theta^2 + \beta_0' e^{-\psi} \theta^0 \Lambda \theta^2, \ldots \ldots \ (1.2.16.c)
\]
\[
d\theta^3 = e^{-\beta} \left( \beta_2' + \frac{1}{r} \right) \theta^2 \Lambda \theta^3 + \beta_0' e^{-\psi} \theta^0 \Lambda \theta^3, \ldots \ldots \ (1.2.16.d)
\]

Now

\[
g(a)(b) = \text{diag}(-1, 1, 1, 1).
\]

\[
W^a_b = g(a)(c) W^c_b \quad \text{and} \quad W^a_b = -W^b_a.
\]

(because \(g(a)(b)\) are constants)

Hence

\[
W^0_0 = W^1_1 = W^2_2 = W^3_3 = 0,
\]

\[
W^1_2 = -W^2_1, \quad W^3_1 = -W^1_3, \quad W^1_0 = -W^0_1,
\]

\[
W^2_3 = -W^3_2, \quad W^0_2 = -W^2_0, \quad W^0_3 = -W^3_0.
\]

Now

\[
W^0_0 = g^{(0)(c)} W^0_0 = g^{(0)(0)} W^0_0 = (-1) W^0_0 = 0.
\]

14
Similarly

\[ w_1 = w_2 = w_3 = 0. \]

\[ w_0 = -w_1^0, \quad w_2^0 = -w_2^0, \quad w_3^0 = -w_3^0. \]

\[ w_2^1 = -w_1^2, \quad w_3^1 = -w_1^3, \quad w_2^3 = -w_3^2. \]

Now from the Cartan's first equation of structure,

\[ d\theta^a = -w^a_b \Lambda \theta^b \]

\[ = -w_0^a \Lambda \theta^b = w_1^0 \Lambda \theta^2 - w_2^1 \Lambda \theta^3 - w_3^0 \Lambda \theta^0. \]

Comparing this result with equation (1.2.16.b) we get

\[ w_0^1 = -w_1^2 = \alpha, \quad e^{-\beta} \theta^1 = -\frac{\omega}{2} \theta^0. \]

Thus the connection 1-form \( w^a_b \) are given by

\[ w_0 = w_1 = w_2 = w_3 = 0. \]

\[ w_0^1 = w_2 = w_3 = 0. \]

\[ w_0^2 = w_1^2 \]
\begin{align*}
    W^1_0 &= -W^0_1 = \sigma_0 e^{-\psi} \theta^1 + \frac{\omega_2}{2} e^\sigma - \beta - \psi \theta^2 \\
    &\quad + \frac{\omega_3}{2} e^\sigma - \beta - \psi \theta^3, \quad \ldots \ldots \quad (1.2.18.d)\\
    W^2_3 &= -W^3_2 = \left[ \frac{1}{r} \sigma_3 \theta^2 - \left( \beta_2 + \frac{1}{r} \right) \theta^3 \right] e^{-\beta}, \quad \ldots \ldots \quad (1.2.18.e)\\
    W^0_2 &= -W^2_0 = \frac{\omega_2}{2} e^{-\beta - \psi} \theta^1 + \psi_2 e^{-\psi} \theta^2 - \psi_2 e^{-\beta} \theta^0, \\
    &\quad \ldots \ldots \quad (1.2.18.f)\\
    W^3_0 &= -W^0_3 = \frac{\omega_3}{2} e^\sigma - \beta - \psi \theta^1 + \beta_0 e^{-\psi} \theta^3 - \frac{\psi_3}{r} e^{-\beta} \theta^0, \\
    &\quad \ldots \ldots \quad (1.2.18.g)
\end{align*}

Now by the definition of the second equation of structure

\begin{align*}
    \Omega^a_b &= \frac{1}{2} R^{(a)}_{(b)(c)(d)} \theta^c \Lambda^d = dW^a_b + W^a_c \Lambda^b_c \\
    &\quad \ldots \ldots \quad (1.2.19)
\end{align*}

The non zero components of Riemann curvature tensor and Ricci tensor in tetrad form for non-static axisymmetric metric are as follows.

\begin{align*}
    R_{(1212)} &= -\left[ \sigma_2 \exp(\sigma - \beta) \right]_{12} e^{-\sigma' - \beta} - \frac{\sigma_3 \beta_3}{r^2} \exp(-2\beta) \\
    &\quad + \sigma_0 \beta_0 \exp(-2\psi) - \frac{1}{4} \left( \omega_2 \right)^2 \exp(-2\psi + 2\sigma - 2\beta)
\end{align*}
\[ R_{(1320)} = -\frac{1}{2} \left\{ \frac{\omega, 3}{r} \exp(\sigma - \beta), 1, 2 \right\} \exp(-\beta - \psi) \]

\[ -\frac{1}{2r} \omega, 2 (2\sigma - \beta - \psi), 3 \exp(\sigma - 2\beta - \psi) \]

\[ R_{(1330)} = -\frac{1}{2r^2} \left[ \omega, 33 + \omega, 3 (3\sigma - \beta - \psi), 3 \right] \exp(\sigma - 2\beta - \psi) \]

\[ -\frac{1}{2} \omega, 2 \left( \beta, 2 + \frac{1}{r} \right) \exp(\sigma - 2\beta - \psi) \]

\[ R_{(1010)} = -\left[ \sigma, 0 \exp(\sigma - \psi) \right], 0 \exp(-\sigma - \psi) \]

\[ + \left[ \sigma, 2 \psi, 2 + \frac{\sigma, 3 \psi, 2}{r^2} \right] \exp(-2\beta) \]

\[ + \frac{1}{4} \left[ (\omega, 2)^2 + \frac{(\omega, 3)^2}{r^2} \right] \exp(2\sigma - 2\psi - 2\beta) \]

\[ R_{(1023)} = -\frac{1}{2r} \left\{ \left(-\omega, 3 \right) \exp(\sigma - \psi), 1, 2 + \left(\omega, 2 \right) \exp(\sigma - \psi), 1, 3 \right\} \exp(-2\beta) \]

\[ R_{(1020)} = -\frac{1}{2} \left[ \omega, 20 + \omega, 2 (3\sigma - \beta - \psi), 0 \right] \exp(\sigma - \beta - 2\psi) \]

\[ R_{(1030)} = -\frac{1}{2r} \left[ \omega, 30 + \omega, 3 (3\sigma - \beta - \psi), 0 \right] \exp(\sigma - \beta - 2\psi) \]

\[ R_{(2323)} = -\left\{ \frac{\beta, 3}{r}, 1, 3 + \frac{(\beta, 2 + \frac{1}{r} r), 2}{2} \right\} \frac{1}{r} \exp(-2\beta) + (\beta, 0)^2 \exp(-2\psi) \]

\[ R_{(2320)} = -\frac{1}{r} \left[ \beta, 02 - \beta, 0 \psi, 2 \right] \exp(-\beta - \psi) \]

\[ R_{(2330)} = \left[ \beta, 02 - \beta, 0 \psi, 2 \right] \exp(-\beta - \psi) \]
\[ R_{(2020)} = \left[ - (\beta, 0 \exp(\beta - \varphi)),_0 + (\varphi, 2 \exp(\varphi - \beta)),_2 \right] \exp(-\beta - \varphi) \]

\[ + \frac{\beta, 3 \varphi, 3}{r^2} \exp(-2\beta) - \frac{3}{4} (\omega, 2)^2 \exp(2\sigma - 2\beta - 2\varphi) \]

\[ R_{(2030)} = \left[ \psi, 32 + \psi, 2(\varphi - \beta),_3 - \psi, 3 \left[ \beta, 2 + \frac{1}{r} \right] \right] \frac{1}{r} \exp(-2\beta) \]

\[ - \frac{3}{4r} \omega, 2 \omega, 3 \exp(2\sigma - 2\beta - 2\varphi) \]

\[ R_{(3030)} = - r^{-2} \left[ (r^2 \beta, 0 \exp(\beta - \varphi)),_0 - (\varphi, 3 \exp(\varphi - \beta)),_3 \right] \exp(-\beta - \varphi) \]

\[ + \left[ \beta, 2 + \frac{1}{r} \right] \psi, 2 \exp(-2\beta) \]

\[ - \frac{3}{4r^2} (\omega, 3)^2 \exp(2\sigma - 2\beta - 2\varphi) \]

\[ R_{(11)} = - \left[ \sigma, 22 + \frac{\sigma, 2}{r} + \sigma, 2 \psi, 2 + (\sigma, 2)^2 \right] \exp(-2\beta) \]

\[ - \frac{1}{r^2} \left[ \sigma, 33 + \sigma, 3 \psi, 3 + (\sigma, 3)^2 \right] \exp(-2\beta) \]

\[ + \left[ \sigma, 00 - \sigma, 0 \psi, 0 + 2 \sigma, 0 \beta, 0 + (\sigma, 0)^2 \right] \exp(-2\psi) \]

\[ - \frac{1}{2} \left[ (\omega, 2)^2 + \frac{\omega, 3)^2}{r^2} \right] \exp(2\sigma - 2\beta - 2\varphi). \]
\[ R_{(12)} = \frac{1}{2} \left[ \omega, 2 (3\sigma_0 - \psi_0) + \omega, 20 \right] \exp(\sigma - \beta - 2\psi). \]

\[ R_{(13)} = \frac{1}{2r} \left[ \omega, 3 (3\sigma_0 - \psi_0) + \omega, 30 \right] \exp(\sigma - \beta - 2\psi). \]

\[ R_{(10)} = \frac{1}{2} \left[ (3\sigma_2 - \psi_2 + \frac{1}{r}) \omega, 2 + \omega, 22 \right] \exp(\sigma - 2\beta - 2\psi) \]
\[ + \frac{1}{2r^2} \left[ (3\sigma_3 - \psi_3 + \omega, 3 + \omega, 33 \right] \exp(\sigma - 2\beta - 2\psi). \]

\[ R_{(22)} = - \left[ \sigma, 22 - \sigma, 2 \beta, 2 + (\sigma, 2)^2 - \beta, 2 \left( \beta, 2 + \frac{1}{r} \right) \right] \]
\[ + \left( \beta, 2 + \frac{1}{r} \right)^2 - \beta, 2 \psi, 2 \]
\[ + \psi, 22 + (\psi, 2)^2 \right] \exp(-2\beta) \]
\[ - \frac{1}{r^2} \left[ \sigma, 3 \beta, 3 + \beta, 33 + \beta, 3 \psi, 3 \right] \exp(-2\beta) \]
\[ + \left[ \sigma, 0 \beta, 0 + 2 (\beta, 0)^2 - \beta, 0 \psi, 0 + \beta, 00 \right] \exp(-2\beta) \]
\[ + \frac{1}{2} (\omega, 2)^2 \exp(2\sigma - 2\beta - 2\psi). \]

\[ R_{(23)} = - \frac{1}{r} \left[ \sigma, 23 - \sigma, 2 \beta, 3 + \sigma, 2 \sigma, 3 - \sigma, 3 \left( \beta, 2 + \frac{1}{r} \right) \right] \]
\[ - \beta, 3 \psi, 2 + \psi, 23 + \psi, 2 \psi, 3 - \psi, 3 \left( \beta, 2 + \frac{1}{r} \right) \right] \exp(-2\beta) \]
\[ + \frac{1}{2r} \omega, 2 \omega, 3 \exp(2\sigma - 2\beta - 2\psi). \]
\[ R_{(20)} = \left[ -\sigma_{20} + \sigma_{2} \beta_{0} - \sigma_{2} \sigma_{0} + \sigma_{0} \psi_{2} - \beta_{20} + \beta_{0} \psi_{2} \right] \exp(-\beta - \psi). \]

\[ R_{(33)} = -\left[ \sigma_{2} \left( \beta_{2} + \frac{1}{r} \right) - \beta_{2} \left( \beta_{2} + \frac{1}{r} \right) + \left( \beta_{2} + \frac{1}{r} \right) \right] \exp(-2\beta) \]

\[ -\frac{1}{r^2} \left[ \sigma_{33} - \sigma_{3} \beta_{3} + (\sigma_{3})^2 + \beta_{33} - \beta_{3} \psi_{3} \right. \]

\[ + \psi_{33} + (\psi_{3})^2 \exp(-2\beta) + \left[ \sigma_{0} \beta_{0} + 2 (\beta_{0})^2 \right. \]

\[ + \beta_{00} - \beta_{0} \psi_{0} \right] \exp(-2\psi) \]

\[ + \frac{1}{2r^2} (\omega_{3})^2 \exp(2\sigma - 2\beta - 2\psi). \]

\[ R_{(30)} = \frac{1}{r} \left[ -\sigma_{30} + \sigma_{3} \beta_{0} - \sigma_{3} \sigma_{0} + \sigma_{0} \psi_{3} \right. \]

\[ - \beta_{30} + \beta_{0} \psi_{3} \right] \exp(-\beta - \psi). \]
The non zero components of Riemann curvature tensor and Ricci tensor in tetrad form for static axisymmetric metric are as follows.

\[ R_{(00)} = \left[ \sigma',_{2} \psi',_{2} + \psi',_{22} + (\psi',_{2})^{2} + \frac{\psi',_{2}}{r} \right] \exp(-2\beta) \]

\[ + \frac{1}{r^2} \left[ \sigma',_{3} \psi',_{3} + \psi',_{33} + (\psi',_{3})^{2} \right] \exp(-2\beta) \]

\[ - \left[ \sigma',_{00} - \sigma',_{0} \psi',_{0} + (\sigma',_{0})^{2} - 2 \beta',_{0} \psi',_{0} \right. \]

\[ + 2 \beta',_{00} + 2 (\beta',_{0})^{2} \right] \exp(-2\psi) \]

\[ - \frac{1}{2} \left[ (\omega',_{2})^{2} + \frac{(\omega',_{3})^{2}}{r^2} \right] \exp(2\sigma - 2\beta - 2\psi). \]
\[
R_{(1230)} = -\frac{1}{2r} \left\{ \{(\omega, 2)\exp(\sigma - \beta)\}_{1, 3} \right\} \exp(-\beta - \psi) \\
- \frac{1}{2r} \omega, 3 \left\{ 2\sigma, 2 - \beta, 2 - \frac{1}{r} - \psi, 2 \right\} \exp(\sigma - 2\beta - \psi)
\]

\[
R_{(1313)} = \left[ \frac{\sigma, 3}{r} \exp(\sigma - \beta) \right], 3 \exp(-\sigma - \beta) - \sigma, 2 \left\{ \beta, 2 + \frac{1}{r} \right\} \exp(-2\beta) \\
- \frac{(\omega, 3)^2}{4r} \exp(-2\psi + 2\sigma - 2\beta)
\]

\[
R_{(1320)} = -\frac{1}{2} \left\{ \left\{ \frac{\sigma, 3}{r} \exp(\sigma - \beta) \right\}_{1, 2} \right\} \exp(-\beta - \psi) \\
- \frac{1}{2r} \omega, 2 \left( 2\sigma - \beta - \psi \right), 3 \exp(\sigma - 2\beta - \psi)
\]

\[
R_{(1330)} = -\frac{1}{2r^2} \left[ \omega, 33 + \omega, 3 \left( 3\sigma - \beta - \psi \right), 3 \right] \exp(\sigma - 2\beta - \psi) \\
- \frac{1}{2} \omega, 2 \left\{ \beta, 2 + \frac{1}{r} \right\} \exp(\sigma - 2\beta - \psi)
\]

\[
R_{(1010)} = \left[ \sigma, 2 \psi, 2 + \frac{\sigma, 3 \psi, 3}{r^2} \right] \exp(-2\beta) \\
+ \frac{1}{4} \left[ \left( \omega, 2 \right)^2 + \frac{(\omega, 3)^2}{r^2} \right] \exp(2\sigma - 2\psi - 2\beta)
\]

\[
R_{(1023)} = -\frac{1}{2r} \left\{ \left\{ (-\omega, 3)\exp(\sigma - \psi) \right\}_{1, 2} + \left\{ (\omega, 2)\exp(\sigma - \psi) \right\}_{1, 3} \right\} \exp(-2\beta)
\]

\[
R_{(2323)} = - \left\{ \left\{ \frac{\beta, 3}{r} \right\}_{1, 3} + \left\{ (\beta, 2 + \frac{1}{r}) r, 1, 2 \right\} \frac{1}{r} \exp(-2\beta) \right\}
\]

23
\[ R_{(2020)} = \left( \psi_2 \exp(\psi - \beta), 2 \right) \exp(-\beta - \psi) + \beta_3 \psi_3 \exp(-2\beta) \]

\[ - \frac{3}{4} (\omega_2)^2 \exp(2\alpha - 2\beta - 2\psi) \]

\[ R_{(2030)} = \left( \psi_3, 2 \psi - \beta, 3 - \psi, 3 \left[ \beta, 2 + \frac{1}{r} \right] \right) \frac{1}{r} \exp(-2\beta) \]

\[ - \frac{3}{4r} \omega_2 \omega_3 \exp(2\alpha - 2\beta - 2\psi) \]

\[ R_{(3030)} = \frac{1}{r} \left[ \left( \psi, 3 \exp(\psi - \beta) \right), 3 \right] \exp(-\beta - \psi) \]

\[ + \left[ \beta, 2 + \frac{1}{r} \right] \psi_2 \exp(-2\beta) - \frac{3}{4r^2} (\omega_3)^2 \exp(2\alpha - 2\beta - 2\psi) \]

\[ R_{(11)} = - \left[ \sigma_2 \frac{\sigma_2}{r} + \sigma_2 \psi_2 + (\sigma_2)^2 \right] \exp(-2\beta) \]

\[ - \frac{1}{r^2} \left[ \sigma_3 \sigma_3 \psi_3 + (\sigma_3)^2 \right] \exp(-2\beta) \]

\[ - \frac{1}{2} \left[ (\omega_2)^2 + \frac{(\omega_3)^2}{r^2} \right] \exp(2\alpha - 2\beta - 2\psi). \]

\[ R_{(12)} = R_{(13)} = 0 \]

\[ R_{(10)} = \frac{1}{2} \left[ (3\sigma_2 - \psi_2 + \frac{1}{r}) \omega_2 + \omega_22 \right] \exp(\sigma - \beta - 2\psi) \]

\[ + \frac{1}{2r^2} \left[ (3\sigma_3 - \psi_3) \omega_3 + \omega_33 \right] \exp(\sigma - \beta - 2\psi). \]
\[ R_{(22)} = - \left[ \sigma_{,22} - \sigma_{,2} \beta_{,2} + (\sigma_{,2})^2 - \beta_{,2} \left( \beta_{,2} + \frac{1}{r} \right) \right] \\
+ \left[ \beta_{,2} + \frac{1}{r} \right]_2 \left[ \beta_{,2} + \frac{1}{r} \right]^2 - \beta_{,2} \psi_{,2} \\
+ \psi_{,22} + (\psi_{,2})^2 \right\} \exp(-2\beta) \\
- \frac{1}{r^2} \left[ \sigma_{,3} \beta_{,3} + \beta_{,33} + \beta_{,3} \psi_{,3} \right] \exp(-2\beta) \\
+ \frac{1}{2} (\omega_{,2})^2 \exp(2\sigma - 2\beta - 2\psi). \]

\[ R_{(23)} = - \frac{1}{r} \left[ \sigma_{,23} - \sigma_{,2} \beta_{,3} + \sigma_{,2} \sigma_{,3} - \sigma_{,3} \left( \beta_{,2} + \frac{1}{r} \right) \right] \\
- \beta_{,3} \psi_{,2} + \psi_{,23} + \psi_{,2} \psi_{,3} - \psi_{,3} \left( \beta_{,2} + \frac{1}{r} \right) \right\} \exp(-2\beta) \\
+ \frac{1}{2r} \omega_{,2} \omega_{,3} \exp(2\sigma - 2\beta - 2\psi). \]

\[ R_{(20)} = 0 \]

\[ R_{(33)} = - \left[ \sigma_{,2} \left( \beta_{,2} + \frac{1}{r} \right) - \beta_{,2} \left( \beta_{,2} + \frac{1}{r} \right) + \left( \beta_{,2} + \frac{1}{r} \right)_2 \right] \\
+ \left[ \beta_{,2} + \frac{1}{r} \right]^2 + \left[ \beta_{,2} + \frac{1}{r} \right] \psi_{,2} \right\} \exp(-2\beta) \\
- \frac{1}{r^2} \left[ \sigma_{,33} - \sigma_{,3} \beta_{,3} + (\sigma_{,3})^2 + \beta_{,33} - \beta_{,3} \psi_{,3} \right] \\
+ \psi_{,33} + (\psi_{,3})^2 \right\} \exp(-2\beta) + \frac{1}{2r^2} (\omega_{,3})^2 \exp(2\sigma - 2\beta - 2\psi). \]
\[ R_{(30)} = 0 \]

\[ R_{(00)} = \left[ \sigma_{,2} \psi_{,2} + \psi_{,22} + (\psi_{,2})^2 + \frac{\psi_{,2}}{r} \right] \exp(-2\beta) \]

\[ + \frac{1}{r^2} \left[ \sigma_{,3} \psi_{,3} + \psi_{,33} + (\psi_{,3})^2 \right] \exp(-2\beta) \]

\[ - \frac{1}{2} \left[ (\omega_{,2})^2 + \frac{(\omega_{,3})^2}{r^2} \right] \exp(2\alpha - 2\beta - 2\psi). \]

1.3 Spherical Symmetric Collapse:

Let us investigate the issue of the final fate of a gravitationally collapsing massive star. Chandrasekhar [1] had proved that when the star is heavier than a few solar masses, it could undergo an endless gravitational collapse without achieving any equilibrium state. This happens when the star has exhausted its internal nuclear fuel which provides the outwards pressure against the inward gravitational pull. In this article we will study the axially symmetric gravitational collapse. The metric for spherical symmetric space-time in comoving coordinate system is

\[ dr^2 = dt^2 - U(r,t) \, dr^2 - V(r,t) \left( d\theta^2 + \sin^2\theta \, d\phi^2 \right). \] \[ \ldots \ldots \quad (1.3.1) \]
The energy-momentum tensor for a fluid of negligible pressure is given by equation

$$T^\mu_\nu = \rho \, u^\mu \, u^\nu,$$  \hspace{1cm} \ldots \ldots \ldots (1.3.2)$$

where $\rho(r,t)$ is the proper energy density and $u^\mu$ the velocity four vector. For comoving coordinate system

$$u^\alpha = 0 \quad \text{for } \alpha = r, \theta, \phi$$

$$= 1 \quad \text{for } \alpha = t$$  \hspace{1cm} \ldots \ldots \ldots (1.3.3)$$

and the Einstein field equations can be written as

$$R_{\mu\nu} = -8\pi \, G \, S_{\mu\nu},$$  \hspace{1cm} \ldots \ldots \ldots (1.3.4)$$

where $S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda_\lambda$

$$\rho \left[ \frac{1}{2} g_{\mu\nu} + u^\mu \, u^\nu \right]:$$  \hspace{1cm} \ldots \ldots \ldots (1.3.5)$$

Using the Ricci-tensor for the metric (1.3.1) and equations \((1.3.2) - (1.3.4)\) yields four field equations

$$\frac{1}{U} \left[ \frac{V''}{V} - \frac{V^2}{2V^2} - \frac{U'V'}{2UV} \right] - \frac{U}{2U} + \frac{U^2}{4U^2} - \frac{4UV}{2UV} = -4\pi G \rho, \quad \ldots \ldots \ldots (1.3.6.a)$$

$$- \frac{1}{V} + \frac{1}{U} \left[ \frac{V''}{2V} - \frac{U'V'}{4UV} \right] - \frac{V}{2V} - \frac{U}{4UV} = -4\pi G \rho, \quad \ldots \ldots \ldots (1.3.6.b)$$

$$\frac{U'}{2U} + \frac{U^2}{4UV} - \frac{V^2}{2V} = -4\pi G \rho, \quad \ldots \ldots \ldots (1.3.6.c)$$
For the simplification of our model even further let us assume that $\rho$ is independent of position. We can now find a separable solution with

$$U = R^2(t) f(r) \quad \text{and} \quad V = S^2(t) g(r).$$

Now the equation (1.3.6.d) implies

$$\frac{R}{R} = \frac{S}{S}$$

Therefore $R(t) = S(t)$

Again redefining $r$ we have

$$U = R^2(t) f(r) \quad \text{and} \quad V = R^2(t) r^2.$$

Hence the equations (1.3.6.a) and (1.3.6.b) become

$$- \frac{f'}{rf^2} = R \frac{R}{r} + 2R^2 - 4\pi G R^2 \rho$$

$$\left[ - \frac{1}{r^2} + \frac{1}{r^2 f} - \frac{f'}{2rf^2} \right] = R \frac{R}{r} + 2R^2 - 4\pi G R^2 \rho.$$
where $k$ is constant. The unique solution of these equations is

$$f(r) = (1 - kr^2)^{-1}.$$ 

Hence the metric (1.3.1) reduces to the form

$$dr^2 = dt^2 - R(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2 \right], \quad \ldots \ (1.3.8)$$

where $k$ is an arbitrary constant. This metric is spatially homogeneous as well as isotropic and for that reason it will provide the kinematic framework for our treatment of relativistic cosmology. Now to calculate the functions $\rho(t)$ and $R(t)$, using equation (1.3.7) and from equations (1.3.6.a) and (1.3.6.b) we get

$$\frac{f'(r)}{r^3} - \frac{R'(t)}{R(t)} R(t) - 2R''(t) = -4\pi G R(t)^2 \rho(t). \quad \ldots \ (1.3.9)$$

The energy conservation equation for the metric (1.3.1) is

$$\frac{\partial}{\partial t} (\rho \sqrt{U}) = 0, \quad \ldots \ldots \ (1.3.10)$$

Substituting values of $U$ and $V$, we find that $\rho(t)R^3(t)$ is constant. We normalize the radial coordinate $r$ so that

$$R(0) = 1 \quad \ldots \ldots \ (1.3.11)$$

and therefore

$$\rho(t) = \rho(0) R^{-3}(t). \quad \ldots \ldots \ (1.3.12)$$
The field equations (1.3.9) and (1.3.10) are now ordinary differential equations

\[-2k - R''(t)R(t) - 2R^2(t) = -4\pi G\varrho(0)R^{-1}(t)\] \hspace{1cm} (1.3.13)

\[R''(t)R(t) = -(4\pi G/3) \varrho(0) R^{-1}(t).\] \hspace{1cm} (1.3.14)

By adding these two equations we get

\[R^2(t) = -k + (8\pi G/3) \varrho(0)R^{-1}(t).\] \hspace{1cm} (1.3.15)

Let us assume that the fluid is at rest (in standard coordinate system) at \(t = 0\), so

\[R'(0) = 0\] \hspace{1cm} (1.3.16)

and therefore equations (1.3.11) and (1.3.15) give

\[k = (8\pi G/3) \varrho(0).\] \hspace{1cm} (1.3.17)

Thus equation (1.3.15) can be written as

\[R^2(t) = k [R^{-1}(t) - 1].\] \hspace{1cm} (1.3.18)

The solution is given by the parametric equations of a cycloid

\[t = (\psi + \sin \psi)/(2\sqrt{k}),\]

\[R = (1/2) (1 + \cos \psi).\] \hspace{1cm} (1.3.19)
Note that $R(t)$ vanishes when $\varphi = \pi$ and hence when $t = T$, where

$$T = \pi/2\sqrt{\frac{3}{8\pi G(\rho)}} \left[ \frac{1}{2} \right]. \quad \ldots \ldots \ldots \quad (1.3.20)$$

Thus a fluid sphere of initial density $\rho(0)$ and zero pressure will collapse from rest to a state of infinite proper energy density in the finite time $T$. Although the collapse is complete at a finite coordinate time $t=T$, any light signal coming to a distant observer from the surface of the sphere will be delayed by its gravitational field, so observer on the earth will not see the star disappearing suddenly. To give specific physical meaning, let us find the metric outside the star. The Birkhoff theorem implies that it is always possible to find a standard coordinate system $r, \theta, \phi$ and $t$ in which the metric outside the sphere of mass $M$ become

$$dr^2 = (1 - 2MG/r) \ dt^2 - (1 - 2MG/r)^{-1} \ dr^2 - r^2 \ d\theta^2 - 2 \ r \ \sin^2 \theta \ d\phi^2. \quad \ldots \ldots \ldots \quad (1.3.21)$$

This metric is not in the Gaussian normal form hence let us convert the interior solution into standard coordinates. The metric (1.3.8) implies that the standard spatial coordinates $r, \theta$ and $\phi$ must be chosen as

$$\bar{r} = r \ R(t), \ \bar{\theta} = \theta, \ \bar{\phi} = \phi.$$
Let us define standard time coordinate $t$ such that $\text{d}t^2$ does not contain a cross term $\text{d}r \text{d}t$. This may be achieved by integrating factor technique which gives

$$
t = \left( \frac{1 - k a^2}{k} \right)^{1/2} \int S(r,t) \frac{dR}{(1 - k a^2/R)^{1/2}} \left( \frac{R}{1 - R} \right)^{1/2}, \ldots . \quad (1.3.22)
$$

where $S(r,t) = 1 - \left( \frac{1 - k r^2}{1 - k a^2} \right)^{1/2} (1 - R(t))$.

The arbitrary constant $a$ may be chosen as the radius of the sphere in comoving coordinates. Now the interior metric takes the standard form

$$
\text{d}r^2 = B(r,t) \text{d}t^2 - A(r,t) \text{d}r^2 - r^2 \text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2,
$$

where $B = \frac{R}{S} \left( \frac{1 - k r^2}{1 - k a^2} \right)^{1/2} \left( \frac{1 - k a^2/S}{1 - k r^2/R} \right)^2$,

$$A = (1 - k r^2/R)^{-1}.$$

On the boundary $r = a$ of the star, we have $r = a(t) \equiv a R(t)$

$$B(a,t) = \frac{1}{A(a,t)} = (1 - k a^2/R).$$
The interior and exterior solutions fit continuously at $r = a R(t)$ if $k = \frac{2MG}{a^3}$.

i.e. $M = \frac{4\pi}{3} \rho(0) a^3$. ... ... ... (1.3.23)

Now let us calculate the behaviour of light signals emitted from the surface of the collapsing sphere. The radial velocity of a light signal emitted at a standard time $t$ is

$$\frac{dr}{dt} = (1 - \frac{2MG}{r}).$$ ... ... ... (1.3.24)

The most striking consequence of equation (1.3.21) and (1.3.22) is that both $t$ and $\tau$ approach infinity when

$$R(t) \rightarrow (2GM/a) = ka^2.$$ ... ... ... (1.3.25)

Hence collapse to the Schwarzschild radius appears to an outside observer to take an infinite time and the collapse to $R = 0$ is not observable from outside. Although the collapsing sphere does not suddenly disappear, it does fade out of side, because light from its surface is subject to an increasing red shift. The proper time for a light source on the surface of sphere is just comoving time, so the comoving time interval between
emission of wave crests at the surface equals the natural wavelength $\lambda_0$ that would be emitted by the source in the absence of gravitation. The standard time interval $dt$ between arrivals of wave crests at $r$ is the observed wavelength $\lambda'$; therefore change of wavelength is given by

$$
\frac{\lambda' - \lambda_0}{\lambda_0} = \frac{dt}{dt} - 1 = \frac{dt}{dt} - aR'(t) \left( 1 - \frac{2GM/aR(t)}{} \right)^{-1} - 1
$$

$$
= - R'(t) \left( 1 - \frac{ka^2}{R(t)} \right)^{-1} \left[ \left( \frac{1-ka^2}{k} \right)^{1/2} \left( \frac{R(t)}{1-R(t)} \right)^{1/2} + a \right] - 1.
$$

Using (1.3.18), we get

$$
z = \left( 1 - \frac{ka^2}{R(t)} \right)^{-1} \left[ (1 - ka^2)^{1/2} + a\sqrt{k} \left( \frac{1-R(t)}{R(t)} \right)^{1/2} \right] - 1.
$$

... ... ... (1.3.26)

Now let us calculate variation of red shift $z$ with $t$. Let us assume that the sphere is initially very much larger than its Schwarzschild radius i.e.

$$
ka^2 = 2GM/a \ll 1
$$

... ... ... (1.3.27)

Case: - (i)

If

$$
ka^2/R(t) \ll 1.
$$

... ... ... (1.3.28)
then use of equations (1.3.27) and (1.3.28) in the equation of \( t \), and equation (1.3.22) and (1.3.26) gives (with \( r \geq a \))

\[
t \equiv t.
\]

\[
t \equiv t + r - aR(t).
\]

\[
t \equiv t + r.\]

\[
z \equiv a\sqrt{k} \left( \frac{1 - R(t)}{R(t)} \right)^{1/2}.
\]

\[
z \equiv a\sqrt{k} \left( \frac{1 - R(t - r)}{R(t - r)} \right)^{1/2}.
\]

... ... ... (1.3.29)

Case :- (ii)

If

\[
ka^2/R(t) \rightarrow 1
\]

at a time \( t \) given by (1.3.19) as

\[
t \equiv 1/2\sqrt{k} \left[ \pi - 4/3 (ka^2)^{3/2} \right].
\]

... ... ... (1.3.30)

Equations (1.3.22) and (1.3.27) give

\[
t \equiv -ka^3 \ln \left[ 1 - ka^2/R(t) \right] + \text{constant}
\]

\[
t \equiv t - ka^3 \ln \left[ 1 - ka^2/R(t) \right] + \text{constant}
\]

\[
t \equiv -2ka^3 \ln \left[ 1 - ka^2/R(t) \right] + \text{constant}
\]
therefore from equation (1.3.26) we get

\[ z \approx 2 \left( 1 - \frac{ka^2}{R(t)} \right) \alpha \exp \left( -\frac{t}{2ka^3} \right). \quad \ldots \quad \ldots \quad (1.3.31) \]

From above the red shift \( z \) seen by an observer at \( r \) is zero when the collapse is observed to begin, increases gradually but remains of order \( a\sqrt{k} \ll 1 \) until a time very close to \( T = \pi/2 \sqrt{k} \) has passed, and then grows exponentially with a rate \( (1/2ka^3) \).

When a star has collapsed to the stage where nothing not even light rays can get out, the star has fallen inside its event horizon. A star that has collapsed inside its event horizon still has a finite size, but there are still no physical forces that can stop the collapse. The star continues to shrink in size until all the matter of the star is crushed to a single point, the singularity. At the singularity, the centre of a black-hole, there is infinite pressure, infinite density and infinite curvature of the space-time.

To travel in to a black-hole, first we would pass through the photon sphere, a thin shell of light orbiting above the black-hole. Then we would pass through the event horizon and completely disappear from the outside universe. Finally, in about a hundred thousandth of a second, we would be dragged into the singularity where everything is crushed out of existence by infinitely warped space-time. (see figure: 1) Here we have discussed the spherically symmetric and non-rotating black-hole, which is described by a solution of the Einstein field equations first discovered by K. Schwarzschild in 1916.
Fig. 1: Structure of the Black-hole
All static, spherically symmetric, uncharged, non-rotating solutions to the field equations are essentially equivalent to the Schwarzschild solution. The Schwarzschild black-hole is the simplest possible black-hole.

This discussion of black-holes only scratches the surface one of the most fantastic and bizarre topics in all of modern science. It can be shown that the full geometry of the Schwarzschild black-hole actually connects two separate universes. The development of a black-hole exhibit, an increasing curvature of space, as shown in the figure. (see figure : 2) This is an imbedding diagram of space about a collapsing star on its way to becoming a black-hole. The shaded area shows where the matter of the star is. In Schwarzschild black-hole it is impossible to go from our universe in to the other universe.

1.4 Distinguished surfaces and the rotating Black-hole:

In the Kerr solution

\[
\begin{align*}
\text{d}s^2 &= \left( 1 - \frac{2mp}{\rho^2 + a^2 \cos^2 \theta} \right) c^2 \text{dt}^2 - \frac{\frac{\rho^2 + a^2 \cos^2 \theta}{\rho^2 + a^2 - 2mp}}{\rho^2 + a^2 - 2mp} \text{d}\phi^2 \\
&\quad - \left( \rho^2 + a^2 \cos^2 \theta \right) \text{d}\theta^2 - \left[ \left( \rho^2 + a^2 \sin^2 \theta + \frac{2mp \rho^2 \sin^4 \theta}{\rho^2 + a^2 \cos^2 \theta} \right) \text{d}\phi^2 \\
&\quad - \frac{2mp \rho^2 \sin^2 \theta}{\rho^2 + a^2 \cos^2 \theta} c \text{dt} \text{d}\phi, \quad \cdots \quad \cdots \quad (1.4.1)
\end{align*}
\]
Fig. 2: The embedding diagram of a collapsing star
two surfaces arise which are analogous to the Schwarzschild singular surface and which are of great physical interest for gravitational collapse of a rotating star.

Let us first study the red shift of light emitted from a source at rest in the Kerr geometry. The frequency relation

\[ \nu = \nu_0 \left( \frac{g_{00}(\chi^\mu_s)}{g_{00}(\chi^\mu_0)} \right)^{1/2}, \quad \ldots \ldots \ldots \quad (1.4.2) \]

where \( \nu \) is the frequency of the light observed at \( \chi^\mu_s \) and \( \nu_0 \) is the proper frequency of the light emitted by the source at rest at \( \chi^\mu_0 \).

It is clear from equation (1.4.1) that for large values of \( \rho \), the red shift in the Kerr metric is approximately equal to that in the Schwarzschild metric, since the potential function \( g_{00} \) differ only by terms of second order in \( a/\rho \). Infinite red shift surface is obtained by setting \( g_{00} = 0 \) in the Kerr metric. Hence infinite red shift surfaces of the Kerr space-time are

\[ \rho = m \pm \left( m^2 - a^2 \cos^2 \theta \right)^{1/2}, \quad \ldots \ldots \ldots \quad (1.4.3) \]

In the limit as \( a \rightarrow 0 \) these surfaces reduce to the Schwarzschild surface \( \rho = 2m \), for the plus sign and the origin \( \rho = 0 \) for the minus sign. The surface corresponding to the plus sign is of much greater physical interest, it is an axially symmetric surface with a radius \( 2m \) at the equator and a radius
2 \text{m} + \frac{(m^2 - a^2)^{1/2}}{2} \text{ at the poles. The surface corresponding to the minus sign is completely contained within the surface }

\rho_\alpha = m + \frac{(m^2 - a^2 \cos^2 \theta)^{1/2}}{2}.

To find the one-way membranes of the Kerr solution, we shall introduce and study the concept of a so-called null hyper-surfaces. Consider a smooth hypersurface $S$ defined by the equation

$$u(x^\mu) = \text{constant.} \quad \ldots \quad (1.4.4)$$

The vector $n_\alpha = u_\alpha$ is a normal to $S$, since its inner product with any $dx^\alpha$ contained in $S$ is zero.

$$n_\alpha dx^\alpha = u_\alpha dx^\alpha = du = 0. \quad \ldots \quad (1.4.5)$$

At any point $P$ on $S$ we introduce a locally Lorentzian metric so that the line element is

$$ds^2 = (dx^0)^2 - (dx)^2 - (dy)^2 - (dz)^2 \quad \ldots \quad (1.4.6)$$

and the local light cone, defined as the local hypersurface $ds^2 = 0$ is very simple. Moreover, by a rotation in three-space we can place the three-vector part of $n^\alpha$ along the $x$-axis. Therefore

$$n^\alpha = (n^0, n^1, 0, 0).$$

$$n_\alpha n^\alpha = (n^0)^2 - (n^1)^2. \quad \ldots \quad (1.4.7)$$

39
This form of the normal vector restricts the form of any vector $t^\alpha$ tangent to $S$ at $P$. Since these two vectors must be orthogonal,

$$n^\alpha t^\alpha = n^0 t^0 - n^1 t^1 = 0.$$ 

Therefore

$$t^0 / t^1 = n^1 / n^0.$$ 

Thus $t^\alpha$ must have the form

$$t^\alpha = \lambda (n^1, n^0, a, b),$$

where $\lambda$, $a$ and $b$ are arbitrary. It follows that the norm of $t^\alpha$ is

$$t^\alpha t_\alpha = \lambda^2 \left[ (n^1)^2 - (n^0)^2 - a^2 - b^2 \right],$$

$$= -\lambda^2 (n^\alpha n_\alpha + a^2 + b^2).$$

This simple relation between the norms of the normal and tangent vectors of $S$ leads to a beautiful geometrical result.

Let us consider three cases.

Case (i)

If the surface $S$ given by equation (1.4.4) is space-like i.e. $n^\alpha n_\alpha < 0$, then equation (1.4.10) implies that $t^\alpha t_\alpha$ may be positive, negative or zero. In particular $t^\alpha t_\alpha = 0$ on the circle defined by $a^2 + b^2 = -n^\alpha n_\alpha$. Hence there exist a family of vectors tangential to $S$ and which lie on the local null cone. The
space-like surface $S$ cut the local null cone such that each of past and future null cone regions divides into two regions. As light has a trajectory on light cone and massive bodies have trajectories on with in the light cone from past to future, it is easy to see from the figure 3 that the world line of a massive particle which is on left side of $S$ in past null cone cross the surface at $P$ and it remains to right side of $S$ in the past null cone. Parallely the world line of particle which is on right side of $S$ in the past null cone cross the surface from right to left. Thus massive objects can pass through a space-like hyper surface in either direction.

Case (ii)

If $n^\alpha$ is null, i.e. $n^\alpha n_\alpha = 0$ then $t^\alpha t_\alpha$ is negative in general. $t^\alpha t_\alpha = 0$ only if $a = b = 0$. Hence there exist only one tangent vector to $S$ which lies on the local light cone at $P$. As shown in figure 4, entire past light cone lies in one side of the surface $S$ and the future past of light cone lies on opposite side of $S$. As world lines of particles of photon are from past light cone region to future light cone region the physical object can pass through $S$ in only one direction. This is the critical case of beginning of one-way behaviour. It is identify as one-way membrane.
Fig. 3: Case (i) Space-like surface i.e. $n^a n_a < 0$
Fig. 4: Case (ii) Null surface i.e. $n^a n_a = 0$
Case (iii)

If $n^\alpha$ is time-like, i.e. $n^\alpha n_\alpha > 0$ then $t^\alpha t_\alpha$ is negative so that $t^\alpha$ is space-like. Hence no tangent vector to the surface $S$ can lie on the local light cone lies. As shown in figure 5, entire region of the past light cone lies on one side of $S$. Therefore as in case (ii) the surface $S$ is one-way membrane.

A spherical surface $r = \text{constant}$ in the Schwarzschild geometry has a normal

$$n_\alpha = (0, 1, 0, 0).$$

$$n^\alpha n_\alpha = -(1 - 2m/r).$$ \hspace{1cm} \ldots \ldots \ldots \ (1.4.11)

Thus as $r$ decreases through $2m$ the spherical surface changes from space-like to null to time-like. The null surface $r = 2m$ is one-way membranes of the Schwarzschild space. We now search for the null hypersurfaces of the Kerr geometry. Firstly we will show that the outer infinite red shift surface (1.4.3) is not a null hyper surface. The normal vector $n_\alpha$ of the surface is

$$n_\alpha = \left[ 0, 1, -\frac{a^2 \cos \theta \sin \theta}{(m^2 - a^2 \cos^2 \theta)^{1/2}}, 0 \right].$$

Therefore the norm of $n_\alpha$

$$n^\alpha n_\alpha = -\frac{\rho^2 + a^2 - 2m\rho + \left(\frac{a}{m} \cos^2 \theta \sin^2 \theta\right)}{\rho^2 + a^2 \cos^2 \theta}. \hspace{1cm} \ldots \ldots \ldots \ (1.4.12)$$
Fig. 5: Case (iii) Time-like surface i.e. $n^a n_a > 0$
is clearly negative. Hence the infinite red shift surface will pass physical objects in both directions and is not a one-way membrane. Let us find an axially symmetric and time-dependent null hypersurface

\[ u(r, \theta) = \text{constant}. \]

The normal vector to this surface is

\[ n_{\alpha} = (0, \partial u/\partial r, \partial u/\partial \theta, 0). \] \hspace{1cm} \ldots \ldots \ldots \ (1.4.13) \]

As the surface is null hypersurface norm of its normal vector \( u^\alpha \) is zero. Therefore

\[ n_{\alpha} n^\alpha = 0 \]

This implies that

\[ (\rho^2 - 2m\rho + a^2) (\partial u/\partial \rho)^2 + (\partial u/\partial \theta)^2 = 0. \] \hspace{1cm} \ldots \ldots \ldots \ (1.4.14) \]

Let us solve this differential equation by method of separation of variables taking

\[ u(\rho, \theta) = R(\rho) \Theta(\theta) \] \hspace{1cm} \ldots \ldots \ldots \ (1.4.15) \]

Now the equation (1.4.14) becomes

\[ - (\rho^2 - 2m\rho + a^2) \left( \frac{\partial R}{\partial \rho} \right)^2 = \left( \frac{\partial \Theta}{\partial \theta} \right)^2. \] \hspace{1cm} \ldots \ldots \ldots \ (1.4.16) \]
Since the left side of this equation is a function of $\rho$ alone and the right side a function of $\theta$ alone, both must be equal to a positive constant, which may be taken as $\lambda$, thus

$$\frac{\partial \Theta}{\partial \theta} = \sqrt{\lambda} \theta.$$ 

A simple solution of this equation is

$$\Theta = A \exp(\sqrt{\lambda} \theta), \quad \ldots \ldots \quad (1.4.17)$$ 

where $A$ is an arbitrary constant. This it is not periodic in $\theta$ and therefore does not correspond to a real surface except $\lambda = 0$. In the case $\lambda = 0$, $\Theta$ becomes constant and equation for $R$ is

$$\left(\frac{\partial R}{\partial \rho}\right)_R^2 (\rho^2 - 2m \rho + a^2) = 0. \quad \ldots \ldots \quad (1.4.18)$$

Which implies that $\partial R/\partial \rho = 0$ or $\rho^2 - 2m \rho + a^2 = 0$.

$\partial R/\partial \rho = 0$ not acceptable because it implies that $U$ is not function of $\rho$ and $\Theta$ both.

The two solutions of second equation are

$$\rho_\pm = m \pm (m^2 - a^2)^{1/2}. \quad \ldots \ldots \quad (1.4.19)$$

These are spherical surfaces. Let us note that these are well defined if \(|a| < m\). In the limit of $a \to 0$ these two surfaces reduce to the Schwarzschild surface $\rho = 2m$ and the origin $\rho = 0$.

The outer one-way membrane divides the region of the Kerr...
space-time into accessible and not accessible surfaces from the
distance exterior

\[ \rho_+ = m + (m^2 - a^2)^{1/2} \]

The inner infinite red-shift surface \( \rho_- = m - (m^2 - a^2 \cos^2 \theta)^{1/2} \), entirely contained within the outer one-way membrane, and it is not physical significance. (see figure :6)

The infinite red shift surface is not a barrier to either massive test bodies or light; both may cross the surface in either direction, except at the special points \( \theta = 0 \) and \( \pi \), where infinite red shift surface and the one-way membrane coincide. The infinite red shift surface refers explicitly to sources at rest in the Kerr metric. A star whose surface approaches the infinite red shift surface will not be considered a black-hole analogous to the Schwarzschild black-hole; light may escape from the surface, dependent upon its actual motion.

The nature of the one-way membrane \( \rho = \rho_+ \) is very different from the infinite red shift surface. It is a surface from which no light ray may emanate, regardless of the motion of the source. Therefore radius \( \rho_+ \) is the true critical radius for which a rotating star becomes a black-hole, analogous to the Schwarzschild black-hole.
Fig. 6: The Distinguished surfaces for the Kerr metric ($a = m/2$)

1. Inner infinite Red shift surface
2. $\rho_-$ Inner null Hyper surface
3. $\rho_+$ Infinite Red Shift Surface
4. $\rho_+$ Null Hyper surface, the black hole.
REFERENCES: