CHAPTER 1

ALGEBRAS OF VECTOR VALUED FUNCTIONS

INTRODUCTION AND PRELIMINARIES

Let $A$ be a function algebra on $X$. Let $A \otimes B$ denote the completion of the algebraic tensor product $A \otimes B$ of $A$ with a Banach algebra $B$ with respect to the least cross-norm $\Lambda$. Now $A \otimes B$ can be looked upon as a subalgebra of $C(X;B)$, the algebra of all $B$-valued functions on $X$. In lemma 1.3, which is crucial for our study, we show that $\Lambda$ coincides with the sup. norm on $A \otimes B$. Hence $A \otimes B$ can be looked upon as a 'vector-valued function algebra $A \otimes B$ on $X$', where $A \otimes B$ denotes the uniform closure of $A \otimes B$ in $C(X;B)$. In this chapter we study both the aspects of $A \otimes B$, as a tensor product of two Banach algebras $A$ and $B$ and also as a vector-valued function algebra on $X$.

**Definition 1.1**: Let $X$ be a compact Hausdorff space and $B$ be a commutative Banach algebra with identity $e$. 
Suppose $C(X; B)$ denotes the set of all $B$-valued continuous functions on $X$. Then $C(X; B)$ is a commutative Banach algebra under pointwise operations, possessing an identity. For $f \in C(X; B)$ define

$$|| f || = \sup_{x \in X} || f(x) ||_B$$

Then $|| \cdot ||$ defines a norm on $C(X; B)$ making it into a Banach algebra. For $f \in C(X)$ and $b \in B$, the function $f.b$ defined by $(f.b)(x) = f(x)b$ is in $C(X; B)$. Thus $C(X)$ and $B$ both are embedded in $C(X; B)$ and hence $C(X; B)$ separates the points of $X$ and contains all constant functions. Taking $B = \mathbb{C}$, we get the standard algebra $C(X)$.

**Definition 1.2:** Let $K$ be a closed subalgebra of $C(X; B)$. $K$ is called a vector function algebra on $X$ (or a $B$-function algebra on $X$) if it separates the points of $X$ and contains all constant functions.

For example let $A$ be a function algebra on $X$. For $f \in A$ and $b \in B$, $f.b$ as defined above is in $C(X; B)$.
Let $A \Box B$ be the space of all finite linear combinations of such functions and $A \hat{\Box} B$ be its closure in $C(X; B)$. Then $A \hat{\Box} B$ is a vector function algebra on $X$.

Throughout the chapter $A$ will denote a function algebra on $X$.

**Lemma 1.3**: $A \hat{\Box} B$ is isometrically isomorphic to $A \hat{\otimes} B$ where $\lambda$ is the least cross-norm.

**Proof**: Define a mapping $\Theta : A \times B \to A \Box B$ by $\Theta(f, b) = f \cdot b$. Then $\Theta$ is a bilinear mapping. Hence there exists a unique linear mapping ([8]; p.232) $\sigma$ from $A \hat{\otimes} B$ to $A \Box B$ such that $\sigma(f \hat{\otimes} b) = f \cdot b$.

$\sigma$ is clearly onto. $\sigma$ is an isometry if we give the norm $\lambda$ to $A \hat{\otimes} B$. For $x \in X$, $\phi_x$ is a linear functional on $A$ of norm 1 and hence if

$$\sum_{i=1}^{n} f_i \hat{\otimes} b_i \in A \hat{\otimes} B$$

then
\[ \lambda(\sum_{i=1}^{n} f_i \otimes b_i) = \text{Sup.} \left\{ \left| \sum_{i=1}^{n} \phi(f_i) \psi(b_i) \right| : \phi \in A^*, \psi \in B^*, \|\phi\| \leq 1, \|\psi\| \leq 1 \right\} \]

\[ \geq \text{Sup.} \left\{ \left| \sum_{i=1}^{n} \phi_x(f_i) \psi(b_i) \right| : x \in X, \psi \in B^*, \|\psi\| \leq 1 \right\} \]

\[ = \text{Sup.} \left\{ \left| \sum_{i=1}^{n} f_i(x) \psi(b_i) \right| : x \in X, \psi \in B^*, \|\psi\| \leq 1 \right\} \]

\[ = \text{Sup.} \left( \sup_{x \in X} \left\{ \left| \sum_{i=1}^{n} f_i(x) b_i \right| \right\} : \psi \in B^*, \|\psi\| \leq 1 \right\} \]

Thus

\[ (1) \quad \lambda(\sum_{i=1}^{n} f_i \otimes b_i) \geq \| \sigma(\sum_{i=1}^{n} f_i \otimes b_i) \| \]
Conversely, let $\Phi \in U(A^*)$ and $\Psi \in U(B^*)$, where $U(A^*)$ denotes the unit ball of $A^*$. Then

$$\left| \sum_{i=1}^{n} \Phi(f_i) \Psi(b_i) \right| = \left| \Phi \left( \sum_{i=1}^{n} \Psi(b_i) f_i \right) \right|$$

$$\leq \left\| \sum_{i=1}^{n} \Psi(b_i) f_i \right\|$$

$$= \left| \sum_{i=1}^{n} f_i(x_o) \Psi(b_i) \right|$$

for some $x_o \in$ for some $x_o \in$

$$= \Psi \left( \sum_{i=1}^{n} f_i(x_o) b_i \right)$$

$$\leq \left\| \sum_{i=1}^{n} f_i(x_o) b_i \right\|_B$$

$$\leq \left\| \sum_{i=1}^{n} f_i \cdot b_i \right\|$$

Then by taking sup. over $\Phi$ and $\Psi$ we get

$$\lambda \left( \sum_{i=1}^{n} f_i \otimes b_i \right) \leq \left\| \sigma \left( \sum_{i=1}^{n} f_i \otimes b_i \right) \right\|$$
Combining (1) and (2) we get that \( \sigma \) is an isometry. Also it can be easily verified that \( \sigma \) is an algebra homomorphism. Thus \( \sigma \) is an isometric isomorphism of \( A \otimes B \) on-to \( A \boxtimes B \) and hence it has a unique extension as an isometric isomorphism from \( A \otimes B \) onto \( A \hat{\boxtimes} B \).

In view of this lemma we can identify \( A \boxtimes B \) with \( A \otimes B \). This we shall denote henceforth by \( A \hat{\otimes} B \).

Hausner [34] has shown that \( C(X) \hat{\boxtimes} B \) is equal to \( C(X;B) \). Hence in our notations \( C(X) \hat{\otimes} B = C(X;B) \).

Because of this identification, propositions 3.4 and 3.5 of [26] follow immediately if we apply the following well known results regarding tensor product.

(3) If \( L \) and \( M \) are Banach spaces then \( L \hat{\otimes} M \) is separable if and only if \( L \) and \( M \) are separable.
( [60] , Lemma 2.4).

(4) If \( L \) and \( M \) are finite dimensional normed linear spaces then

\[
\dim(L \hat{\otimes} M) = \dim L \times \dim M \quad ( [8] , \text{lemma 5} , \text{p.231})
\]
We note in passing that \( f \in C(X; B) \) is invertible if and only if \( f(x) \in B^{-1} \) for every \( x \in X \). Consequently

\[
(5) \quad \sigma_{C(X;B)}(f) = \bigcup_{x \in X} \sigma_B(f(x))
\]

where \( \sigma_B(b) \) denotes the spectrum of an element \( b \) in \( B \).

When \( X \) is locally compact, the analogue of \( C(X; B) \) is \( C_0(X; B) \), the algebra of \( B \)-valued functions vanishing at infinity. Fields has established (5) for \( C_0(X; B) \) ([26], lemma 3.13).

**POINT DERIVATIONS ON \( A \hat{\otimes} B \).**

In the study of any commutative Banach algebra the characterization of two special classes of linear functionals are useful and important viz. the class of nonzero complex homomorphisms (equivalently the space of maximal ideals) and the class of point derivations.

As we have seen the sup norm on \( A \hat{\otimes} B \) coincides with the least cross-norm \( \Lambda \) and it is an algebra norm. Hence by a result of Robbins [58], and a remark there in, we get
Proposition 1.4: The maximal ideal space $m(A \hat{\otimes} B)$ of $A \hat{\otimes} B$ is homeomorphic to $m(A) \times m(B)$, the product of maximal ideal spaces of $A$ and $B$ respectively.

Each $\phi \in m(A \hat{\otimes} B)$ is related to the corresponding pair $(\phi_A, \phi_B)$ in $m(A) \times m(B)$ by

$$\phi\left( \sum_{i=1}^{n} f_i \cdot b_i \right) = \sum_{i=1}^{n} \phi_A(f_i) \phi_B(b_i) \quad (f_i \in A, b_i \in B).$$

we shall identify $\phi$ and the corresponding pair $(\phi_A, \phi_B)$.

As an immediate consequence we get

Proposition 1.5: If $B$ is semisimple then $A \hat{\otimes} B$ is semisimple.

Proof: Let $f \in A \hat{\otimes} B$ be in the radical of $A \hat{\otimes} B$ i.e. $\phi(f) = 0$ for all $\phi \in m(A \hat{\otimes} B)$. In particular $(\phi_A, \phi_B)(f) = 0$ for all $x \in X$, $\phi_B \in m(B)$. But $(\phi_A, \phi_B)(f) = \phi_B(f(x))$. Therefore $\phi_B(f(x)) = 0$ for every $x \in X$ and $\phi_B \in m(B)$. Since $B$ is semisimple, $f(x) = 0$ for all $x$ in $X$ i.e. $f \equiv 0$. Thus $A \hat{\otimes} B$ is semisimple.
The converse 'if $A \hat{\otimes} B$ is semisimple then $B$ is semisimple' is trivial. $A$ is always semisimple as it is a function algebra.

Remark 1.6: Proposition 1.5 does not hold for the tensor product of two arbitrary Banach algebras ([8], p. 237).

Remark 1.7: By a result of Gelbaum [30], if $\Gamma(A)$ denotes the Silov boundary of $A$ then, $\Gamma(A \hat{\otimes} B)$ is homeomorphic to $\Gamma(A) \times \Gamma(B)$, the homeomorphism being the restriction of the homeomorphism mentioned in prop. 1.4.

Since $\Gamma(A) \subset X$, $\Gamma(A \hat{\otimes} B) \subset X \times m(B)$. Using this we can prove that

(6) Radical of $A \hat{\otimes} B = \{ f \in A \hat{\otimes} B : f(X) \subset \text{Radical of } B \}$

Prop. 1.5 can be deduced from (6) also. The result (6) for $C_0(X;B)$, where $X$ is locally compact, was proved by Fields ([26], Lemma 3.11). He proved that

$\text{Rad} (C_0(X;B)) = \{ f \in C_0(X;B) : f(X) \subset \text{Rad } (B) \}$.

Next we characterize the bounded point derivations on $A \hat{\otimes} B$. 
Theorem 1.8: Let \( \Phi = (\Phi_A, \Phi_B) \in \mathfrak{m}(A \hat{\otimes} B) \). Then \( D \) is a bounded point derivation on \( A \hat{\otimes} B \) at \( \Phi \) if and only if there exist bounded point derivations \( D_A \) on \( A \) at \( \Phi_A \) and \( D_B \) on \( B \) at \( \Phi_B \) such that \( D = D_1 + D_2 \) where

\[
\begin{align*}
D_1 \left( \sum_{i=1}^{\eta} f_i \cdot b_i \right) &= \sum_{i=1}^{\eta} D_A(f_i) \Phi_B(b_i), \\
D_2 \left( \sum_{i=1}^{\eta} f_i \cdot b_i \right) &= \sum_{i=1}^{\eta} \Phi_A(f_i) D_B(b_i)
\end{align*}
\]

Proof: Suppose \( D_A \) is a bounded point derivation on \( A \) at \( \Phi_A \). Define \( D_1 \) on \( A \hat{\otimes} B \) by

\[
D_1 \left( \sum_{i=1}^{\eta} f_i \cdot b_i \right) = \sum_{i=1}^{\eta} D_A(f_i) \Phi_B(b_i).
\]
$D_1$ is well defined. For if $D_A = 0$ then $D_1 = 0$.

If $D_A \neq 0$, let

$$\sum_{i=1}^{n} f_i \cdot b_i = \sum_{j=1}^{m} g_j \cdot c_j,$$

then

$$\sum_{i=1}^{n} f_i \cdot b_i - \sum_{j=1}^{m} g_j \cdot c_j = 0.$$  

By lemma 1.3 we have

$$\lambda \left( \sum_{i=1}^{n} f_i \cdot b_i + \sum_{j=1}^{m} (-g_j) \cdot c_j \right) = 0.$$

Hence

$$\left\| \frac{1}{\| D_A \|} \left( \sum_{i=1}^{n} D_A(f_i) \cdot \Phi_B(b_i) - \sum_{j=1}^{m} D_A(g_j) \cdot \Phi_B(c_j) \right) \right\| \leq \lambda \left( \sum_{i=1}^{n} f_i \cdot b_i - \sum_{j=1}^{m} g_j \cdot c_j \right) = 0.$$
Thus
\[
\sum_{i=1}^{\eta} D_A(f_i) \Phi_B(b_i) = \sum_{j=1}^{m} D_A(g_j) \Phi_B(c_j),
\]
which proves that \( D_1 \) is well defined. It is easy to check that \( D_1 \) is linear on \( A \otimes B \). Now if \( f, g \in A \otimes B \) with
\[
f = \sum_{i=1}^{\eta} f_i \cdot b_i, \quad g = \sum_{j=1}^{m} g_j \cdot c_j,
\]
then
\[
D_1(fg) = D_1 \left( \sum_{j=1}^{m} \sum_{i=1}^{\eta} f_i g_j \cdot b_i c_j \right)
\]
\[
= \sum_{j=1}^{m} \sum_{i=1}^{\eta} D_A(f_i g_j) \Phi_B(b_i c_j)
\]
\[
\sum_{j=1}^{m} \sum_{i=1}^{n} \left[ D_A(f_i) \Phi_A(g_j) + \Phi_A(f_i) D_A(g_j) \right] \Phi_B(b_i) \Phi_B(c_j)
\]

\[
\sum_{j=1}^{m} \sum_{i=1}^{n} \left[ D_A(f_i) \Phi_B(b_i) \Phi_A(g_j) \Phi_B(c_j) + \Phi_A(f_i) \Phi_B(b_i) D_A(g_j) \Phi_B(c_j) \right]
\]

\[
\sum_{j=1}^{m} \sum_{i=1}^{n} \left[ D_1(f_i b_i) \Phi(g_j c_j) + \Phi(f_i b_i) D_1(g_j c_j) \right]
\]

\[
\Phi \left( \sum_{j=1}^{m} g_j c_j \right) D_1 \left( \sum_{i=1}^{n} f_i b_i \right) + \Phi \left( \sum_{i=1}^{n} f_i b_i \right) D_1 \left( \sum_{j=1}^{m} g_j c_j \right)
\]

\[
(9) = \Phi(g) D_1(f) + \Phi(f) D_1(g)
\]

Also, \[
\left| D_1 \left( \sum_{i=1}^{n} f_i b_i \right) \right| \leq \left\| D_A \right\| \lambda \left( \sum_{i=1}^{n} f_i b_i \right)
\]

\[
= \left\| D_A \right\| \left\| \sum_{i=1}^{n} f_i b_i \right\|
\]

\[
(10)
\]
by definition of \( \lambda \) and by lemma 1.3. From (9), (10) and linearity of \( D_1 \), it follows that \( D_1 \) is a continuous point derivation on \( A \otimes B \) at \( \Phi \). Its unique continuous linear extension to \( A \hat{\otimes} B \), which we again denote by \( D_1 \), is then a bounded point derivation on \( A \otimes B \) at \( \Phi \) which satisfied (7).

Similarly it follows that if \( D_B \) is a bounded point derivation on \( B \) at \( \Phi_B \) then there exists a unique bounded point derivation \( D_2 \) on \( A \hat{\otimes} B \) at satisfying (8). Then \( D = D_1 \times D_2 \) is also a bounded point derivation on \( A \hat{\otimes} B \) at \( \Phi \).

Conversely, let \( D \) be a bounded point derivation on \( A \hat{\otimes} B \) at \( \Phi \). Define \( D_A \) and \( D_B \) on \( A \) and \( B \) respectively by \( D_A(f) = D(f, e) \) and \( D_B(b) = D(1, b) \), where \( e \) is the identity of \( B \). Then it can be easily checked that \( D_A \) and \( D_B \) are bounded point derivations on \( A \) and \( B \) at \( \Phi_A \) and \( \Phi_B \) respectively. Let \( D_1 \) and \( D_2 \) be the bounded point derivations at \( \Phi \) on \( A \hat{\otimes} B \).
corresponding to \( D_A \) and \( D_B \) respectively, as defined in the first part of the proof and let \( \psi = D_1 + D_2 \).

Then for

\[
f = \sum_{i=1}^{n} f_i b_i \text{ in } A \otimes B,
\]

\[
\psi(f) = (D_1 + D_2) \left( \sum_{i=1}^{n} f_i b_i \right)
\]

\[
= \sum_{i=1}^{n} \left\{ D_A(f_i) \phi_B(b_i) + \phi_A(f_i) D_B(b_i) \right\}
\]

\[
= \sum_{i=1}^{n} \left\{ D(f_i \cdot e) \phi(1 \cdot b_i) + \phi(f_i \cdot e) D(1 \cdot b_i) \right\}
\]

\[
= \sum_{i=1}^{n} D(f_i \cdot e) (1 \cdot b_i)
\]

\[
= \sum_{i=1}^{n} D(f_i \cdot b_i)
\]

\[
= D \left( \sum_{i=1}^{n} f_i b_i \right)
\]

\[
= D(f)
\]
Therefore, $\Psi = D$ on $A \otimes B$ and since both are continuous
$\Psi = D$ on $A \otimes B$. Thus $D = D_1 + D_2$ as was to be proved.

Remark 1.9: It can be seen from the above proof that if $B_1$ and $B_2$ are any two Banach algebras with identities
and $\alpha$ is a cross-norm on $B_1 \otimes B_2$ which is also an algebra
norm and if $\lambda \leq \alpha \leq \rho$ ($\lambda$ - the least cross-norm,
$\rho$ - the greatest cross-norm) then we have a similar char­
acterization for bounded point derivations on $B_1 \otimes B_2$.

Corollary 1.10: Let $x \in X$. Then a linear functional
$D$ on $C(X; B)$ is a bounded point derivation on $C(X; B)$
at $(x, \Phi_B)$ if and only if there exists a bounded point
derivation $D_B$ on $B$ at $\Phi_B$ such that $D(f) = D_B(f(x))$
for each $f$ in $C(X; B)$.

Proof: Since $C(X; B) = C(X) \otimes B$, by the above theorem
$D = D_1 + D_2$ where $D_1$ and $D_2$ are as defined in the theorem.
But there is no nonzero point derivation on $C(X)$ at $x$.
([10], [11]). Therefore $D_A = 0$ and hence $D_1 = D$.
Thus $D = D_2$. 
If \( f \in \mathcal{C}(X;B) \) then by continuity of \( D \),
\[
D(f) = \lim_{n \to \infty} D^2(f_n)
\]
where \( \{f_n\} \) is a sequence in \( \mathcal{C}(X) \otimes B \) converging to \( f \). For any element
\[
g = \sum_{j=1}^{m} g_j \cdot c_j \text{ in } \mathcal{C}(X) \otimes B
\]

\[
D_2\left( \sum_{j=1}^{m} g_j c_j \right) = \sum_{j=1}^{m} \phi_x(g_j) D_B(c_j)
\]

\[
= \sum_{j=1}^{m} g_j(x) D_B(c_j)
\]

\[
= D_B\left( \sum_{j=1}^{m} g_j(x) c_j \right)
\]

\[
= D_B(g(x))
\]

Hence \( D(f) = \lim_{n \to \infty} D^2(f_n) \),
\[
= \lim_{n \to \infty} D_B(f_n(x))
\]

\[
= D(f(x))
\]
for \( f_n \rightarrow f \) in \( C(X; B) \) implies \( f_n(x) \rightarrow f(x) \) in \( B \) and \( D_B \) is continuous on \( B \).

**Corollary 1.11:** If \( \phi = (\phi_A, \phi_B) \in \mathfrak{m}(A \otimes B) \) then
\[
M^2 = M \quad \text{if and only if} \quad M_A^2 = M_A \quad \text{and} \quad M_B^2 = M_B,
\]
where
\[
M_A = \ker \phi_A, \quad M_B = \ker \phi_B, \quad M = \ker \phi \quad \text{and} \quad M_A^2 \quad \text{denotes the ideal generated by} \quad \{fg : f, g \in M_A \} \quad \text{in} \quad A.
\]

**Proof:** This follows from Thm.1.8 and the result on p.64 in [11].

**Definition 1.12:** Let \( B \) be a commutative Banach algebra with identity and \( M \in \mathfrak{m}(B) \). \( M \) is said to have an approximate identity if there exists a constant \( K \) such that for every \( \varepsilon > 0 \), every \( f_1, f_2, \ldots, f_n \) in \( M \) there exists \( e \in M \) with \( ||e|| \leq K \) such that \( ||ef_j - f_j|| < \varepsilon \) for \( j = 1, 2, \ldots, n \).

Equivalently, \( M \) is said to have an approximate identity if there exists a bounded net \( \{e_\alpha \} \) in \( M \) such that \( e_\alpha f \rightarrow f \) for every \( f \) in \( M \).
Before proving the next proposition we shall discuss the extension of functions in $A \hat{\otimes} B$ to $m(A)$.

Let \( f = \sum_{i=1}^{n} f_i \cdot b_i \) be in $A \hat{\otimes} B$. Define \( \tilde{f} \)
on $m(A)$ by
\[
\tilde{f}(\phi_A) = \sum_{i=1}^{n} \hat{f}_i(\phi_A) b_i.
\]
Then
\[
\tilde{f} \in C(m(A); B). \ \ \tilde{f} \text{ is well defined and is an extension of } f. \text{ It is routine to check that } \| \tilde{f} \| = \| f \|.
\]
Hence if \( f \in A \hat{\otimes} B \), we define \( \tilde{f} = \lim_{n \to \infty} \tilde{f}_n \) where \( \{ \tilde{f}_n \} \) is a sequence in $A \hat{\otimes} B$ converging to $f$. We shall denote by $\tilde{A} \hat{\otimes} \tilde{B}$ the set \( \{ \tilde{f} : f \in A \hat{\otimes} B \} \), which is a vector function algebra on $m(A)$. In fact $\tilde{A} \hat{\otimes} \tilde{B}$ is isometrically isomorphic to $\tilde{A} \hat{\otimes} \tilde{B}$, where $\tilde{A}$ denotes the Gelfand transform of $A$.

**Proposition 1.13**: Suppose \( \phi = (\phi_A, \phi_B) \in m(\tilde{A} \hat{\otimes} \tilde{B}) \). If ker $\phi$ has an approximate identity then each of ker $\phi_A$ and ker $\phi_B$ has an approximate identity.
Proof: Let \( \{f_\alpha\} \) be an approximate identity in \( \ker \Phi \)
i.e. there is \( M > 0 \) such that \( \|f_\alpha\| \leq M \), \( \Phi(f_\alpha) = 0 \)
for each \( \alpha \) and \( \|f_\alpha f - f\| \to 0 \) for all \( f \in \ker \Phi \).

Now \( \Phi_B \circ f_\alpha \) is in \( A \), actually in \( \ker \Phi_A \).

Let \( g_\alpha = \Phi_B \circ f_\alpha \).

Suppose \( g \in \ker \Phi_A \). Then \( g \cdot e \in \ker \Phi \) and
\[
g = \Phi_B \circ (g \cdot e)
\]
Also
\[
\|g_\alpha g - g\| = \|\Phi_B \circ f_\alpha \left( \Phi_B(g \cdot e) \right) - \Phi_B(g \cdot e)\|
\]
\[
\leq \|f_\alpha(g \cdot e) - g \cdot e\| \to 0
\]

Thus \( g \) is an approximate identity for \( \ker \Phi_A \).

Similarly it can be verified that \( \{\tilde{f}_\alpha(\Phi_A)\} \) is an
approximate identity for \( \ker \Phi_B \), where \( \tilde{f}_\alpha \) is the
extension of \( f_\alpha \) on \( m(A) \).
Proposition 1.14: Let $\Phi = (\Phi_A, \Phi_B) \in \text{m}(A \widehat{\otimes} B)$.

Then $M^2 = M$ implies $M_A^2 = M_A$ and $M_B^2 = M_B$, where

$M = \ker \Phi$, $M_A = \ker \Phi_A$ and $M_B = \ker \Phi_B$.

Proof: Suppose $M^2 = M$. Take $b \in M_B$. Then $1.b \in M$.

Hence, as $M^2 = M$, there exist $g_i, h_i \in M$ $i=1,2,...,n$ such that $1.b = \sum_{i=1}^{n} g_i \cdot h_i$. Then $\tilde{g}_i(\Phi_A)$, $\tilde{h}_i(\Phi_A)$ are in $M_B$ for $i=1,2,...,n$ and $b = \sum_{i=1}^{n} \tilde{g}_i(\Phi_A) \cdot \tilde{h}_i(\Phi_A)$.

Similarly it can be shown that $M_A^2 = M_A$.

Remark 1.15: It is well known that $M^2 = M$ if and only if there is no nonzero point derivation at $\Phi$ ([11]; p.64).

However, while we have characterized bounded point derivations on $A \widehat{\otimes} B$, no such characterization is possible for all point derivations on $A \widehat{\otimes} B$. Consequently, while it has been possible to deduce the coro. 1.11 from thm. 1.8, prop.1.14
cannot be proved in a similar fashion. But we can reverse the process and use the prop. 1.14 to get information about point derivations on $A \hat{\otimes} B$.

**Corollary 1.16**: If there is no nonzero point derivation on $A \hat{\otimes} B$ at $\phi = (\phi_A, \phi_B)$ then there is no nonzero point derivation on $A$ at $\phi_A$ and on $B$ at $\phi_B$.

The consideration of converse of the above corollary leads to the following general question whose answer we do not know.

**QUESTION**: If a point derivation vanishes on a dense subalgebra of a commutative Banach algebra with identity, is it necessarily zero?

**AUTOMORPHISMS OF $A \hat{\otimes} B$**

As in the case of complex homomorphisms and bounded point derivations one would expect that every automorphism of $A \hat{\otimes} B$ will be determined by a pair of automorphism on $A$ and $B$. But this is not true in general even for $C(X;B)$.
as will be shown later. The result holds one way: given an automorphism of $A$ and an automorphism of $B$ we get an automorphism of $A \otimes B$ if $B$ is semisimple (Prop. 1.17). The converse holds for automorphisms satisfying certain conditions. Throughout our discussion regarding automorphisms, we assume that $B$ is semisimple.

**Proposition 1.17:** Let $T_A$ and $T_B$ be automorphisms of $A$ and $B$ respectively. Define $T$ on $A \otimes B$ by

$$T(\sum_{i=1}^{n} f_i \otimes b_i) = \sum_{i=1}^{n} T_A(f_i) T_B(b_i) \quad f_i \in A, b_i \in B, \quad 1 \leq i \leq n.$$ 

Then $T$ defines a continuous automorphism on $A \otimes B$ and hence has a unique extension to an automorphism on $A \otimes B$.

**Proof:** Suppose $T_A(f_i) = g_i \quad i=1,2,\ldots,n$.

Then

$$\| \sum_{i=1}^{n} T_A(f_i) T_B(b_i) \| = \sup_{x \in X} \| \sum_{i=1}^{n} g_i(x) T_B(b_i) \|_B$$

$$= \sup_{x \in X} \| T_B \left( \sum_{i=1}^{n} g_i(x) b_i \right) \|_B$$

$$= \| T_B \| \sup_{x \in X} \| \sum_{i=1}^{n} g_i(x) b_i \|_B$$
because $B$ is semisimple implies $T_B$ is continuous

([44] ; p.135).

Since $\sum_{i=1}^{n} g_i b_i$ is a $B$-valued continuous function, there exists $x_0 \in X$

Such that $\sup_{x \in X} \left\| \sum_{i=1}^{n} g_i(x) b_i \right\|_B = \left\| \sum_{i=1}^{n} g_i(x_0) b_i \right\|_B$

Thus $\left\| \sum_{i=1}^{n} T_A(f_i) T_B(b_i) \right\| \leq \left\| T_B \right\| \left\| \sum_{i=1}^{n} g_i(x_0) b_i \right\|_B$

Now

$\left\| \sum_{i=1}^{n} g_i(x_0) b_i \right\|_B = \sup \left\{ \left\| \sum_{i=1}^{n} g_i(x_0) \psi(b_i) \right\| : \psi \in U(B^*) \right\}$
\[ = \sup_{i=1}^{\eta} \left\{ \left| \sum_{i=1}^{\eta} (T_A(f_i))(x_0) \psi(b_i) \right| : \psi \in U(B^*) \right\} \]

\[ = \sup_{i=1}^{\eta} \left\{ \left| (T_A(\sum_{i=1}^{\eta} \psi(b_i)f_i))(x_0) \right| : \psi \in U(B^*) \right\} \]

\[ \leq \sup_{i=1}^{\eta} \left\{ \left\| T_A \left( \sum_{i=1}^{\eta} \psi(b_i)f_i \right) \right\| : \psi \in U(B^*) \right\} \]

\[ \leq \left\| T_A \right\| \sup_{\psi \in U(B^*)} \sup_{x \in X} \left\| \sum_{i=1}^{\eta} f_i(x) \psi(b_i) \right\| \]

\[ \leq \left\| T_A \right\| \left\| \left( \sum_{i=1}^{\eta} f_i \cdot b_i \right) \right\| \]

\[ = \left\| T_A \right\| \left\| \sum_{i=1}^{\eta} f_i \cdot b_i \right\| \]

Hence

\[ \left\| \sum_{i=1}^{\eta} T_A(f_i)T_B(b_i) \right\| \leq \left\| T_A \right\| \left\| T_B \right\| \left\| \sum_{i=1}^{\eta} f_i \cdot b_i \right\| \]
Thus $T$ is well defined and continuous on $A \otimes B$.

Let us denote the unique extension of $T$ also by $T$. It is trivial to verify that $T$ defines an automorphism on $A \otimes B$, which completes the proof.

We shall denote the above $T$ by $(T_A, T_B)$.

**Proposition 1.18**: Let $T$ be an automorphism of $A \otimes B$.
If for each $m \in m(B)$ there exists $m' \in m(B)$ such that

$$
(f(m(A))) \subseteq m \quad (\widetilde{f}(m(A))) \subseteq m'.
$$

then such an $m'$ is unique and $T(1 \otimes B) = 1 \otimes B$, i.e. $T$ maps the set of vector constant functions onto itself.

**Proof**: First to show uniqueness, suppose that $m', m'' \in m(B)$ such that $\widetilde{f}(m(A)) \subseteq m'$ if and only if $\widetilde{f}(m(A)) \subseteq m$ and $\widetilde{f}(m(A)) \subseteq m''$ if and only if $\widetilde{f}(m(A)) \subseteq m$. Let $b \in m'$.

Then $(\widetilde{1.b})(m(A)) = \{b\} \subseteq m'$ and hence

$$T^{-1}(\widetilde{1.b})(m(A)) \subseteq m'.\$$

But then

$$T(T^{-1}(\widetilde{1.b})(m(A))) \subseteq m''.$$  Thus $(\widetilde{1.b})(m(A))$ is contained in $m''$ or $b \in m''$. It follows that $m' \subseteq m''$ and hence $m' = m''$. This proves the uniqueness of $m'$. 

Now for $\phi = (\phi_A, \phi_B) \in m(A) \times m(B)$ define a function on $A \otimes B$ by $r \mapsto \phi_B(\tilde{r}(\phi_A))$. Then it defines a complex homomorphism on $A \otimes B$. Therefore there exists $(\psi_A, \psi_B) \in m(A) \times m(B)$ such that

$$\phi_B(\tilde{r}(\phi_A)) = \psi_B(\tilde{r}(\psi_A)).$$

Here $\psi_A$ and $\psi_B$ depend on both $\phi_A$ and $\phi_B$.

Claim: If $\ker \phi_B = M'$ then $\ker \psi_B = M$

where $M'$ and $M$ are as given in the hypothesis i.e. $\psi_B$ is independent of $\phi_A$.

Let $b \in M$. Then $1 \circ b(m(A)) \subseteq M$.

Therefore $\frac{1}{b}(m(A)) \subseteq M' = \ker \phi_B$.

Hence $\phi_B(\frac{1}{b}(\phi_A))(\phi_A) = 0$ but

$$\phi_B(\frac{1}{b}(\phi_A))(\phi_A) = \psi_B(\frac{1}{b}(\psi_A)) = \psi_B(b).$$

Thus $b \in \ker \psi_B$ or $M \subseteq \ker \psi_B$. Since both $M$ and $\ker \psi_B$ are maximal ideals, we must have $M = \ker \psi_B$.

Hence the claim.
Thus for each $\phi_A \in m(A)$, fixing $\phi_B$, we get $\psi_A \in m(A)$ depending on $\phi_A$, $\phi_B$ with $\Phi_B(\tilde{f}(\phi_A)) = \psi_B(\tilde{r}(\psi_A))$.

Choose $a \in B$. Then $1_a \in 1 \otimes B$. Now if $\phi_A', \phi_A'' \in m(A)$ and $\phi_B \in m(B)$ then

$\Phi_B(\tilde{T}(1_a)(\phi_A') - \tilde{T}(1_a)(\phi_A''))$

$= \Phi_B(\tilde{T}(1_a)(\psi_A')) - \Phi_B(\tilde{T}(1_a)(\psi_A''))$

$= \psi_B((\tilde{1_a})(\psi_A')) - \psi_B((\tilde{1_a})(\psi_A''))$

$= \psi_B(a) - \psi_B(a)$

$= 0$.

Thus $\tilde{T}(1_a)(\phi_A') - \tilde{T}(1_a)(\phi_A'') \in \ker \Phi_B$

for every $\phi_B \in m(B)$. 
By semisimplicity of $B$,

$$\tilde{T}(1.a)(\phi'_A) = \tilde{T}(1.a)(\phi''_A).$$

Since $\phi'_A, \phi''_A$ are arbitrary elements in $m(A)$, we have

$$T(1.a)(x_1) = T(1.a)(x_2)$$

for every $x_1, x_2$ in $X$.

It follows that $T(1.a)$ is a constant function.

Therefore $T(1 \otimes B) \subset 1 \otimes B$. Similarly, $T^{-1}(1 \otimes B) \subset 1 \otimes B$ and hence $T(1 \otimes B) = 1 \otimes B$.

**Proposition 1.19:** Let $T$ be an automorphism of $A \otimes B$.

If for each $\phi_A \in m(A)$ there exists $\psi_A \in m(A)$ such that

$$\tilde{T}f(\phi_A) = 0 \text{ if and only if } \tilde{T}f(\psi_A) = 0$$

then such a $\psi_A$ is unique and $T(A \otimes e) = A \otimes e$.

**Proof:** The uniqueness of $\psi_A$ can be easily verified.

Now for $(\phi_A, \phi_B) \in m(A) \times m(B)$, $f \mapsto \phi_B(\tilde{T}f(\phi_A))$

defines a complex homomorphism on $A \otimes B$. Therefore there
exists \((n_A, n_B) \in m(A) \times m(B)\) such that

\[(12) \quad \Phi_B(\tilde{Tf(\Phi_A)}) = n_B(\tilde{f}(n_A))\]

for all \(f\) in \(\hat{A} \otimes B\). We show that \(n_A\) is independent of \(\Phi_B\) and in fact \(n_A = \psi_A\) (corresponding to \(\Phi_A\) as in the hypothesis).

Suppose \(n_A \neq \psi_A\). Then there exists \(f \in A\) such that \(\tilde{f}(n_A) = 1\) and \(\tilde{f}(\psi_A) = 0\). Then \(\tilde{f}\cdot e(\psi_A) = 0\).

Hence \(\tilde{T(f\cdot e)(\Phi_A)} = 0\) by hypothesis. Now

\[n_B(\tilde{f}\cdot e(n_A)) = n_B(e) = 1\]

while

\[\Phi_B(\tilde{T(f\cdot e)(\Phi_A)} = 0\]

as \(\tilde{T(f\cdot e)(\Phi_A)} = 0\). But

\[\Phi_B(\tilde{T(f\cdot e)(\Phi_A)} = n_B(\tilde{f}\cdot e(n_A))\]

as \(f\cdot e \in A \otimes B\). But then

\(0 = 1\) which is absurd. Hence \(n_A = \psi_A\).

Thus for any \(\Phi_A \in m(A)\) and \(f \in A\)

\[\Phi_B(\tilde{T(f\cdot e)(\Phi_A)} = n_B(\tilde{f}\cdot e(\psi_A)) = \tilde{f}(\psi_A)\]

for every \(\Phi_B \in m(B)\).
Suppose for a fixed \( f \in A \), \( \overline{T(f \cdot e)}(\varphi_A) = b \) where \( b \in B \). Then \( b = m + \lambda e \) where \( m \in \ker \Phi_B \) and \( \lambda \) is a scalar, \( \lambda = \Phi_B(b) \).

Then \( \lambda = \Phi_B(b) = \Phi_B(\overline{T(f \cdot e)}(\varphi_A)) \)

i.e. \( \Phi_B(b) = \hat{f}(\psi_A) \), and this holds for any \( \Phi_B \in m(B) \).

Hence \( b - \hat{f}(\psi_A) \cdot e = 0 \) as \( B \) is semisimple.

Therefore \( b = \hat{f}(\psi_A) \cdot e \) or \( \overline{T(f \cdot e)} \) is a scalar function.

Clearly, then \( T(f \cdot e) \) is also a scalar function.

Since \( T \) maps \( A \hat{\otimes} B \) onto \( A \hat{\otimes} B \) and since \( T(f \cdot e) \) is a scalar function, \( T(f \cdot e) \in A \hat{\otimes} e \). Thus \( T(A \hat{\otimes} e) \subseteq A \hat{\otimes} e \).

Similarly we get \( T^{-1}(A \hat{\otimes} e) \subseteq A \hat{\otimes} e \) and hence \( T(A \hat{\otimes} e) = A \hat{\otimes} e \).

**Proposition 1.20:** Let \( T \) be an automorphism of \( A \hat{\otimes} B \) satisfying the hypothesis of Propositions 1.18 and 1.19.

Then there exist automorphisms \( T_A \) and \( T_B \) on \( A \) and \( B \) respectively such that \( T = (T_A, T_B) \).

**Proof:** Since \( T \) satisfies the hypothesis of Propositions 1.18 and 1.19 we have \( T(1 \hat{\otimes} B) = 1 \hat{\otimes} B \) and \( T(A \hat{\otimes} e) = A \hat{\otimes} e \).

Hence \( T|_A \) and \( T|_B \) define automorphisms on \( A \) and \( B \) respectively. Let \( T|_A = T_A \) and \( T|_B = T_B \).
Since $B$ is semisimple, $A \hat{\otimes} B$ is semisimple (Prop. 1.5). Hence $T_A$, $T_B$ and $T$ are continuous. Let $T_1$ be the automorphism on $A \otimes B$ defined by $T_1 = (T_A, T_B)$. Then

$$T_1 \left( \sum_{i=1}^{n} f_i \cdot b_i \right) = \sum_{i=1}^{n} T_A(f_i) \cdot T_B(b_i)$$

$$= \sum_{i=1}^{n} T(f_i \cdot a) \cdot T(1 \cdot b_i)$$

$$= \sum_{i=1}^{n} T(f_i \cdot b_i)$$

$$= T \left( \sum_{i=1}^{n} f_i \cdot b_i \right)$$

Thus $T_1 = T$ on $A \otimes B$ and hence by continuity on $A \hat{\otimes} B$ also, which completes the proof.

Since the automorphisms of $C(X)$ are determined by homeomorphisms of $X$, we have the following:
Corollary 1.21: Given a homeomorphism $\tau$ of $X$ and an automorphism $T_B$ of $B$, we get an automorphism $T$ on $\mathcal{C}(X; B)$ defined by

$$(T(f))(x) = T_B(f(\tau(x)))$$

for all $x \in X$ and $f \in \mathcal{C}(X; B)$.

Conversely if $T$ is an automorphism of $\mathcal{C}(X; B)$ satisfying the hypothesis of Thom.1.20 then $T$ is determined by a homeomorphism of $X$ and an automorphism of $B$.

The following example shows that not every automorphism of $\mathcal{C}(X; B)$ is determined by a homeomorphism of $X$ and an automorphism of $B$.

Example 1.22: Let $X = \{a, b\}$ be a two element set with discrete topology and $B = \mathbb{C}^2$, the algebra of all ordered pair of complex numbers with co-ordinatewise operations and norm defined by $\| (\alpha, \beta) \| = \max(|\alpha|, |\beta|)$. Then any $f$ in $\mathcal{C}(X; B)$ can be looked upon as pair $(f_1, f_2)$ where $f_1, f_2 \in \mathcal{C}(X)$ such that $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$ where $p_1$ and $p_2$ are projections in the
first and second co-ordinates respectively. Again to each $h \in C(X)$ a pair $(h(a), h(b))$ in $\mathcal{C}$ is uniquely associated.

Since $X$ has only two automorphisms namely the identity $I$ and $\tau$ defined by $\tau(a) = b$, $\tau(b) = a$, there are only two automorphisms on $C(X)$. Also $B = \mathcal{C}$ has precisely two automorphisms, the identity $I$ and $\sigma$ defined by $\sigma(\alpha, \beta) = (\beta, \alpha)$. Thus there can be four automorphisms on $C(X; B)$ determined by homeomorphisms of $X$ and automorphisms of $B$ viz

$$T_1 = (i, I), T_2 = (\tau, I), T_3 = (i, \sigma) \text{ and } T_4 = (\tau, \sigma).$$

Now define $T$ on $C(X; B)$ by $T(f_1, f_2) = (\phi_1, \phi_2)$ where $\phi_1$ and $\phi_2$ are defined by

$$\phi_1(a) = f_1(b), \quad \phi_2(a) = f_1(a) \quad \phi_1(b) = f_2(b), \quad \phi_2(b) = f_2(a).$$

Then $T$ can be easily seen to be an automorphism on $C(X; B)$, which does not coincide with any of the $T_i$, $i = 1, 2, 3, 4$. 
In the above example \( X \) is disconnected and the automorphism \( T \) does not map constant functions to constant functions. This cannot happen if \( X \) is connected.

In fact we get

**Proposition 1.23**: If \( m(A) \) is connected then every automorphism of \( A \otimes \mathbb{C}^2 \) maps vector constant functions to vector constant functions.

**Proof**: Let \( T \) be an automorphism of \( A \otimes \mathbb{C}^2 \). Let \( f \equiv (\alpha, \beta) \) be a constant function on \( X \). Suppose that \( Tf = (g_1, g_2) \) where \( g_1, g_2 \in A \). If \( Tf \) is not constant, at least one of \( g_1, g_2 \) is not constant. For definiteness assume that \( g_1 \) is not constant. Then we have \( x, y \in X, x \neq y \) such that \( g_1(x) \neq g_1(y) \). Let \( \Phi_1 \) and \( \Phi_2 \) be complex homomorphisms on \( \mathbb{C}^2 \) defined by \( \Phi_1(p,q) = p \) and \( \Phi_2(p,q) = q \). It is easy to see that these are the only nonzero complex homomorphisms on \( \mathbb{C}^2 \).

Now for \( x \in X, g_1(x) = \Phi_1((Tf)(x)) \). But as noted in the proof of prop.1.18, \( \Phi_1((Tf)(x)) = \Psi(f(\Psi_x)) \)
where \( \psi_x \in \mu(A) \) and \( \psi \in \mu(\mathcal{C}^\ast) \). Hence
\[
g_1(x) = \psi(\tilde{f}(\psi_x))
\]
Since \( f \) is constant on \( X \), so is \( \tilde{f} \) on \( \mu(A) \). Also \( \psi = \phi_1 \) or \( \phi_2 \).
Hence \( g_1(x) = \alpha \) or \( \beta \). Thus \( g_1 \) takes at most two values on \( X \). By the same argument \( \hat{g}_1 \) takes at most two values on \( \mu(A) \). Since \( \mu(A) \) is connected and \( \hat{g}_1 \) is continuous on \( \mu(A) \), \( \hat{g}_1 \) is constant on \( \mu(A) \). Hence \( g_1 \) is constant on \( X \). Similarly \( g_2 \) is constant on \( X \). Thus \( T_f \) is a constant function.

Remark 1.24 (i) We note that '\( \mu(A) \) is connected' is a weaker hypothesis than '\( X \) is connected' as \( X \) is connected implies \( \mu(A) \) is connected ([45], p.112).

(ii) Prop.1.23 can be generalized for \( \mathcal{C}^n \).

SEPARATION PROPERTIES

It is well known that the tensor product of two regular Banach algebras is regular and conversely ([49]). Thus \( A \hat{\otimes} B \) is regular on \( \mu(A) \times \mu(B) \) if and only if \( A \)
and $B$ are regular on $m(A)$ and $m(B)$ respectively. It can also be shown that $A \hat{\otimes} B$ is regular on $\times m(B)$ if and only if $A$ is regular on $X$ and $B$ is regular on $m(B)$.

Now we shall discuss a stronger separation property viz strong regularity for tensor products (see chapter 0 for definition). We extend the idea of strong regularity for any Banach algebra as follows:

**Definition 1.25**: Let $B$ be a Banach algebra and $F$ be any closed subset of $m(B)$. $B$ is said to be strongly regular on $F$ if given $\phi \in F$, $b \in B$ and $\varepsilon > 0$ there exist an open neighbourhood $U$ of $\phi$ in $F$ and $a \in B$ such that $a(\psi) = b(\phi)$ for $\psi \in U$ and $\|a - b\|_B < \varepsilon$.

If $B$ is strongly regular on $m(B)$ we simply say that $B$ is strongly regular. If $B_1$ is a subalgebra of $B$ and if given $b \in B_1$, $\phi \in F$ and $\varepsilon > 0$ we get a neighbourhood $U$ of $\phi$ in $F$ and $a \in B$ such that $a(\psi) = b(\phi)$ for $\psi \in U$ and $\|a - b\|_B < \varepsilon$ then $B_1$ is said to be strongly regular on $F$. 

It is easy to see that $C(X)$ is strongly regular.

The following results are immediate:

(13) $B$ is strongly regular if and only if $\overline{J_\phi} = M_\phi$ for all $\phi \in \mathfrak{m}(B)$ where

$$J_\phi = \{ a \in B : a \text{ vanishes in some neighbourhood of } \phi \}$$

$$M_\phi = \text{Ker } \phi \text{ and bar denotes closure in } B$$

(14) If $B$ is strongly regular then $B$ is regular.

The following lemma will enable us to show that just as in the case of regularity, $A \hat{\otimes} B$ is strongly regular if and only if the factor algebras are strongly regular.

Lemma 1.26: Let $B_1$ be a dense subalgebra of $B$ containing identity. If $B_1$ is strongly regular on $\mathfrak{m}(B)$ then $B$ is strongly regular.

Proof: Let $\phi \in \mathfrak{m}(B), b \in B$ such that $\hat{b}(\phi) = \phi(b) = 0$.

Then given $\varepsilon > 0$ there exists $b_1 \in B_1$ such that

$\phi(b_1) = 0$ and $\| b_1 - b \| < \frac{\varepsilon}{2}$. For, first choose $b_1' \in B_1$ such that $\| b_1' - b \| < \frac{\varepsilon}{4}$. Take $b_1 = b_1' - \phi(b_1')$.e.
Then
\[ \| b_1 - b \| = \| (b'_1 - b) - \phi(b'_1) + \phi(b_1) \| \leq 2 \| b_1 - b \| \leq \frac{\varepsilon}{2}. \]

Now since \( b_1 \in B_1 \) and \( \phi(b_1) = 0 \), by strong regularity of \( B_1 \), there exist a neighbourhood \( U \) of \( \phi \) and \( a \in B_1 \) such that \( \hat{a} = 0 \) in \( U \) and \( \| a - b_1 \| < \frac{\varepsilon}{2} \). Clearly \( a \in B \) and \( \| a - b \| < \varepsilon \).

Thus \( B \) is strongly regular using (13) above.

**Theorem 1.27**: \( A \boxtimes B \) is strongly regular if and only if \( A \) and \( B \) are strongly regular.

**Proof**: Suppose first that \( A \) and \( B \) are strongly regular. By the above lemma it suffices to show that \( A \boxtimes B \) is strongly regular on \( m(A) \times m(B) \).

Let \( \phi = (\phi_A, \phi_B) \in m(A) \times m(B) \) and

\[ f = \sum_{i=1}^{n} f_i \cdot b_i \in A \boxtimes B. \] Then \( \phi(f) = \sum_{i=1}^{n} \phi_A(f_i) \phi_B(b_i) \)
Since $A$ and $B$ are strongly regular, given $\varepsilon > 0$ there exist neighbourhoods $U_i$ and $V_i$ of $\Phi_A$ and $\Phi_B$, $g_i \in A$ and $a_i \in B$ ($i = 1, 2 \ldots n$) such that

$$\hat{g}_i(\psi_A) = \hat{f}_i(\Phi_A), \psi_A \in U_i$$

$$\|g_i - f_i\| < \frac{\varepsilon}{2n \|b_i\|} \quad i = 1, 2, \ldots, n$$

and

$$\hat{a}_i(\psi_B) = \hat{b}_i(\Phi_B), \psi_B \in V_i$$

$$\|a_i - b_i\| < \frac{\varepsilon}{2n \|g_i\|} \quad i = 1, 2, \ldots, n.$$ 

Then

$$\sum_{i=1}^{n} g_i \cdot a_i \in A \otimes B.$$ 

Take $U = \bigcap_{i=1}^{n} U_i$

and $V = \bigcap_{i=1}^{n} V_i$. Then $U \times V$ is a neighbourhood of $(\Phi_A, \Phi_B)$ and for $\Psi = (\psi_A, \psi_B)$ in $U \times V$
\[
\left( \sum_{i=1}^{n} g_i \cdot a_i \right) (\Psi) = \sum_{i=1}^{n} \hat{g}_i(\Psi_A) \hat{a}_i(\Psi_B)
\]

\[
= \sum_{i=1}^{n} \phi_A(f_i) \phi_B(b_i),
\]

Also

\[
\left\| \sum_{i=1}^{n} g_i \cdot a_i - \sum_{i=1}^{n} f_i \cdot b_i \right\| =
\]

\[
\left\| \sum_{i=1}^{n} g_i \cdot a_i - \sum_{i=1}^{n} g_i \cdot b_i + \sum_{i=1}^{n} g_i \cdot b_i - \sum_{i=1}^{n} f_i \cdot b_i \right\|
\]

\[
\leq \left( \left\| g_i \right\| \left\| a_i - b_i \right\| + \left\| b_i \right\| \left\| g_i - f_i \right\| \right)
\]

\[
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}
\]

\[
= \varepsilon
\]

This shows that \(A \otimes B\) is strongly regular on \(m(A) \times m(B)\)

which proves the theorem one way.
Conversely suppose $A \hat{\otimes} B$ is strongly regular and $\phi_A \in \text{m}(A)$ and $f \in A$. Choose any $\phi_B \in \text{m}(B)$. Then $\phi(f \cdot e) = \phi_A(f)$ where $\phi = (\phi_A, \phi_B)$. Given $\epsilon > 0$ by the strong regularity of $A \hat{\otimes} B$ there exist $g \in A \hat{\otimes} B$ and a neighbourhood $U$ of $(\phi_A, \phi_B)$ such that $\hat{g}(\psi) = \phi_A(f)$ for all $\psi \in U$ and $\| g - f \cdot e \| < \epsilon$.

Consider neighbourhoods $U_A$ and $U_B$ of $\phi_A$ and $\phi_B$ respectively such that $U_A \times U_B \subseteq U$.

Taking $h = \phi_B \circ g$, we get $h \in A$ and for any $\psi_A \in U_A$, $\hat{h}(\psi_A) = \phi_A(f)$ and $\| h - f \| \leq \| g - f \cdot e \| < \epsilon$.

Thus $A$ is strongly regular. Similarly it can be shown that $B$ is strongly regular.

Again the result holds for the tensor product of any two Banach algebras with an algebra cross-norm $\alpha$ satisfying $\lambda \leq \alpha \leq p$.

**CLOSED IDEALS OF $C(X;B)$**

The closed ideals of $C(X;B)$ have been described by Kaplansky ([53], p.336). It follows from his result that if $I$ is a closed ideal of $C(X;B)$ then for each $x \in X$,
\[ I_x = \{ f(x) : f \in I \} \]

is a closed ideal of \( B \) and

\[ I = \{ f \in C(X;B) : f(x) \in I_x, x \in X \} . \]

When \( I \) is a proper ideal, at least one \( I_x \) is proper.

From this characterization we get

**Proposition 1.28**: \( C(X;B) \) has the ideal intersection property (i.e. every closed ideal is the intersection of all maximal ideals containing it) if and only if \( B \) has that property.

**Proof**: Suppose \( B \) has the ideal intersection property.

Let \( I \) be a closed ideal of \( C(X;B) \). If \( I \) is contained in the maximal ideal \( M = (x, M_B) \) of \( C(X;B) \) then

\[ I_x \subseteq M_x \subseteq M_B . \]

Conversely, if \( (x, M_B) \) is a maximal ideal of \( C(X;B) \) such that \( I_x \subseteq M_B \) then for \( f \in I \),

\[ f(x) \in I_x \subseteq M_B \]

and hence \( f \in (x, M_B) \). Hence

\[ I \subseteq (x, M_B) . \]

Thus maximal ideals of \( C(X;B) \) which contain \( I \) are precisely those of the form \( (x, M_B) \), where \( I_x \subseteq M_B \).
Then if \( J \) is the intersection of maximal ideals of \( C(X;B) \) containing \( I \), we have

\[
J = \bigcap \{ (x, \mathfrak{m}_B) : I_x \subsetneq \mathfrak{m}_B, \ x \in X, \ \mathfrak{m}_B \in \mathfrak{m}(B) \}
\]

\[
= \{ f \in C(X;B) : f(x) \in \mathfrak{m}_B, \ I_x \subsetneq \mathfrak{m}_B, \ x \in X, \ \mathfrak{m}_B \in \mathfrak{m}(B) \}
\]

Since \( B \) has the ideal intersection property,

\[
\bigcap \mathfrak{m}_B = I_x
\]

\[
\mathfrak{m}_B \in \mathfrak{m}(B)
\]

\[
I_x \subsetneq \mathfrak{m}_B
\]

Hence \( J = \{ f \in C(X;B) : f(x) \in I_x, \ x \in X \} \)

\[
= I
\]
It follows that $C(X; B)$ has the ideal intersection property.

Conversely suppose that $C(X; B)$ has the ideal intersection property. Let $I$ be a proper closed ideal of $B$. Take

$$J = \bigcap \{ m_B : I \subseteq m_B \in m(B) \}.$$  

Then $J$ is a closed ideal of $B$ and $I \subseteq J$. Consider $C(X) \otimes I$ which is a proper closed ideal of $C(X; B)$.

In fact

$$(15) \quad C(X) \otimes I = \{ f \in C(X; B) : f(x) \subseteq I \}.$$  

Clearly, $C(X) \otimes I \subseteq \text{RHS of } (15)$.

Now it can be easily verified that

$C(X) \otimes I \subseteq (x, m_B)$ for each $x \in X$ if and only if $I \subseteq m_B$.

Since $C(X; B)$ has the ideal intersection property.

$$C(X) \otimes I = \bigcap \{(x, m_B) : x \in X, I \subseteq m_B \}$$  

$$= \{ f \in C(X; B) : f(x) \subseteq m_B \text{ where } f(x) \subseteq m_B \}$$  

$$= \{ f \in C(X; B) : f(x) \subseteq J \}.$$  

- 59 -
But as $I \subseteq J$

\[(16) \quad \{ f \in C(X;\mathcal{B}) : f(X) \subseteq J \} \subseteq \{ f \in C(X;\mathcal{B}) : f(X) \subseteq I \} \]

combining (15) and (16) we obtain

\[(17) \quad C(X) \otimes I = \{ f \in C(X;\mathcal{B}) : f(X) \subseteq I \} \]

\[= \{ f \in C(X;\mathcal{B}) : f(X) \subseteq J \} \]

Now let $b \in J$. Then $1.b \in C(X) \otimes I$ by (17).

But then again by the same equality $b \in I$, which shows that $J \subseteq I$ and hence $J = I$. Thus $B$ has the ideal intersection property and the proof is complete.

Note: There exist Banach algebras other than $C(X)$, which satisfy the ideal intersection property, e.g. $AC[0,1]$, the algebra of all absolutely continuous functions on $[0,1]$ (see [65], [66]).

Definition 1.29: An ideal $P$ in a Banach algebra is said to be a primary ideal if it is continued in a unique maximal ideal.
It is obvious that the closure of a primary ideal is a primary ideal.

Definition 1.30: A Banach algebra is said to have the primary ideal intersection property if every closed ideal is the intersection of all closed primary ideals containing it.

The examples of such algebras are $\text{Lip}(X,d)$ and $\mathcal{C}^k(\Omega)$ where $\text{Lip}(X,d)$ is the algebra of all Lipschitz functions on a compact metric space $(X,d)$ and $\mathcal{C}^k(\Omega)$ is the algebra of $k$-times continuously differentiable functions on a domain $\Omega$ of $\mathbb{R}^n$ ([62], [73]).

Theorem 1.31: Let $\mathcal{P}$ be a closed primary ideal of $B$ and $x \in X$. Then $(x, \mathcal{P}) = \{ f \in C(X;B) : f(x) \in \mathcal{P} \}$ is a closed primary ideal of $C(X;B)$. Conversely, if $I_x$ is any closed primary ideal of $C(X;B)$ then each $I_x$, which is proper is a closed primary ideal of $B$.

Consequently, $C(X;B)$ has the primary ideal intersection property if and only if $B$ has that property.
Proof: Let $P$ be a closed primary ideal of $B$ and $x \in X$. Then $(x, P) = \{ f \in C(X; B) : f(x) \in P \}$ is a closed ideal of $C(X; B)$. If $M_B$ is the unique maximal ideal of $B$ containing $P$ then $(x, P) \subseteq (x, M_B)$. To show that $(x, P)$ is primary, we have to show that $(x, M_B)$ is the only maximal ideal containing $(x, P)$. Suppose $(x, P) \subset (x', M_B')$. If $x' \neq x$ then there exists $f \in C(X)$ such that $f(x') = 1$ and $f(x) = 0$. Then $f \cdot e \in (x, P)$ but $f \cdot e \notin (x', M_B')$ as $f(x').e = e \notin M_B'$, which is a contradiction and hence $x' = x$. Again if $M_B \neq M_B'$ then there is $b \in P$ such that $b \notin M_B'$ as $P$ is primary. Then $1 \cdot b \in (x, P)$ but $1 \cdot b \notin (x, M_B')$ which is a contradiction. Hence $M_B = M_B'$ and $(x, P)$ is a closed primary ideal.

Now let $I$ be a closed primary ideal of $C(X; B)$. Then for each $x \in X$ there is a closed ideal $I_x$ of $B$ such that $I = \{ f \in C(X; B) : f(x) \in I_x \}$ where at least one $I_x$ is proper. We want to show that each $I_x$, which is proper, is a primary ideal.
Suppose \((x_0, \mathcal{M}_B)\) is the unique maximal ideal containing \(I\). Clearly \(I_{x_0} \subseteq \mathcal{M}_B\), and hence is proper. We first show that if \(x \neq x_0\), then \(I_x\) cannot be proper. For let \(I_x\) be proper. Then \(I_x \subseteq \mathcal{M}_B'\) for some \(\mathcal{M}_B' \in \mathcal{m}(B)\). But then \(I \subseteq (x, \mathcal{M}_B')\), which implies that \(x = x_0\) and \(\mathcal{M}_B' = \mathcal{M}_B\). This also shows that \(I_{x_0}\) is primary as the only maximal ideal containing \(I_{x_0}\) is \(\mathcal{M}_B\).

Thus \(I = (x_0, I_{x_0})\). Hence we have a one-to-one correspondence between the family of closed primary ideals of \(C(X; B)\) and \(X \times (\text{family of closed primary ideals of } B)\).

Now the proof of the rest of the theorem follows as on Prop. 1.28.

We shall now study some special types of closed ideals of \(C(X; B)\). Let \(I\) and \(J\) be closed ideals of \(C(X)\) and \(B\) respectively and \(K = I \otimes J\). Then \(K\) is a closed ideal of \(C(X; B)\). Since \(I\) is a closed ideal of \(C(X)\) there is a closed subset \(F\) of \(X\) such that \(I = \{f \in C(X) : f|_F = 0\}\).
If $x \in F$ and $M_B \in m(B)$ is any maximal ideal of $B$ then $K$ is contained in $(x, M_B)$, for if
\[ f = \sum_{i=1}^{n} f_i \cdot b_i \] is in $I \otimes J$ then $f(x) = \sum_{i=1}^{n} f_i(x) b_i = \in \in M$.

Thus $I \otimes J \subseteq (x, M_B)$ and hence $I \otimes J \subseteq (x, M_B)$. Similarly it can be shown that if $J \subseteq M_B$ then for each $x \in X$, $I \otimes J \subseteq (x, M_B)$.

We show here that these are the only maximal ideals containing $K$. For let $K \subseteq (x_0, M_0)$ and suppose that $x_0 \notin F$. Then there exists $f \in I$ such that $f(x_0) = 1$. Take $b \in J$. Then $f \cdot b \in K$. Therefore $f(x_0) \cdot b \in M_0$ or $b \in M_0$, which implies that $J \subseteq M_0$.

Now let us determine the family $\{ K_x : x \in X \}$ for this $K$.

If $x \in F$ then $K_x = \{0\}$ as $f(x) = 0$ for all $f \in K$ and $x \in F$. If $x \in F$ then there is $f \in I$ such that $f(x) = 1$. Then for any $b \in J$, $f \cdot b \in K$.

Thus $b \in K_x$ or $J \subseteq K_x$. Conversely, if $b \in K_x$ then there is $f \in K$ such that $f(x) = b$. Now if
\[ g \in I \otimes J, \quad g = \sum_{i=1}^{n} g_i \cdot b_i \quad \text{for some } g_i \in I, \ b_i \in J \]
then \[ g(x) = \sum_{i=1}^{n} g_i(x) \cdot b_i \in J. \]
Hence, since \( J \) is closed, by taking limits we can prove that \( g(x) \in J \)
for any \( g \in K \). In particular \( b = f(x) \in J \). Hence \( K_x \subseteq J \).

Thus \( K_x = \{0\} \) if \( x \in F \) and \( K_x = J \)
if \( x \in F^c \). If \( B \), (and hence \( C(X;B) \)) has the ideal
intersection property then \( K = \{ f \in C(X;B) : f(x) \subseteq J \text{ and } f(F) = \{0\} \} \).

As special cases we can consider the ideals \( C(X) \otimes J \) or \( I \otimes B \) of \( C(X;B) \). If \( S = I \otimes B \) then
\( S_x = \{0\} \) for each \( x \in F \) and \( S_x = B \) for \( x \in F^c \),
and if \( T = C(X) \otimes J \) then \( T_x = J \) for all \( x \in X \). Further,
if \( B \) has the ideal intersection property then
\[ S = \{ f \in C(X;B) : f|_F = 0 \} \]
and \[ T = \{ f \in C(X;B) : f(x) \subseteq J \} \].
So far we have discussed those properties of $A \otimes B$ which can be discussed for any Banach algebra. The corresponding properties of $A \otimes B$ depend on those of $A$ and $B$. Now we shall study $A \otimes B$ as a $B$-valued function algebra (Def. 1.2). We can generalize the ideas of peak set, set of antisymmetry etc. for a vector-valued function algebra and as one would expect many of those notions for $A$ and $A \otimes B$ correspond. Stout [63] and Oberlin [56] have studied peak interpolation sets and interpolation sets for $A \otimes B$.

Here we introduce another vector function algebra closely related with $A$ and $B$, which we call the "slice product" of $A$ and $B$.

**Definition 1.32**: Let $A$ be a function algebra on $X$ and $B$ be a Banach algebra. Let

$$A \neq B = \left\{ f \in C(X; B) : \phi \text{ of } A \text{ for all } \phi \in \text{m}(B) \right\}.$$  

Then $A \neq B$, which is called the slice product of $A$ and $B$, is a closed subalgebra of $C(X; B)$. It can be easily checked
that \( A \otimes B \) is contained in \( A \# B \). Hence \( A \# B \) is a vector function algebra on \( X \).

If \( A \) and \( B \) are function algebras, the slice product as defined above coincides with the usual slice product of function algebras, \( B \) being regarded as a function algebra on \( m(B) \). (For the definition of slice product of function algebras, see \([6]\) or \([23]\)). Now let \( C \) be any vector function algebra on \( X \). Define

\[
A_1 = \{ f \in C(X) : f \circ e \in C \}
\]
and

\[
A_2 = \{ \Phi_B \circ f : f \in C, \Phi_B \in m(B) \}.
\]

Then \( A_1 \) and \( A_2 \) are subalgebras of \( C(X) \) containing constants. \( A_1 \) is uniformly closed and \( A_1 \otimes B \subset C \).

If \( A_1 = A_2 = A \) then \( A \otimes B \subset C \subset A \# B \).

If \( C = A \otimes B \) or \( A \neq B \), the condition \( A_1 = A_2 \) does hold. But there exist vector function algebras which are neither tensor products nor slice products.

**Example 1.33**: Let \( X = \{ z \in \mathbb{C} : |z| \leq 1 \} \)

and \( B = C([0,1]) \).
Let \( \mathcal{C} = \{ f \in C(X; B) : f_0 \in A(X) \} \)

where \( f_0(y) = \hat{f}(y, 0) \), \( y \in X \) and \( A(X) \)

is the function algebra of all continuous functions on \( X \) which are analytic in \( \text{int} \ X \). \( \hat{f} \) denotes the Gelfand transform of \( f \in C(X; B) \) on \( X : [0, 1] = m(C(X; B)) \).

Then \( \mathcal{C} \) is a vector function algebra on \( X \).

Clearly \( A(x) \otimes B = A(X) \neq B \subset \mathcal{C} \)

(The first equality holds as \( B \) has the approximation property \([9]\)).

For any vector function algebra we can define the separation properties, peakset etc. exactly as we define for a function algebra replacing absolute value by norm whenever needed. Also pervasiveness, analyticity etc. can be defined in a similar fashion.

Now onwards we will assume that \( \mathcal{C} \) is a vector function algebra satisfying the condition \( A_1 = A_2 \) and \( B \) is semisimple. For such algebras we get the following straightforward results:
(i) \( C \) is regular (normal, approximately regular) on \( X \) if and only if \( A \) is regular (normal, approximately regular) on \( X \).

(ii) If \( C \) is strongly regular then \( A \) and \( A \otimes B \) are strongly regular.

(iii) \( C \) is pervasive (analytic) if and only if \( A \) is pervasive (analytic).

(iv) Let \( F \) be a closed subset of \( X \). Then \( F \) is a peak set (peak set in the weak sense) for \( C \) if and only if it is a peak set (peak set in the weak sense) for \( A \).

Also, for any vector function algebra \( C \) on \( X \)
the following results regarding peak sets and peak sets in the weak sense hold exactly as in case of function algebras e.g.

(i) \([11], \text{Lemma 2.3.1, p.96}\). If \( F \) is a peak set in the weak sense and a \( G \), then \( F \) is a peak set. In particular, the intersection of a countable family of peak sets is a peak set.
(ii) ([11], Thm. 2.4.1, p. 102). Suppose $F$ is a peak set for $C$ and $g \in C$ is invertiable on $F$. Then there exists $f \in C$ such that $f|_F = g|_F$ and $\|f(x)\| < \|f\|$ for all $x \in X - F$.

(iii) ([11], Thm. 2.4.2, p. 104). Let $F$ be a peak set in the weak sense for $C$ and let $g \in C$ such that $g|_F$ is invertible. If $L$ is any $G_b$ set containing $F$, there exists $f \in C$ with $f|_F = g|_F$ and $\|f(x)\| < \|f\|$ for all $x \in X - L$.

(iv) ([11], Cor. 2.4.3, p. 104). Let $F$ be a peak set in the weak sense for $C$. Then $C|_F$ is closed in $C(F; B)$.

It is well known that $C(X; B)^*$ equals $M(X, B^*)$ where $M(X, B^*)$ denotes the space of all $B^*$-valued regular Borel measures on $X$ ([22]).

**Definition 1.33**: Let $K$ be a subspace of $C(X; B)$ and $\Lambda$ be a continuous linear functional on $K$. Then by Hahn-Banach theorem $\Lambda$ can be extended to the whole of $C(X; B)$. 

- 70 -
Hence there exists a \( \mathcal{M} \in \mathcal{M}(X;B^*) \) such that
\[
\Lambda(f) = \int_X f \, d\mathcal{M}
\]
for all \( f \) in \( K \) and \( \|\Lambda\| = \|\mathcal{M}\| \).
Such a \( \mathcal{M} \) is called a representing measure for \( \Lambda \) with respect to \( K \). It is clear that \( \Lambda \) may have more than one representing measures.

By a result of Johnson [42] every extreme point of the unit ball of \( K^* \) is of the form
\[
(x, \Phi_B) \quad \text{where} \quad x \in X
\]
and \( \Phi_B \) is an extreme point of \( \mathcal{U}_B^* \), the unit ball of \( B^* \). From this we can have the following:

**Proposition 1.35**: Every extreme point of \( S_K^* \), the unit sphere of \( K^* \), has a unique representing measure.

The proof is similar to that of an analogous result for a subspace of \( C(X) \) ([17], p.92).

We conclude this chapter with some questions:

(i) If \( A \) is strongly regular, can we say that a vector function algebra \( C \), for which \( A_1 = A_2 = A \) is strongly regular or \( A \# B \) is strongly regular?
(ii) If \( x \in \text{ch}(A) \), the Choquet boundary of \( A \), and \( \Phi_B \in \text{Ext}(U_B^*) \), is \( (x, \Phi_B) \) an extreme point of \( U(A \hat{\otimes} B)^* \) ?

(iii) If \( \Phi_A \in \mathcal{m}(A) \) has unique representing measure and \( \Phi_B \in \mathcal{m}(B) \) is an extreme point of \( U_B^* \), does \( (\Phi_A, \Phi_B) \) possess a unique representing measure?