In Chapter 3, an optimal pole assignment (OPA) theorem has been enunciated. Using this theorem optimal pole region (OPR) has been delineated. A recursive procedure has been used to carry out optimal pole assignment. At each recursion one or two poles have been assigned. In this chapter a method for multiple real pole assignment at each recursion has been developed. In this process, physical interpretation for Riccati equation solution matrix \( P \) has been given. An algorithm for multiple pole assignment has been presented. This has been followed by a numerical example.

4.1 LEMMA

When \( P_k \) (\( \leq m \)) poles are optimally assigned using OPA theorem, the leading principal minors \( P_{ii} \) of \( P_{KL} \) are directly proportional to pole shift.

Proof

The reduced order system matrices at kth recursion are

\[
A_{KL} = \begin{bmatrix}
  a_{11} & \beta_{12} & \beta_{13} & \cdots & \beta_{1p} \\
  0 & a_{22} & \beta_{23} & \cdots & \beta_{2p} \\
  \cdots & \cdots & \cdots & \cdots & \cdots \\
  0 & 0 & \cdots & 0 & \alpha_{pp}
\end{bmatrix},
\]
Then from (3.2) we have

$$P_{KL} = (B_{KL}B_{KL}^T)^{-1} (A_{KL} - \bar{A}_{KL})$$

(4.1)

From (4.1) we get

$$P_{11} = b_{11} (a_{11}-\bar{a}_{11}) > 0$$

(4.2)

and

$$p_{12} = b_{12} (a_{11}-\bar{a}_{11}) = b_{11}(\beta_{12} - \bar{\beta}_{12}) + b_{12}(\alpha_{22}-\bar{\alpha}_{22})$$

substituting for ($\beta_{12} - \bar{\beta}_{12}$) from this equation in (4.1) and simplifying
\[ P_{22} = \left( \begin{pmatrix} b_{12} & \pi_{11} \\ \pi_{11} & p_{11} \end{pmatrix} \right)^2 - p_{12} + \frac{\Delta_{22}}{b_{11}} P_{11} (a_{22} - \bar{a}_{22}) \]  

(4.3)

where

\[ \Delta_{22} = (b_{11} b_{22} - b_{12}^2) > 0 \]  

(4.4)

Now equations (4.2) and (4.3) are equations of straight line. Similarly it can be shown that the \( \bar{a}_{11} \) is directly proportional to \( p_{11} \). This has been illustrated in the Figure 4.1.

![Figure 4.1: Relation between principal minor and closed loop poles.](image)

Therefore principal minors of \( P_{KL} \) are directly proportional to pole shift.

This Lemma leads us to the following definition.
Definition

Since the leading principal minors of $P_{kL}$ are proportional to pole shift in RSF plane, the algebraic Riccati equation solution matrix $P$ is called pole shift matrix.

$P_k(\leq m)$ closed loop poles $(\bar{p}_{ii})$ are assigned progressively such that $P_{ii} > 0$ and $Q_{ii}$ (leading principal minor of $Q_{kL}$) $\geq 0$. This has been illustrated by a numerical example involving one recursion.

4.2 ALGORITHM FOR MULTIPLE POLE ASSIGNMENT

1) Transform $A$ and $B$ to RSF

$$A_0 = U_0^T A U_0$$
$$B_0 = U_0^T B$$

where $U_0$ is the unitary similarity transformation matrix that transforms $A$ to RSF $A_0$

2) Choose $q$ the number of recursions necessary to carry out pole assignment and the order in which the poles are to be assigned. Set $k=0$ and $\bar{A}_0 = A_0$

3) Set $k = k+1$

4) Obtain $A_k = U_k^T \bar{A}_{k-1} U_k$ and $B_k = U_k^T B_{k-1}$

5) Set $i = 0$

6) Set $i = i+1$ and choose $P_{ii} = a > 0$

7) Calculate $p_{ii}$ by solving $P_{ii} = a$

8) Solve the simultaneous equations for $P_{ii}$, $P_{i1}$, $P_{i2}$, ..., $P_{ii}$ and determine $\bar{a}_{ii}$

9) Draw the straight line joining $a_{ii}$ and $\bar{a}_{ii}$. Choose desired $\bar{a}_{ii}$ on this straight line and read $P_{ii}$

10) Determine $p_{ii}$ from $P_{ii}$

11) If $i=1$ go to step (13) otherwise go to step (12)
12) Solve the simultaneous equations for $p_{1i}$, $p_{2i} ... p_{(i-1)i}$ and determine $\beta_{1i}, \beta_{2i}, \ldots \beta_{(i-1)i}$

13) Set $j = i$

14) Set $j = j + 1$

15) Calculate $p_{ij}$ if $j = p_k$ go to step (16) otherwise go to step (14)

16) Check for $Q_{ii}$; if $Q_{ii} > 0$ go to step (17) or else put $\bar{a}_{ii} = \bar{a}_{ii} - b$ ($b > 0$), read $p_{ii}$ on straight line and go to step (10)

17) If $i = p_k$ go to step (18) otherwise go to step (6)

18) Calculate $K_{kL}$

19) If $k = q$ go to step (20) otherwise go to step (3)

20) Calculate $P, K, Q$ and $\bar{A}$ in original system coordinates

4.3 ILLUSTRATIVE EXAMPLE

Consider

\[
\begin{bmatrix}
-4 & 2 & 1 \\
0 & -2 & 0 \\
0 & 1 & -1
\end{bmatrix}
\]

By RSF transformation we get

\[
A_0 = \begin{bmatrix}
-4 & -0.707 & -2.121 \\
0 & -2 & -1 \\
0 & 0 & -1.0
\end{bmatrix}, \quad B_0 = \begin{bmatrix}
2.0 & 0 & 1 \\
-0.707 & 0 & -0.707 \\
0.707 & 1.414 & 2.121
\end{bmatrix}
\]
where

\[
U_0 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0.707 & -0.707 \\
0 & 0.707 & 0.707
\end{bmatrix}
\]

The open loop poles are at -4, -2, -1 and \(q = 1\).

Since \(U_1 = I_3\) the identity matrix we have, \(A_{1L} = A_1 = A_0\) and \(B_{1L} = B_1 = B_0\). To shift the o.l. pole at -4, choose \(a = -0.5\).

\[
P_{1L} = (B_{1L}P_{1L})^{-1} (A_{1L} - \bar{A}_{1L})
\]

\[
= \begin{bmatrix}
3 & 7.707 & 0.707 \\
7.777 & 22.5 & 2.5 \\
0.707 & 2.5 & 0.5
\end{bmatrix}
\begin{bmatrix}
-4 - \bar{a}_{11} & -0.707 - \bar{a}_{12} & 2.121 - \bar{a}_{13} \\
0 & -2 - \bar{a}_{22} & -1 - \bar{a}_{23} \\
0 & 0 & -1 - \bar{a}_{33}
\end{bmatrix}
\]

Now \(P_{11} = P_{1L} = 3(-4 - \bar{a}_{11}) = 0.5\) which implies \(\bar{a}_{11} = -4.167\).

A straight line is drawn through the points -4 and -4.167 as in Figure 4.2.

Choosing \(\bar{a}_{11} = -4.333\), from Figure 4.2 we get \(P_{11} = P_{11} = 1\). Then

\[
P_{1L} = \begin{bmatrix}
1 & 2.593 & 0.236 \\
2.593 & P_{22} & P_{23} \\
0.236 & P_{23} & P_{33}
\end{bmatrix}
\], \(P_{11} > 0\)
Fig. 4.2: CLOSED LOOP POLES OF ILLUSTRATIVE EXAMPLE

and

\[ Q_{1L} = \begin{bmatrix} 8.333 & 17.128 & 1.728 \\ 17.128 & q_{22} & q_{23} \\ 1.728 & q_{23} & q_{33} \end{bmatrix} , \quad Q_{11} > 0 \]

Now to shift the open loop pole at -2, \( P_{22} = 0.5 \) gives

\[ P_{22} = 7.222 \text{ and } \bar{a}_{22} = -2.214 \]

A straight line is drawn through -2 and -2.214 in Figure 4.2. Choosing \( \bar{a}_{22} = -2.548 \), we have \( P_{22} = 1.278 \) and \( P_{22} = 8.0 \). Solving for \( P_{12} \), we have \( \bar{P}_{12} = -0.151 \). Thus,

\[ P_{1L} = \begin{bmatrix} 1 & 2.593 & 0.236 \\ 2.593 & 8.0 & 0.976 \\ 0.236 & 0.976 & P_{33} \end{bmatrix} , \quad P_{22} > 0 \]
Finally to shift the o.l. pole at $-1$, $P_{33} = 0.5$, $p_{33} = 0.551$, and $a_{33} = -3.739$.

Choosing $\bar{a}_{33} = -2.681$ on the straight line joining the points $-1$ and $-3.739$ in Figure 4.2, we get $P_{33} = 0.307$ and $p_{33} = 0.4$.

Solving for $p_{13}$ and $p_{23}$ we get $p_{13} = 1.599$ and $p_{23} = -0.676$. Then

$$ P_{1L} = \begin{bmatrix} 1 & 2.593 & 0.236 \\ 2.593 & 8.0 & 0.976 \\ 0.236 & 0.976 & 0.4 \end{bmatrix}, \quad P_{33} > 0 $$

and

$$ Q_{1L} = \begin{bmatrix} 8.333 & 17.128 & 1.728 \\ 17.128 & 38.607 & 5.999 \\ 1.728 & 5.999 & q_{33} \end{bmatrix}, \quad Q_{33} > 0 $$

We thus have

$$ \bar{A}_{1L} = \begin{bmatrix} -4.333 & -0.151 & 1.599 \\ 0 & -2.548 & -0.676 \\ 0 & 0 & -2.681 \end{bmatrix} $$
Therefore

\[ K_{1L} = B_{1L}^T P_{1L} = \begin{bmatrix} 0.333 & 0.219 & 0.064 \\ 0.333 & 1.380 & 0.566 \\ -0.333 & -0.993 & 0.394 \end{bmatrix} \]

Now referred to original system coordinates

\[ \bar{A} = \begin{bmatrix} -4.333 & 1.024 & 1.238 \\ 0 & -2.952 & -0.405 \\ 0 & 0.271 & -2.276 \end{bmatrix} \]

\[ K = \begin{bmatrix} 0.333 & 0.2 & -0.109 \\ 0.333 & 1.376 & -0.576 \\ -0.333 & -0.424 & 0.981 \end{bmatrix} \]

\[ P = \begin{bmatrix} 1.0 & 1.2 & -1.667 \\ 1.2 & 5.176 & -3.8 \\ -1.667 & -3.8 & 3.224 \end{bmatrix} \]

\[ Q = \begin{bmatrix} 8.333 & 13.333 & -10.889 \\ 13.333 & 26.418 & -18.188 \\ -10.889 & -18.188 & 14.420 \end{bmatrix} \]