CHAPTER 6

OPTIMAL CONTROL AND EIGENSTRUCTURE ASSIGNMENT IN STATE SPACE SYSTEMS

6.1 INTRODUCTION

Optimal control problems discussed in this chapter consider minimization of linear quadratic impulse (LQI) cost function (and $H_2$ norm) and upper bound of $H_\infty$ norm to improve robustness and linear quadratic regulator cost functions to meet performance specifications along with eigenstructure assignment in state space systems.

Controllers that assign the given eigenstructure are important in decoupling of errors, reducing the sensitivity to parameter changes, improving the integrity with respect to stability for failures in the actuators, improving damping factor etc. But such controllers will not provide performance robustness in the presence of disturbances and measurement noise. Hence achievement of robust performance along with eigenstructure assignment yields a better design.

Unlike in $H_2$ and $H_\infty$ optimal control, the LQI and LQR optimal control designs allow arbitrary weighting of variables in the performance measure that is being minimized. This finds application specially when covariances of disturbance and measurement noise are not unit matrices. In LQI and LQR optimal controls presented in this chapter, assignment of eigenstructure is possible with any arbitrary choice of weighting matrices in the performance measure to be minimized. In literature several methods are available for optimal control (Saif 1989), wherein the quadratic weights are selected to give prescribed set of eigenvalues.
With full order controllers Doyle et al. (1989) present, a decoupled design of the controller and the compensator, with fixed closed loop pole locations. This chapter presents design of LQI optimal controller and $H_2$ optimal controller along with eigenstructure assignment for state space systems. $H_\infty$ controller design, minimizes the upper bound of the $H_\infty$ norm defined in terms of Hankel singular values. These designs need solutions of nonlinear minimizations of the objectives which are functions of free elements of the allowable eigenspace. A robust controller is designed for a satellite altitude control problem with multiple design objectives.

This chapter is divided into seven sections including this section. Section 2 gives formulation of all optimal control problems. Section 3 gives solution on $H_2$ optimal control. Section 4 presents the results on $H_\infty$ optimal control. Section 5 discusses LQ eigenstructure assignment regulators. Section 6 includes numerical examples in all these eigenstructure assignment problems. Section 7 summarizes the results of this chapter.

6.2 PROBLEM FORMULATION

Let us define a state space system in the form

$$\dot{X} = AX + BU + D_\theta \theta$$  \hspace{1cm} (6.1)

$$Y = CX$$  \hspace{1cm} (6.2)

where $\theta \in \mathbb{R}^{m\theta}$ is an exogenous input like reference inputs and disturbances and $D_\theta$ is a compatible constant matrix.

Let $Y_m$ be the measurement made at the output with

$$Y_m = C_m X + D_\mu \mu$$  \hspace{1cm} (6.3)
where $\mu \in \mathbb{R}^n$ is the measurement noise and $C_m$ is assumed to have full rank.

Now our aim is to apply a feedback $U$ given by

$$
U = K Y_m + H X_c
$$

(6.4)

where

$$
\dot{X}_c = F X_c + S Y_m
$$

(6.5)

is a dynamic compensator of order $\eta$, which minimizes

(i) the linear quadratic impulse (LQI) cost function or $H_2$ norm

$$(or)$$

(ii) the upper bound $H_\infty$ norm of the transfer function between regulated outputs and disturbance inputs

$$(or)$$

(iii) the linear quadratic regulator cost function (LQ)

and simultaneously assigns the given eigenstructure to the closed loop system.

Let the closed loop system be

$$
\dot{X} = \bar{A} X + \bar{B} U
$$

(6.6)

$$
\bar{Y} = \bar{C} X + \bar{D} U
$$

(6.7)

where the variables and matrices of the composite system are defined in the same manner as in section 5.3. Let

$$
V_r = [v_1 \ v_r2 \ \ldots \ v_{r(n+\eta)}] \quad \text{and}
$$

$$
V_l = [v_{l1}^T \ v_{l2}^T \ \ldots \ v_{l(n+\eta)}^T]^T
$$

be the matrices containing the right and left eigenvectors respectively. Let $\{\lambda_i\}$ and $\{v_i\}$ be the set of eigenvalues and eigenvectors to be assigned. Let $v_i(:,k)$ be the $k$-th column of $V_r$ and $v_i(k,:)$ be the $k$-th row of $V_l$. 

Let us define \( \tilde{n}_1(q_i) \) as the nullspace of \( \lambda_i^{-1}A_{\eta}^{-1}B_{\eta} \)

and \( W = [w_1 \ w_2 \ \ldots \ w_{(r+\eta)}] \) where \( w_i \in \text{span} \{\tilde{n}_2(q_i)\} \).

Let the augmented matrices be defined by

\[
A_{\eta} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} ; \quad B_{\eta} = \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix} ; \quad C_{m_{\eta}} = \begin{bmatrix} C_m & 0 \\ 0 & 1 \end{bmatrix} ; \quad p + \eta, n + \eta \quad n + \eta, m + \eta
\]

\[
K_{\eta} = \begin{bmatrix} K & H \\ S & F \end{bmatrix} ; \quad \text{and} \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n+\eta})
\]

(1) Let us define TF as the transfer function between \( \bar{Y} \) and \( \bar{U} \). Let \( Q_D \) and \( Q_m \) be the covariances of disturbances and measurement noise respectively. Let \( Q_s \) and \( Q_l \) be the weighting matrices for regulated outputs \( Y \) and control inputs \( U \). Let \( Q_l \) be the matrix containing \( Q_D \) and \( Q_m \) and \( Q_2 \) be the matrix containing \( Q_s \) and \( Q_l \) as their diagonal blocks. Then LQ1 cost function may be defined as

\[
\text{Tr} (L_c \bar{C}^* Q_2 \bar{C}) + 2 \text{Tr} (\bar{B} \bar{D}^* \bar{C}) + p \text{Tr} (\bar{D}^* \bar{D})
\]

where \( L_c \) is the solution of

\[
\bar{A} L_c + L_c \bar{A}^* + \bar{B} Q_l \bar{B}^* = 0
\]

Expression for \( H_2 \) norm is obtained when \( Q_D, Q_m, Q_s \) and \( Q_l \) are unit matrices of appropriate dimensions.
(2) The upper bound of $H_\infty$ norm is given by

$$\min_q \bar{\sigma}(\bar{D}) + \sum_{i=1}^{n+\eta} \sqrt{\lambda_i(L_c L_\alpha)}$$

where $L_c$ and $L_\alpha$ are the solutions of the following Lyapunov equations

$$\bar{A}L_c + L_c \bar{A}^* + \bar{B}\bar{B}^* = 0$$
$$L_\alpha \bar{A} + \bar{A}^* L_\alpha + \bar{C}^* \bar{C} = 0$$

(3) The linear quadratic regulator cost function is defined by

$$C_\phi = \int_0^\infty (X(t)^T C_s^T C X(t) + U(t)^T Q_l U(t)) \, dt$$

Let $\phi = \sum_{k=1}^{m+\eta} \{w_k(w_k)^T w_k(C_{\eta l} V)^T [C_{\eta l} V(C_{\eta l} V)^T]^{-1} C_{\eta l} V(w_k)^T\}$

It is assumed that in the case of complete eigenstructure assignment, the controller’s eigenstructure design is carried out prior to minimization of performance measure and hence $W \subset C_{\eta l} V_r$. The freedom in $V_r$ is reduced to give all controllers as given in chapter 2. Then $K_\eta$ takes the form given below and $\phi$ will always be zero with respect to the reduced vector of free elements provided $V_r$ is non-singular.

$$K_\eta = W (C_{\eta l} V_t)^T (C_{\eta l} V_r (C_{\eta l} V_r)^T)^{-1}$$

In the case of partial eigenstructure assignment $\phi = 0$ constraint is to be checked.

Let $Z = W(C_{\eta l} V_t)^T [C_{\eta l} V_t(C_{\eta l} V_t)^T]^{-1} = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}$
\[ B' = \begin{bmatrix} D_0 B \mu \ 0 \end{bmatrix}; \quad C' = \begin{bmatrix} C & 0 \\ Z_1 C_m & Z_2 \end{bmatrix} \]

### 6.3 LQI OPTIMAL CONTROL

**Theorem 6.1**

An LQI optimal controller is function of \( q \) that simultaneously minimizes \( \phi \) (to zero) and minimizes the following expression.

\[
\begin{aligned}
\min_{\{q\}} \quad & \sum_{i=1}^{n+\eta} \left[ \Sigma \nu_i(\cdot,:)B' \begin{bmatrix} 2D'^* - Q_1B'^* \end{bmatrix} \Sigma \begin{bmatrix} \nu_i^*(\cdot,:),\nu_i^*(\cdot,k) \\ C'^*Q_2 \end{bmatrix} C'\nu_i(:,i) \right] \\
or & \sum_{k=1}^{n+\eta} \frac{\nu_i^*(\cdot,k)}{(\lambda_i + \lambda_k^*)} + \text{Tr} p^{-1}(D'^*D) \\
\text{subject to det} \ V = \neq 0
\end{aligned}
\] (6.13)

where \( p^{-1} \) is a large positive constant

**Proof**

When \( V_r \) is non-singular \( V_0 \) can be obtained. In addition, minimization of \( \phi \) (to zero) ensures that eigenstructure assignment constraints are satisfied. Then \( B = \bar{B}, C' = \bar{C}, D' = \bar{D} \) and \( A = V_r \Lambda V_i \). Let us replace the summation over \( i \) of equation (6.13) by trace operation, to get

\[
\begin{aligned}
\min_{\{q\}} \text{Tr} \left( V_r \bar{B} 2D'^* - Q_1 \bar{B}'^* \right) \sum_{k=1}^{n+\eta} \frac{\nu_i^*(\cdot,:)\nu_i^*(\cdot,k)}{(\lambda_i + \lambda_k^*)} \bar{C}'Q_2 \bar{C}V_r + p^{-1} \text{Tr}(D'^*D) \\
\end{aligned}
\]
or

\[
\min \left[ 2 \text{Tr} (\tilde{B} \tilde{D}^* \tilde{C}) - \text{Tr}(V_i \tilde{B} Q_1 \tilde{B}^*) \sum_{k=1}^{n+\eta} \frac{[v_i(\cdot, :)v_i^*(\cdot, :)C^*Q_2C V_i]}{(\lambda_i + \lambda_k^*)} + p \sim \text{Tr}(\tilde{D}^* \tilde{D}) \right]
\]

The second term can be rearranged as

\[
- \text{Tr} V_i \tilde{B} Q_1 \tilde{B}^* \sum_{k=1}^{n+\eta} \frac{[v_i(\cdot, :)v_i^*(\cdot, :)C^*Q_2C V_i]}{(\lambda_i + \lambda_k^*)}
\]

or

\[
- \text{Tr} \begin{bmatrix}
    \frac{v_i(1,:)\tilde{B} Q_1 \tilde{B}^* v_i(1,:)^*}{(\lambda_1 + \lambda_1^*)} & \ldots & \frac{v_i(1,:)\tilde{B} Q_1 \tilde{B}^* v_i(n+\eta,:)^*}{(\lambda_1 + \lambda_{n+\eta}^*)} \\
    \vdots & \ddots & \vdots \\
    \frac{v_i(n+\eta,:)\tilde{B} Q_1 \tilde{B}^* v_i(1,:)^*}{(\lambda_{n+\eta} + \lambda_1^*)} & \ldots & \frac{v_i(n+\eta,:)\tilde{B} Q_1 \tilde{B}^* v_i(n+\eta,:)^*}{(\lambda_{n+\eta} + \lambda_{n+\eta}^*)}
\end{bmatrix}
\times
\begin{bmatrix}
    v_i(:,1)^*C^*Q_2C v_i(:,1) & \ldots & v_i(:,1)^*C^*Q_2C v_i(:,n+\eta) \\
    \vdots & \ddots & \vdots \\
    v_i(:,n+\eta)^*C^*Q_2C v_i(:,1) & \ldots & v_i(:,n+\eta)^*C^*Q_2C v_i(:,n+\eta)
\end{bmatrix}
\]

which is equivalent to

\[
\text{Tr} \left( L_c^* V_i^* C^*Q_2C V_i \right)
\]

where \( L_c \) is the solution of the Lyapunov equation

\[
\Lambda L_c + L_c \Lambda^* + V_i \tilde{B} Q_1 \tilde{B}^* V_i^* = 0
\]

The equation mentioned above is equivalent to equation(6.8), if \( V_i L_c^* V_i^* \) is replaced by \( L_c \) that satisfies the following equation

\[
\overline{\Lambda} L_c + L_c \overline{A}^* + \tilde{B} Q_1 \tilde{B}^* = 0
\]
Hence minimization of equation (6.13) is equivalent to minimization of the LQI cost function given by equation (6.8).

\[ \text{6.4 } H_\infty \text{ OPTIMAL CONTROL} \]

The \( H_\infty \) optimal controller presented in this section is an extension of similar type of controller design presented for descriptor systems in chapter 5.

**Theorem 6.2**

The \( H_\infty \) controller that assigns the given eigenstructure and minimizes the upper bound of \( H_\infty \) norm is a function of \( q \) that simultaneously minimizes \( \phi \) (to zero) and the following function.

\[
\begin{align*}
\min_{q} & \quad \alpha(D') + 2 \sum_{i=1}^{n+\eta} \sqrt{\lambda_i(G)} \\
\text{subject to } & \quad \det(V) \neq 0
\end{align*}
\]  

(6.14)

where

\[
g(i,j) = v_i(:,i)B' B'^* \sum_{k=1}^{n+\eta} \frac{[v_i^*(k,:) v_i^*(i,k)] C^* C v_i(:,j)}{(\lambda_i + \lambda_k^*) (\lambda_j^* + \lambda_k^*)} \]  

(6.15)

**Proof**

When \( \phi \) is zero for a non-singular \( V \), eigenstructure assignment conditions exist and \( B' = \tilde{B} D' = \tilde{C} D' = \tilde{D} \) and \( \tilde{A} = V_r \Lambda V_l \). From the definition of \( g(i,j) \), \( G \) can be constructed as
which is equivalent to

\[
\begin{bmatrix}
\nu_1(1,:) B^* B \nu(1,:) \* \\
(\lambda_1 + \lambda_1^*) \\
\vdots \\
\nu_1(n+\eta,:) B^* B \nu(1,:) \* \\
(\lambda_{n+\eta} + \lambda_1^*) \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\nu_1(1,:) B^* B \nu(1,:) \* \\
(\lambda_1 + \lambda_{n-\eta}^*) \\
\vdots \\
\nu_1(n+\eta,:) B^* B \nu(1,:) \* \\
(\lambda_{n-\eta} + \lambda_{n-\eta}^*) \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\nu_i(:,1) C^* C \nu_i(:,1) \\
(\lambda_1^* + \lambda_1) \\
\vdots \\
\nu_i(:,n+\eta) C^* C \nu_i(:,1) \\
(\lambda_{n+\eta}^* + \lambda_1) \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\nu_i(:,1) C^* C \nu_i(:,1) \\
(\lambda_1^* + \lambda_{n-\eta}) \\
\vdots \\
\nu_i(:,n+\eta) C^* C \nu_i(:,1) \\
(\lambda_{n-\eta}^* + \lambda_{n-\eta}) \\
\end{bmatrix}
\]

which is equivalent to

\[
\begin{bmatrix}
\nu_1(1,:) B^* B \nu(1,:) \* \\
(\lambda_1 + \lambda_1^*) \\
\vdots \\
\nu_1(n+\eta,:) B^* B \nu(1,:) \* \\
(\lambda_{n+\eta} + \lambda_1^*) \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\nu_1(1,:) B^* B \nu(1,:) \* \\
(\lambda_1 + \lambda_{n-\eta}^*) \\
\vdots \\
\nu_1(n+\eta,:) B^* B \nu(1,:) \* \\
(\lambda_{n-\eta} + \lambda_{n-\eta}^*) \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\nu_i(:,1) C^* C \nu_i(:,1) \\
(\lambda_1^* + \lambda_1) \\
\vdots \\
\nu_i(:,n+\eta) C^* C \nu_i(:,1) \\
(\lambda_{n+\eta}^* + \lambda_1) \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\nu_i(:,1) C^* C \nu_i(:,1) \\
(\lambda_1^* + \lambda_{n-\eta}) \\
\vdots \\
\nu_i(:,n+\eta) C^* C \nu_i(:,1) \\
(\lambda_{n-\eta}^* + \lambda_{n-\eta}) \\
\end{bmatrix}
\]

where \( L'_c \) and \( L'_o \) are the solutions of the lyapunov equations

\[
\begin{align*}
\Lambda L'_c + L'_c \Lambda^* + \nu_i B^* B \nu_i^* &= 0 \\
L'_o \Lambda + \Lambda^* L'_o + \nu_i^* C^* C \nu_i^* &= 0
\end{align*}
\]

Since eigenvalues are invariant by similarity transformations we may write

\[
\{ \lambda (L'_c L'_o) \} = \{ \lambda (\nu_i^* L'_c L'_o \nu_i) \} = \{ \lambda (L_c L_o) \}
\]

(6.16)

where \( L_c \) and \( L_o \) are solutions of

\[
\begin{align*}
\bar{A} L_c + L_c \bar{A}^* + B B^* &= 0 \\
L_o \bar{A} + \bar{A}^* L_o + C^* C &= 0
\end{align*}
\]

Hence minimization (6.16) is equivalent to minimization of the \( \Gamma_{ub} \) with \( K_\eta \) defined by equation (6.9) provided \( \phi = 0 \) and \( \det (V) \neq 0. \)
6.5 LINEAR QUADRATIC EIGENSTRUCTURE ASSIGNMENT REGULATORS

Let $C^T Q S C = Q$ and $Q_J$ be the weighting matrices chosen to weight the state variables and input variables in the linear quadratic regulator problems. The eigenstructure assignment problem discussed in chapter 2 is applied first and all the controllers are obtained. Then the optimality condition is imposed.

6.5.1 Results on LQ regulators

Theorem 6.3

The output feedback $K$ that assigns the given eigenstructure and minimizes an LQR performance measure iff the following is minimized.

$$\text{Min } (X^*(0) P X(0)) \quad (6.17)$$

subject to

$$(A+BKC)^T P + P (A+BKC) + C^T K^T R K C + C^T Q_S C = 0 \quad (6.18)$$

Proof (Sufficiency)

When equation (6.18) is true then $P$ can be solved and substituted in (6.17) to get

$$(X^*(0) P X(0))$$

$$= X(0)^T \int_0^\infty [ \exp^{\lambda t} [ C^T Q_S C + C^T K^T Q_J K C ] \exp^{\lambda t} dt ] X(0)$$

$$= \int_0^\infty [ \exp^{\lambda t} X(0)]^T [ Q + C^T K^T Q_J K C ] \exp^{\lambda t} X(0) dt \quad (6.19)$$

where $A' = A + BK C$. 
But \( \exp^{A^t} X(0) \) is the solution of the state equation

\[
\dot{X} = (A + B K C) X
\]

Hence substituting for \( \exp^{A^t} X(0) \) in the above integral we get

\[
(X(0)^* P X(0))
\]

\[
= \int_0^\infty [X(t)]^T [Q + C^T K^T Q_l K C] X(t) \, dt \tag{6.20}
\]

\[
= \int_0^\infty ([X(t)]^T Q X(t) + X(t)^T C^T K^T Q_l K C X(t)) \, dt
\]

Since \( K \) is an output feedback matrix \( K C X(t) \) can be replaced by \( U(t) \) if \( Y = C X \) is output equation of the state space system.

Then

\[
\min (X(0)^* P X(0)) = \min \int_0^\infty (X(t)^T Q X(t) + U(t)^T Q_l U(t)) \, dt
\]

with

\[
X = (A + B K C) X
\]

and

\[
Y = C X
\]

**Necessity**: Consider an LQR performance measure. Replace \( U(t) \) by \( K C X(t) \) to apply an output feedback. Since \( X(t) \) should satisfy the state equations, it can be replaced by the solution of the homogeneous state equations to get equation(6.19). Taking the constant terms outside the integral, the remaining terms can be obtained as a solution of Lyapunov type equation given by the equation(6.18). Hence minimization of an LQR performance measure is equivalent to minimization of equation(6.17).
Note: The K matrix will be defined by equation (6.12) which is a function of q provided $\phi = 0$ and $\det(V) \neq 0$.

6.5.2 Numerical Procedure

Equations (6.17) and (6.18) can be replaced by the following for numerical solution of this problem.

$$\text{Min } (X(0)^* (V^{-1})^T P^* V^{-1} X(0))$$  \hspace{1cm} (6.21) $$

where

$$\Lambda^* P^* + P^* \Lambda + V^T [ C^T K^T Q_1 K C + Q ] V = 0$$  \hspace{1cm} (6.22) $$

For any random $X(0)$ with the expected value of $X(0) X(0)^T$ given by

$$E[X(0)X(0)^T] = I,$$

the following function can be used for minimization instead of equation (6.17).

$$\text{Min } \text{tr } ((V^{-1})^T P^* V^{-1})$$  \hspace{1cm} (6.23) $$

This minimization can be carried out for all possible $K$’s defined by (6.12). Moreover, equation (6.22) can be solved easily exploiting the diagonal structure of $\Lambda$.

The numerical procedures explained in chapter 5 can be adopted for obtaining the solutions of these problems with suitable modifications. The following section gives some comparative numerical results and solutions for a practical design problem.
6.6 NUMERICAL EXAMPLES

Example 6.6.1

To illustrate the given method let us consider the following example. The system should have a triple pole at -1.

\[ A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} ; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} ; C = [1 0]; C_m = [1 -1]; D_m = 1; \]

\[ D_\theta = \begin{bmatrix} 1 & 1 \\ -15.066 & 0.0664 \end{bmatrix} \]

(i) H_2 controller

(a) The results of minimization of gama and norm(K量产) with compensator of order 2 is given as gama = 462 and norm^2(K量产) = 29. The closed loop poles are -1, -1, -1 and -4. With the same order controller the results obtained, by applying Doyle et.al(1989) method, the minimum H_2 norm is given by 462 and square of the feedback matrix norm is given by 730 with the closed loop poles at -1, -1, -0.8835 and -16.

\[ K_{\eta} = \begin{bmatrix} 0.0 & -1.212 & 0.729 \\ 1.209 & -3.269 & -0.445 \\ -0.733 & -0.443 & -3.731 \end{bmatrix} ; \quad V = \begin{bmatrix} -0.69 & -0.72 & -0.73 & -0.72 \\ 0.69 & 0.29 & 0.49 & 0.36 \\ 1.0 & 1.0 & 1.0 & 1.0 \\ 0.08 & -0.3 & -0.03 & -0.16 \end{bmatrix} \]

(b) With a reduced order controller the results are presented below

\[ K_{\eta} = \begin{bmatrix} 0.0 & -1.6544 \\ 1.209 & -3.0 \end{bmatrix} \]

\[ V = \begin{bmatrix} -0.69 & -0.72 & -0.73 & -0.72 \\ 0.69 & 0.29 & 0.49 & 0.36 \\ 0.08 & -0.3 & -0.03 & -0.16 \end{bmatrix} \]
The closed loop system has a triple pole at -1 for the same $H_2$ norm and $\text{norm}^2(K_\eta) = 13.2$.

(ii) $H_\infty$ controller

With $\Gamma_{ub}$ and $\text{norm}(K)$ minimization the results are given below.

$\Gamma_{ub} = 32.2$ and $\text{norm}^2(K) = 16.88$ for the same closed loop eigenvalues. The $H_\infty$ norm of the open loop and the closed loop system are given by

$$K_\eta = \begin{bmatrix} 0.691 & -1.516 & -0.292 \\ 2.189 & -3.191 & 2.522 \\ 1.697 & -1.268 & -4.500 \end{bmatrix} ; \quad \nu = \begin{bmatrix} 1.18 & -3.23 & 2.68 & -0.71 \\ -1.18 & 2.15 & -1.34 & 0.28 \\ 2.59 & -7.05 & 6.34 & -1.87 \\ 0.20 & -0.06 & -0.49 & 0.35 \end{bmatrix}$$

Example 6.6.2

Consider the following satellite altitude control problem (Kaplan 1970). The system matrices are given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ YA & YB & 0 & 0 & YC & YD \\ RA & RB & 0 & RC & 0 & RD \\ PA & PB & 0 & PC & PD & 0 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/\lambda_{XX} & 0 \\ 0 & 1/\lambda_{YY} & 0 \\ 0 & 0 & 1/\lambda_{ZZ} \end{bmatrix}$$

with

$$YA = [ I_{xx} - I_{yy} ] \omega_o^2 / \lambda_{xx}$$

$$YB = -[\omega_o^2 \cdot I_{yz} ] / \lambda_{xx}$$

$$YC = -[( I_{xx} - I_{zz} + I_{yy} ) \cdot \omega_o ] / \lambda_{xx}$$

$$YD = -2( I_{yz} \cdot \omega_o / \lambda_{xx}$$

$$RA = ( \omega_o^2 I_{xy} / \lambda_{xy}$$

$$RB = -[\omega_o^2 ( -I_{zz} + I_{xx} ) ] / \lambda_{yy}$$
\[
\begin{align*}
RC & = -(l_{xx} - l_{zz} + l_{yy}) \omega_o / l_{yy} \\
RD & = -2(l_{xz} \omega_o) / l_{yy} \\
PA & = [\omega_o^2 l_{zz} / l_{zz} \\
PB & = [\omega_o^2 l_{yz} / l_{zz} \\
PC & = 2(l_{yz} \omega_o) / l_{zz} \\
PD & = -2(l_{zx} \omega_o) / l_{zz}
\end{align*}
\]

where \( I \) stands for the moments of inertia and cross moments of inertia with respect to Roll, Pitch and Yaw axes. It is given in the matrix form as

\[
\begin{bmatrix}
1200 & -25 & -25 \\
-25 & 1200 & -25 \\
-25 & -25 & 500
\end{bmatrix}
\]

and \( \omega_o \) is the orbital velocity given by \( 0.0011/\text{second} \).

The first three state variables are the errors in angular velocities in the three axes respectively (in radians). The remaining three state variables are taken to be the rates of errors (in radians/\( \text{sec} \)). The control inputs are the applied torques in the three axes. Let \( I_n \) be the unit matrix of dimension \( n \). Then

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix} ; \quad D_\theta = B \quad \text{and} \quad D_\mu = I_3.
\]

\[
Q_D = 1.0E-4 I_3; \quad Q_m = 1.75E-5 I_3; \quad Q_4 = 4.9386E+5 I_3;
\]

\[
Q_s = 1.0E+6 \begin{bmatrix}
4 & 0 & 0 \\
0 & 7.1 & 0 \\
0 & 0 & 0.3
\end{bmatrix}
\]

A controller should be designed to meet the following requirements.

(i) Dominant modes based on the control specification for the satellite are -0.015, -0.02 and -0.01
(ii) The three eigenvectors corresponding to the dominant eigenvalues should be of the form \([1 \ 0 \ 0 \ x \ x \ x]^T\), \([0 \ 1 \ 0 \ x \ x \ x]^T\) and \([0 \ 0 \ 1 \ x \ x \ x]^T\) to obtain decoupling of errors and wfs corresponding to free eigenvalues should take the structure \([x \ 0 \ 0]^T\), \([0 \ x \ 0]^T\) and \([0 \ 0 \ x]^T\) to obtain asymptotic decoupling of actuators.

(iii) Robust with respect to external disturbances and measurement noises.

Linear quadratic impulse cost function is chosen to obtain robustness. The robust controller that provides the required design features is given by

\[
\begin{bmatrix}
-1.0e-09 & 3.0e-08 & 1.2e-09 & 2.4e-00 & -3.0e-5 & 5.3e-15 & -2.52e+02 & 2.09e+00 & -5.5e-02 \\
-1.3e-15 & -2.1e-09 & -9.1e-09 & 3.0e-05 & -4.5e+00 & 5.3e-15 & 2.09e+00 & -5.18e+02 & -5.5e-02 \\
8.5e-09 & 1.2e-09 & 2.8e-15 & 3.0e-05 & 3.0e-05 & -3.0e+00 & -5.5e-02 & -5.5e-02 & -1.6e-02 \\
7.2e-01 & -1.7e-03 & 4.6e-05 & -7.2e-01 & 1.7e-03 & 4.6e-05 & 1.0e+00 & 0 & 0 \\
-1.7e-03 & 8.2e-01 & 4.6e-05 & -1.7e-03 & -8.2e-01 & -4.6e-05 & 0 & 1.0e+0 & 0 \\
-1.1e-04 & 1.1e-04 & 7.2e-01 & 1.1e-04 & -1.1e-04 & -7.2e-01 & 0 & 0 & 1.0e+0 \\
1.3e-01 & -1.4e-03 & 3.3e-05 & -1.3e-01 & 1.4e-03 & -3.3e-05 & -2.0e-01 & -1.4e-14 & -4.5e-16 \\
-1.3e-03 & 1.7e-01 & 3.3e-05 & -1.2e-03 & -1.7e-01 & -3.3e-05 & -1.7e-14 & -2.6e-01 & -4.5e-16 \\
-7.9e-05 & 9.0e-05 & 1.2e-01 & 7.9e-05 & -9.0e-05 & -1.3e-01 & -1.1e-15 & 1.2e-15 & -3.2e-01
\end{bmatrix}
\]

with the required structure of V and W. The closed loop poles are found to be -0.01, -0.015, -0.02, -0.2, -0.25, -0.27, -0.3, -0.35, -0.37, -0.4, -0.42 and -0.45.

6.7 SUMMARY

Most of the practical design problems need particular choices of eigenvalues and predefined structure for eigenvectors to satisfy stability. In this chapter, theorems are presented that help to minimize linear quadratic impulse cost function, \(H_2\) norm, upper bound of \(H_\infty\) norm and linear quadratic regulator cost functions besides eigenstructure assignment in statespace systems. In linear quadratic impulse and linear quadratic regulator problems arbitrary choice of pole locations and weighting matrices is possible. Then the covariances of disturbances and measurement noise and the Q and R weighting matrices will no more be the
deciding factors for pole assignment. These optimal control designs need minimization of a non-linear function. In addition, these controllers can be designed with a specified order of compensators. Design of a robust controller for satellite altitude control that decouples the errors, actuators, assigns dominant modes ensuring robustness and stability is presented to illustrate the importance of the proposed design. Another numerical example has also been presented to illustrate the $H_2$ and $H_\infty$ optimal designs.