CHAPTER 4

METRIC OF AXIALLY SYMMETRIC RADIATION ZONE

Introduction: -

Stars, Nebulas and Supernovas are cloud of gases. It is known that these objects rotate about some axis. During the evolution star expands and becomes supernova. At certain stage of expansion, it explode and radiate energy. It might then blow off enough matter in to space, this matter converts in to radiation energy. Due to rotation of supernova the enveloping energy form a axially symmetric radiation zone. After the explosion interior core of the supernova begins to contract due to gravitational collapse. During this gravitational collapse fusion starts and a star gets equilibrium position. It may start expanding and again becomes supernova and radiate out some of its energy. This oscillation process continue until the star attains die down state i.e. after this stage fusion is not possible. In this process of evolution of a star, it radiate out sufficient amount of energy. As stars are rotating, the radiation given by the stars during its evolution may be axially symmetric. Thus one may get the situation where whole space may be axially symmetric radiation zone. Therefore it is important to study the gravitational field of axially symmetric radiation zone. Vaidya [4] had obtained metric
for a spherically symmetric radiating star. We have used here his techniques, to define energy momentum tensor for axially symmetric radiation zone. Here in this paper we have obtained metric for such axially symmetric radiation zone by assuming "Energy distribution in the enveloping radiation zone is axially symmetric at all time". Landau-Lifshitz complex is used to obtain conserved total mass energy of the radiation zone.

4.1 Metric and Field equations:

According to Vaidya the energy momentum tensor for the enveloping radiation zone is

\[ T^\alpha_\beta = q_k k^\alpha k^\beta, \]  \[ \ldots \ldots \ldots (4.1.1) \]

Where \( k^\alpha k_\alpha = 0, \) \[ \ldots \ldots \ldots (4.1.2) \]

\( (k^\alpha)_;\beta k^\beta = 0, \) \[ \ldots \ldots \ldots (4.1.3) \]

and \( q \) is the energy flux density.

As we are interested in studying physical properties of a radiation zone, which is axially symmetric at all time, the suitable metric for the space-time is that of Chandrasekhar and Friedman [11]

\[ ds^2 = -e^{2\psi} dt^2 + e^{2\sigma} (d\phi - \omega dt)^2 + e^{2\beta} (dr^2 - r^2 d\theta^2), \ldots \ldots (4.1.4) \]

Where \( \psi, \sigma, \omega \) and \( \beta \) are functions of \( r, \theta \) and \( t \) only. As enveloping radiation zone is axially symmetric at all time, the stationary
flow of radiation in \( \phi \) direction will be allowed but there may not be any flow of radiating energy along \( r \) and \( \theta \) direction. Hence \( k^2 = k^3 = 0 \). Then null vector which is solution of equations (4.1.2) and (4.1.3) is of the form

\[
k^\alpha = (e^{-\psi}, -e^{-\sigma} + \omega e^{-\psi}, 0, 0), \quad \ldots \ldots \ldots (4.1.5)
\]

To obtain the field equations we will use local orthogonal tetrad

\[
e^{(a)} = \begin{bmatrix}
e^{-\psi} & 0 & 0 & 0 \\
-\omega e^{-\sigma} & e^{-\sigma} & 0 & 0 \\
0 & 0 & e^{\beta} & 0 \\
0 & 0 & 0 & re^{\beta}
\end{bmatrix}, \quad \ldots \ldots \ldots (4.1.6)
\]

\[
g^{(ab)} = \text{diag}(-1, 1, 1, 1). \quad \ldots \ldots \ldots (4.1.7)
\]

In tetrad formulation, null vector \( k^{(a)} \) is obtained by the formula

\[
k^{(a)} = e^{(a)}k^{\beta}. \quad \ldots \ldots \ldots (4.1.8)
\]

Therefore

\[
k^{(a)} = (1, -1, 0, 0) \quad \text{and} \quad k^{(a)} = (-1, -1, 0, 0). \quad \ldots \ldots \ldots (4.1.9)
\]

Now

\[
T^{(ab)} = q k^{(a)}k^{(b)} \quad \text{and} \quad T = T^{(a)} = q k^{(a)}k^{(a)} = 0.
\]
Now the Einstein's field equations are

\[ R_{(ab)} = -8\pi \left[ T_{(ab)} - (1/2)g_{(ab)}T(\lambda) \right] \]

\[ = -8\pi T_{(ab)}. \] \[ \cdots \cdots . (4.1.10) \]

Here we have taken units such that \( G = c = 1 \). Calculations of the Ricci tensor \( R_{(\alpha\beta)} \) for the axially symmetric time dependent metric are given in the appendix. The Einstein field equations for the radiation zone are as follows.

\[
\left[ \sigma,_{2}\psi,_{2} + \psi,_{22} + (\psi,_{2})^{2} + (\psi,_{2}/r) \right] \exp(-2\beta) + (1/r^{2}) \left[ \sigma,_{3}\psi,_{3} + \psi,_{33} + (\psi,_{3})^{2} \right] \exp(-2\beta) - \left[ \sigma,_{00} - \sigma,_{0}\psi,_{0} + \sigma,_{0}^{2} - 2\beta,_{0}\psi,_{0} + 2\beta,_{00} + 2\beta,_{0}^{2} \right] \exp(-2\beta) - (1/2) \left[ \omega,_{2}^{2} + (\omega,_{3}/r) \right] \exp(-2\beta) - (1/2) \left[ \omega,_{2}^{2} + (\omega,_{3}/r) \right] \exp(-2\beta) - (1/2) \left[ \omega,_{2}^{2} + (\omega,_{3}/r) \right] \exp(-2\beta) - (1/2) \left[ \omega,_{2}^{2} + (\omega,_{3}/r) \right] \exp(-2\beta) - (1/2) \left[ \omega,_{2}^{2} + (\omega,_{3}/r) \right] \exp(-2\beta) = -8\pi q, \]

\[ \cdots \cdots . . . . . (4.1.11) \]

\[
(1/2) \left[ (3\sigma,_{2} - \psi,_{2} + 1/r)\omega,_{2} + \omega,_{22} \right] \exp(\sigma - 2\beta - \psi) + (1/2r^{2}) \left[ (3\sigma,_{3} - \psi,_{3})\omega,_{3} + \omega,_{33} \right] \exp(\sigma - 2\beta - \psi) = -8\pi q, \]

\[ \cdots \cdots . . . . . (4.1.12) \]

\[
- \left[ \sigma,_{22} + (\sigma,_{2}/r) + \sigma,_{2}\psi,_{2} + \sigma,_{2}^{2} \right] \exp(-2\beta) - (1/r^{2}) \left[ \sigma,_{33} + \sigma,_{3}\psi,_{3} + \sigma,_{3}^{2} \right] \exp(-2\beta) + \left[ \sigma,_{00} - \sigma,_{0}\psi,_{0} + 2\sigma,_{0}\beta,_{0} + \sigma,_{0}^{2} \right] \exp(-2\psi) - (1/2) \left[ \omega,_{2}^{2} + (\omega,_{3}/r) \right] \exp(2\sigma - 2\beta - 2\psi) = -8\pi q, \]

\[ \cdots \cdots . . . . . (4.1.13) \]
\[
\begin{align*}
\begin{cases}
3\sigma_0 - \psi_0 \bigl( \omega_2 + \omega_{20} \bigr) = 0, \\
3\sigma_0 - \psi_0 \bigl( \omega_3 + \omega_{30} \bigr) = 0,
\end{cases} \quad \ldots \ldots \ (4.1.14)
\end{align*}
\]
\[
\begin{align*}
- \left[ \sigma_{22} - \sigma_{2}^2 + (\beta_2/r) + \beta_{22} - \beta_{2}^2 \psi_{2} + \psi_{22} + \psi_{2}^2 \right] \\
- (1/r^2) \left[ \sigma_{3}^3 + \beta_{33} + \beta_{3}^2 \psi_{3} \right] + \left[ \sigma_{0}^2 + 2(\beta_0)^2 - \beta_{0} \psi_{0} \right] \\
p_{00} \exp(2\beta - 2\psi) + (1/2) \omega_{2}^2 \exp(2\sigma - 2\psi) = 0, \\
& \ldots \ldots \ (4.1.16)
\end{align*}
\]
\[
\begin{align*}
- \left[ \sigma_{23} - \sigma_{2}^2 + (\beta_2/r) + \beta_{2}^2 \psi_{2} + \psi_{23} \right] \\
+ \left[ \omega_{2}^2 \psi_{3} - (\beta_2/r) \psi_{3} \right] + (1/2) \omega_{2} \omega_{3} \exp(2\sigma - 2\psi) = 0, \\
& \ldots \ldots \ (4.1.17)
\end{align*}
\]
\[
\begin{align*}
- \left[ \sigma_{20} + \sigma_{2}^2 + \sigma_{0}^2 + \sigma_{0} \psi_{2} - \beta_{20}^2 \psi_{2} \right] = 0, \\
& \ldots \ldots \ (4.1.18)
\end{align*}
\]
\[
\begin{align*}
- \left[ \sigma_{2}^2 + (\sigma_{2}^2 + \beta_2/r) + \beta_{22} + \beta_{2}^2 \psi_{2} + \psi_{22}/r \right] - (1/r^2) \left[ \sigma_{3}^3 \\
- \sigma_{3}^2 + \sigma_{3}^2 + \beta_{33}^2 - \beta_{3}^2 \psi_{3} + \psi_{33} + \psi_{3}^2 \right] + \left[ \sigma_{0} \beta_{0} + 2\beta_{0}^2 \\
+ \beta_{00} - \beta_{0} \psi_{0} \right] \exp(2\beta - 2\psi) + (1/2r^2) \omega_{3}^2 \exp(2\sigma - 2\psi) = 0, \\
& \ldots \ldots \ (4.1.19)
\end{align*}
\]
4.2 Solutions of Einstein Field Equations for Radiation Zone:

The field equations (4.1.11) and (4.1.13) give

\[
\begin{bmatrix}
- \sigma_{30} + \sigma_3 \beta_{0} - \sigma_0 \beta_{30} + \sigma_0 \psi_{3} - \beta_{1} \psi_{3} - \beta_{3} \psi_{1}
\end{bmatrix} = 0.
\] ......... (4.1.20)

Now the equations (4.1.18) and (4.2.1) suggest a solution

\[\sigma + \psi = 0,\] ......... (4.2.2)

and \[\beta = \beta(r, \theta).\] ......... (4.2.3)

Then equation (4.1.17) becomes

\[\omega_{2} \omega_{3} = 4 \psi_{2} \psi_{3} \exp(4\psi).\]

\[\text{i.e. } \frac{\partial \omega}{\partial r} \frac{\partial \omega}{\partial \theta} = \frac{\partial}{\partial r} \left( \exp(2\psi) \right) \frac{\partial}{\partial \theta} \left( \exp(2\psi) \right).\]

Hence we can select a solution of this equation as

\[\omega = \exp(2\psi),\] ......... (4.2.4)
and equations (4.1.16) and (4.1.19) reduce to an identical result

\[ r^2 \beta_{22} + r \beta_{2} + \beta_{33} = 0. \] \hspace{1cm} \ldots \ldots \ldots (4.2.5)

Now equation (4.1.18) becomes

\[- \psi_{20} + 2\psi_2 \psi_0 = 0. \] \hspace{1cm} \ldots \ldots \ldots (4.2.6)

Authors [3] have discussed the solutions for axially symmetric space-time vacuum field equations, in these solutions \( \sigma \) and \( \psi \) are functions of \( r \) and \( \theta \). Hence for axisymmetric radiation zone, let us perturb these quantities and consider that \( \sigma \) is a function of retarded time \( u \) and \( \theta \), where \( u = t - r \). Then, equation (4.2.6) becomes

\[ \frac{\partial^2 \psi}{\partial u^2} - 2 \left( \frac{\partial \psi}{\partial u} \right)^2 = 0. \] \hspace{1cm} \ldots \ldots \ldots (4.2.7)

The general solution of equation (4.2.7) is

\[ \psi = -(1/2) \log(Bu + F), \] \hspace{1cm} \ldots \ldots \ldots (4.2.8)

where \( F \) is an arbitrary function of \( \theta \) and \( B \) is an arbitrary constant. For this solution of \( \psi \) equation (4.1.20) satisfies identically. To solve the equation (4.2.5) let us suppose that \( e^\beta \) is a separable function in two variables \( r \) and \( \theta \).

i.e. \( e^\beta = P(r)Q(\theta) \).
Therefore, \( \beta = \log P(r) + \log Q(\theta) = R(r) + H(\theta) \).

Then equation (4.2.5) becomes

\[
r^2R'' + rR' = -H'' ,
\]

where overhead \( \partial \) represents derivative with respect to its argument. As L.H.S. is a function of \( r \) only and R.H.S is a function of \( \theta \) only, each side must be constant. This implies that

\[
r^2R'' + rR' = k, \quad \ldots \ldots \ldots (4.2.9.a)
\]

\[
H'' + k = 0. \quad \ldots \ldots \ldots (4.2.9.b)
\]

where \( k \) is constant. The solution of equation (4.2.9.b) is

\[
H = \log C_1 + A\theta - (k\theta^2/2),
\]

where \( C_1 \) and \( A \) are arbitrary constants. To solve equation (4.2.9.a) let us substitute \( \xi = \log r \) then equation (4.2.9) transforms to

\[
\frac{\partial^2 R}{\partial \xi^2} = k.
\]

Solution of this equation is

\[
R = \frac{k}{2} \xi^2 + D\xi + \log E = \frac{k}{2} \left( \log r \right)^2 + D \log r + \log E,
\]

where \( D \) and \( E \) are arbitrary constants.
Hence

$$\beta = \frac{k}{2} (\log r)^2 + D \log r + A \theta - \frac{k \theta^2}{2} + \log C, \ldots \ldots (4.2.10)$$

$$2\beta = C^2 e^{2D} k (\log r)^2 + A \theta - \frac{k \theta^2}{2}, \ldots \ldots \ldots (4.2.11)$$

The arbitrary constants may be determined from the physical requirement.

4.3 The Landau-Lifshitz Complex and the Conserved Quantities:

Let us apply Cornish's prescription to obtain the Landau-Lifshitz complex in terms of which the various conserved quantities can be defined. We shall specialize the coordinate system to that of spherical polar coordinates at infinity we shall write

$$ds^2 = -\nu^2 dt^2 + \rho^2 e^{2\delta} \sin^2 \alpha (d\phi - \omega dt)^2 + e^{2K} (d\rho^2 + \rho^2 d\alpha^2), \ldots \ldots \ldots (4.3.1)$$

where $\nu, \delta, \omega$ and $K$ are functions of $t, \rho$ and $\alpha$. Also we assume that the foregoing metric tends to the flat metric

$$ds^2 = -dt^2 + \rho^2 \sin^2 \alpha d\phi^2 + d\rho^2 + \rho^2 d\alpha^2, \ldots \ldots \ldots (4.3.2)$$

According to the Cornish's prescription, we shall call (4.3.2) the b-metric. Here we shall use the coordinates

$$(x^0, x^1, x^2, x^3) \leftrightarrow (t, \phi, \rho, \alpha).$$
To evaluate the Landau-Lifshitz complex, appropriately for the system described by the metric (4.3.1) let us take

\[ h^{ij} = \frac{(-g)^{1/2}}{\rho^2 \sin^2 \alpha} g^{ij} = e^{(\delta + \nu + 2K)} g^{ij}, \quad \ldots \ldots \quad (4.3.3) \]

and define quantities \( T^{ikjl} \) and \( \chi^{ikj} \) in terms of \( h^{ij} \) as

\[ T^{ikjl} = h^{il}h^{kj} - h^{ij}h^{kl}, \quad \ldots \ldots \quad (4.3.4) \]

\[ \chi^{ikj} = T^{ikjl} ;l, \quad \ldots \ldots \quad (4.3.5) \]

where \( ;l \) signifies covariant differentiation with respect to \( x^l \) and b-metric. Then we obtain

\[ \phi^{ij} = (1/2) \chi^{ikj};k. \quad \ldots \ldots \quad (4.3.6) \]

The Landau-Lifshitz pseudo-tensor \( t^{ij} \) and complex \( \theta^{ij} \) are defined by

\[ -8\pi e^{2(\nu + \delta + 2K)} t^{ij} = \phi^{ij} + e^{2(\nu + \delta + 2K)} g^{ij}, \quad \ldots \ldots \quad (4.3.7) \]

\[ \theta^{ij} = e^{2(\nu + \delta + 2K)} (t^{ij} + T^{ij}). \quad \ldots \ldots \quad (4.3.8) \]
Now according to Cornish's prescription, the total energy $M$ is given by

$$M = - \frac{1}{8\pi} \iiint_{s \rightarrow \alpha} \phi^{00} \rho^2 \sin \alpha \, d\rho \, d\alpha \, d\phi,$$

$$= - \frac{1}{16\pi} \iiint_{s \rightarrow \alpha} \chi^{020} \rho^2 \sin \alpha \, d\rho \, d\alpha \, d\phi,$$

where the notation $s \rightarrow \alpha$ under the integral sign means that the integral is first evaluated within a volume enclosed by a sphere of radius $\rho$ or on the surface of a sphere of radius $\rho$ and then the limit of resulting expression is taken by making $\rho \rightarrow \alpha$. Now evaluating the required quantity $\chi^{020}$, we find that

$$\chi^{020} = \left[ 2(\delta + K),_2 + (1/\rho)(1 - e^{2(K-\delta)}) \right] e^{2(\delta+K)}.$$

Using (4.3.9) and (4.3.10), we get the conversed total mass energy

$$M = - \left(1/8\right) \lim_{\rho \rightarrow \alpha} \int_{0}^{\pi} \left( \partial/\partial \rho \right) \left( e^{2(\delta+K)} \right) + \left( e^{2K}/\rho \right) \left( e^{2\delta} - e^{2K} \right) \rho \sin \alpha \, d\alpha.$$

Now to compare metric (4.3.1) and (4.1.4) take transformation

$$\nu = \psi = (-1/2) \left( \log(Bu + F(\theta)) \right), \quad \rho = r, \quad \alpha = \theta.$$
\[
e^{2\delta} = \frac{e^{2\alpha}}{r^2 \sin^2 \theta} = \frac{Bu + F(\theta)}{r^2 \sin^2 \theta} \quad \text{and}
\]

\[
e^{2K} = e^{2\beta} = C^2 \quad \text{2D} \quad e^{k(\log r)^2 + 2A\theta - k\theta^2}, \quad \ldots \ldots \ldots \quad (4.3.12)
\]

Therefore

\[
M = -\left(\frac{1}{8}\right) \lim_{r \to \infty} \int_0^\pi \left( \frac{\partial}{\partial r} \right) \left\{ e^{2\alpha} + \frac{2\beta}{r} \frac{2}{\sin^2 \theta} \right\} \] \[+ \left( \frac{e^{2\beta}}{r} \right) \left( \frac{e^{2\alpha}}{r^2 \sin^2 \theta} - e^{2\beta} \right) \left( r^2 \sin \theta \d\theta \right). \quad \ldots \ldots \ldots \quad (4.3.13)
\]

Now substituting value of \( \beta \) from (4.3.12), one can obtain the total energy \( M \) associate with problem. \( M \) to be non zero and finite this result demands that \( k = 0 \) and \( D = (-1/4) \).

Then

\[
M = (C^4/8) \left[ \frac{e^{4\pi A} + 1}{16A^2 + 1} \right]. \quad \ldots \ldots \ldots \quad (4.3.14)
\]

Now the value of energy flux density \( q \) is determined by the equations (4.1.11) and (4.1.12). straight forward calculations gives

\[
q = \left( \frac{1}{16\pi} \right) \left[ \frac{B/r + F''/r^2}{r^2} \right] \left[ e^{-2\beta}/Bu+F \right]
\]

\[
= \left( \frac{1}{16\pi} \right) \left[ \frac{rB + F''}{r^{3/2}c^2} \right] \quad \ldots \ldots \ldots \quad (4.3.15)
\]
The final form of the metric representing gravitational field of axisymmetric radiation zone is

\[ ds^2 = -2(du + dr)d\phi + \left( \frac{8M(16A^2 + 1)}{r(e^{4\pi A} + 1)} \right)^{1/2} e^{2A\theta} (dr^2 + r^2 d\theta^2) \]

\[ + (Bu + F) d\phi^2, \]

... ... ... (4.3.16)

Where A and B are arbitrary real numbers.

4.4 Conclusion :-

The metric representing gravitational field of axially symmetric radiation zone has a singularity at origin. The total mass energy associated with the axially symmetric radiation zone is shown to be finite. Therefore the metric we have obtained represents gravitational field of solid matter at origin surrounded by axially symmetric radiation energy. On the boundary of null hyper surface \( u = 0 \), the metric match with the axially symmetric empty space-time metric.
REFERENCES:


