Chapter 3

Classes involving Salagean - Ruscheweyh Operator

The objective of the present chapter\footnote{Reference [61] is based on this chapter} is to introduce two new subclasses $\mathcal{V}S_p(\lambda, m, \alpha)$, $\mathcal{V}S_p(\lambda, m, \alpha, \beta)$ of starlike functions $\mathcal{S}_p(\alpha)$ with varying arguments. First we obtain coefficient estimates for functions in these classes. Further we prove a new radii result which unifies radii of close-to-convexity, starlikeness and convexity. We also discuss about the extreme points. For suitable choices of parameters the results deduced coincide with the results obtained by Singh and Tygel \cite{74} and Vijaya and Murugusundaramoorthy \cite{80}.
3.1 Prelude

Goodman introduced the classes of uniformly convex and uniformly starlike functions denoted by $\text{UCV}$ and $\text{UST}$ respectively. Following the work of Goodman [22], Ronning [56], Ma and Minda [33] independently gave analytic characterization of the class $\text{UCV}$ by proving $f \in \text{UCV}$ if and only if

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq \left| \frac{zf''(z)}{f'(z)} \right|,$$

Ronning [56], defined a subclass $\mathcal{S}_p$, namely the class of parabolic starlike functions by

$$\mathcal{S}_p = \left\{ f \in \mathcal{A} : \Re \left\{ \frac{zf'(z)}{f(z)} \right\} \geq \left| \frac{zf'(z)}{f(z)} - 1 \right|, z \in U \right\}$$

Note that, $f \in \text{UCV}$ if and only if $zf' \in \mathcal{S}_p$.

Now using $D^m_\lambda$ [Definition 1.2.23] linear operator, for $0 \leq \alpha < 1$, $m \in \mathbb{N}_0$ and $\lambda > 0$, we define the subclass $\mathcal{S}_p(\lambda, m, \alpha)$.

**Definition 3.1.1.** A function $f$ of the form (1.1.1) is in $\mathcal{S}_p(\lambda, m, \alpha)$ if it satisfies the following

$$\Re \left\{ \frac{z(D^m_\lambda f(z))'}{D^m_\lambda f(z)} - \alpha \right\} \geq \left| \frac{z(D^m_\lambda f(z))'}{D^m_\lambda f(z)} - 1 \right|,$$

where

$$D^m_\lambda f(z) = z + \sum_{n=2}^{\infty} B_\lambda(m, n)a_n z^n.$$
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\[ B_\lambda(m, n) = [(1 - \lambda)n^m + \lambda c(m, n)]. \]

and \( c(m, n) = \frac{\Gamma(n + m)}{\Gamma(n)\Gamma(1 + m)}. \)

For \( 0 \leq \alpha < 1 \), we define \( \mathcal{V}_p(\lambda, m, \alpha) = \mathcal{S}_p(\lambda, m, \alpha) \cap \mathcal{V} \), where \( \mathcal{V} \) is defined as follows.

**Definition 3.1.2.** [71] A function \( f \) of the form (1.1.1) is said to be in the class \( \mathcal{V}(\theta_n) \) if \( f \in \mathcal{A} \) and \( \arg(a_n) = \theta_n \) for all \( n \geq 2 \). If furthermore there exist a real number \( \beta \) such that \( \theta_n + (n - 1)\beta = \pi \pmod{2\pi} \), then \( f \) is said to be in the class \( \mathcal{V}(\theta_n, \beta) \). The union of \( \mathcal{V}(\theta_n, \beta) \) taken over all possible sequences \( \{\theta_n\} \) and all possible real numbers \( \beta \) is denoted by \( \mathcal{V} \).

### 3.2 Principal results

**Theorem 3.2.1.** A function \( f \) of the form (1.1.1) is in \( \mathcal{V}_p(\lambda, m, \alpha) \) if and only if

\[
\sum_{n=2}^{\infty} (2n - 1 - \alpha)B_\lambda(m, n)|a_n| \leq 1 - \alpha \tag{3.2.1}
\]

**Proof.** By definition of the class \( \mathcal{V}_p(\lambda, m, \alpha) \) it suffices to show that

\[
\left| \frac{z(D_{\lambda}^m f(z))'}{D_{\lambda}^m f(z)} - 1 \right| \leq \Re \left\{ \frac{z(D_{\lambda}^m f(z))'}{D_{\lambda}^m f(z)} - \alpha \right\}
\]
That is
\[
\left| \frac{z(D^m f(z))'}{D^m f(z)} - 1 \right| - \Re \left\{ \frac{z(D^m f(z))'}{D^m f(z)} - 1 \right\} \leq 2 \left| \frac{z(D^m f(z))'}{D^m f(z)} - 1 \right| \\
\sum_{n=2}^{\infty} (n-1)B_\lambda(m,n)|a_n||z|^{n-1} \\
\leq 2 \frac{\sum_{n=2}^{\infty} B_\lambda(m,n)|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty} B_\lambda(m,n)|a_n||z|^{n-1}}.
\]

Now the last expression is bounded by \((1 - \alpha)\) and only if
\[
\sum_{n=2}^{\infty} (2n - 1 - \alpha)B_\lambda(m,n)|a_n| \leq 1 - \alpha.
\]
Conversely, if \(f \in \mathcal{VS}_p(\lambda, m, \alpha)\) then by definition,
\[
\left| \frac{z + \sum_{n=2}^{\infty} nB_\lambda(m,n)a_nz^n}{z + \sum_{n=2}^{\infty} B_\lambda(m,n)a_nz^n} - 1 \right| \leq \Re \left\{ \frac{z + \sum_{n=2}^{\infty} nB_\lambda(m,n)a_nz^n}{z + \sum_{n=2}^{\infty} B_\lambda(m,n)a_nz^n} - \alpha \right\}
\]
That is
\[
\left| \frac{\sum_{n=2}^{\infty} (n-1)B_\lambda(m,n)a_nz^{n-1}}{1 + \sum_{n=2}^{\infty} B_\lambda(m,n)a_nz^{n-1}} \right| \leq \Re \left\{ \frac{(1 - \alpha) + \sum_{n=2}^{\infty} (n - \alpha)B_\lambda(m,n)a_nz^{n-1}}{1 + \sum_{n=2}^{\infty} B_\lambda(m,n)a_nz^{n-1}} \right\}
\]
since \(f \in \mathcal{V}\) and \(f\) lies in \(\mathcal{V}(\theta_n, \beta)\) for some sequence \(\{\theta_n\}\) and a real number \(\beta\) such that \(\theta_n + (n-1)\beta \equiv \pi (\text{mod} 2\pi)\) set \(z = re^{i\beta}\) in the
above inequality
\[
\left| \sum_{n=2}^{\infty} (n - 1)B_\lambda(m, n)a_nr^{n-1} \right| \leq \Re \left\{ \frac{(1-\alpha) - \sum_{n=2}^{\infty} (n - \alpha)B_\lambda(m, n)a_nr^{n-1}}{1 - \sum_{n=2}^{\infty} B_\lambda(m, n)a_nr^{n-1}} \right\}
\]

Letting \( r \to 1 \), leads the desired inequality
\[
\sum_{n=2}^{\infty} (2n - 1 - \alpha)B_\lambda(m, n)|a_n| \leq 1 - \alpha.
\]

**Corollary 3.2.2.** If \( f \in \mathcal{VS}_p(\lambda, m, \alpha) \) then \( |a_n| \leq \frac{1 - \alpha}{(2n - 1 - \alpha)B_\lambda(m, n)} \) for \( n \geq 2 \). The equality holds for
\[
f(z) = z + \sum_{n=2}^{\infty} \frac{(1 - \alpha)}{(2n - 1 - \alpha)B_\lambda(m, n)} e^{i\theta_n} z^n
\]
for \( n \geq 2, z \in \mathbb{U} \).

**Theorem 3.2.3.** If \( f \in \mathcal{VS}_p(\lambda, m, \alpha) \), then
\[
\left| \frac{f \ast \Phi}{f \ast \Psi} - 1 \right| < 1 - \delta, \quad \text{in} \quad |z| < r,
\]
with \( \Phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n \), and \( \Psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n \), are analytic in \( \mathbb{U} \) with the conditions \( \lambda_n \geq 0, \mu_n \geq 0, \lambda_n \geq \mu_n \), for \( n \geq 2 \) and
\[
f(z) \ast \Psi(z) \neq 0, \quad \text{where}
\]
\[
r = \inf_n \left[ \frac{(2n - 1 - \alpha)B_\lambda(m, n)(1 - \delta)^{n-1}}{(1 - \alpha)[(\lambda_n - \mu_n) + \mu_n(1 - \delta)]} \right], \quad n \geq 2. \quad (3.2.2)
\]
Proof. Consider,

\[
\left| \frac{f \ast \Phi}{f \ast \Psi} - 1 \right| = \left| \frac{z + \sum_{n=2}^{\infty} \lambda_n a_n z^n}{z + \sum_{n=2}^{\infty} \mu_n a_n z^n} - 1 \right|
\]

\[
\leq \left| \frac{z + \sum_{n=2}^{\infty} \lambda_n a_n z^n - z - \sum_{n=2}^{\infty} \mu_n a_n z^n}{z + \sum_{n=2}^{\infty} \mu_n a_n z^n} \right|
\]

\[
\sum_{n=2}^{\infty} a_n |(\lambda_n - \mu_n)| |z|^{n-1} \leq \sum_{n=2}^{\infty} \mu_n a_n |z|^{n-1} < 1 - \delta.
\]

\[
\sum_{n=2}^{\infty} a_n [(\lambda_n - \mu_n) + (1 - \delta) \mu_n] \leq 1 - \delta, \quad (|z| < r, 0 \leq \delta < 1), \quad (3.2.3)
\]

where \( r \) is given by (3.2.2).

From Theorem 3.2.1, (3.2.3) will be true if,

\[
\frac{[\lambda_n - \mu_n] + (1 - \delta) \mu_n}{1 - \delta} |z|^{n-1} \leq \frac{(2n - 1 - \alpha) B_\lambda(m, n)(1 - \delta)}{(1 - \alpha)[(\lambda_n - \mu_n) + \mu_n(1 - \delta)]},
\]

that is, if

\[
|z| = \left[ \frac{(2n - 1 - \alpha) B_\lambda(m, n)(1 - \delta)}{(1 - \alpha)[(\lambda_n - \mu_n) + \mu_n(1 - \delta)]} \right]^{\frac{1}{n-1}}, \quad (n \geq 2).
\]

As corollaries to the above theorem we get the following results.

By choosing \( \Phi(z) = \frac{z}{(1 - z)^2} \) and \( \Psi(z) = z \), we get
Corollary 3.2.4. Let the function $f$ defined by (1.1.1) be in the class $\mathcal{VS}_p(\lambda, m, \alpha)$. Then $f$ is close-to-convex of order $\delta$ $(0 \leq \delta < 1)$, hence univalent, in the disc $|z| < r_1$, where

$$r_1 = \inf_n \left[ \frac{(2n - 1 - \alpha)B_\lambda(m, n)(1 - \delta)}{(1 - \alpha)n} \right]^{\frac{1}{n-1}}, \quad n \geq 2. \quad (3.2.4)$$

The result is sharp.

For $\Phi(z) = \frac{z}{(1 - z)^2}$ and $\Psi(z) = \frac{z}{1 - z}$, we get

Corollary 3.2.5. Let the function $f$ defined by (1.1.1) be in the class $\mathcal{VS}_p(\lambda, m, \alpha)$. Then $f$ is starlike of order $\delta$ $(0 \leq \delta < 1)$, hence univalent, in the disc $|z| < r_2$, where

$$r_2 = \inf_n \left[ \frac{(2n - 1 - \alpha)B_\lambda(m, n)(1 - \delta)}{(1 - \alpha)(n - \delta)} \right]^{\frac{1}{n-1}}, \quad n \geq 2. \quad (3.2.5)$$

The result is sharp.

If $\Phi(z) = \frac{z + z^2}{(1 - z)^3}$ and $\Psi(z) = \frac{z}{(1 - z)^2}$, then we get

Corollary 3.2.6. Let the function $f$ be defined by (1.1.1) be in the class $\mathcal{VS}_p(\lambda, m, \alpha)$. Then $f$ is convex of order $\delta$ $(0 \leq \delta < 1)$, hence univalent, in the disc $|z| < r_3$, where

$$r_3 = \inf_n \left[ \frac{(2n - 1 - \alpha)B_\lambda(m, n)(1 - \delta)}{n(1 - \alpha)(n - \delta)} \right]^{\frac{1}{n-1}}, \quad n \geq 2. \quad (3.2.6)$$

The result is sharp.
Next we obtain distortion bounds for functions belonging to the class $\mathcal{VS}_p(\lambda, m, \alpha)$.

**Theorem 3.2.7.** Let $f$ of the form (1.1.1) be in the class $\mathcal{VS}_p(\lambda, m, \alpha)$. Then

$$r - \frac{1 - \alpha}{(3 - \alpha)B_\lambda(m, 2)} r^2 \leq |f(z)| \leq r + \frac{1 - \alpha}{(3 - \alpha)B_\lambda(m, 2)} r^2$$

and

$$1 - \frac{2(1 - \alpha)}{(3 - \alpha)B_\lambda(m, 2)} r \leq |f'(z)| \leq 1 + \frac{2(1 - \alpha)}{(3 - \alpha)B_\lambda(m, 2)} r.$$

The result is sharp.

**Proof.** Let $f$ of the form (1.1.1) be in the class $\mathcal{VS}_p(\lambda, m, \alpha)$. By taking absolute value of $f$

$$|f(z)| = \left| z + \sum_{n=2}^\infty a_n z^n \right| \leq |z| + |z|^2 \sum_{n=2}^\infty |a_n|,$$

since $f \in \mathcal{VS}_p(\lambda, m, \alpha)$ and by Theorem 3.2.1, we have

$$(3 - \alpha)B_\lambda(m, 2) \sum_{n=2}^\infty |a_n| \leq \sum_{n=2}^\infty (2n - 1 - \alpha)B_\lambda(m, n) |a_n| \leq 1 - \alpha.$$

Thus

$$|f(z)| \leq |z| + \frac{1 - \alpha}{(3 - \alpha)B_\lambda(m, 2)} |z|^2.$$

That is

$$|f(z)| \leq r + \frac{1 - \alpha}{(3 - \alpha)B_\lambda(m, 2)} r^2,$$

similarly we get

$$|f(z)| \geq r - \frac{1 - \alpha}{(3 - \alpha)B_\lambda(m, 2)} r^2.$$
On the other hand  \( f'(z) = 1 + \sum_{n=2}^\infty na_n z^{n-1} \),
and  \( |f'(z)| = 1 + \sum_{n=2}^\infty n|a_n||z|^{n-1} \leq 1 + |z| \sum_{n=2}^\infty n|a_n| \),
since  \( f \in \mathcal{VS}_p(\lambda, m, \alpha) \).

Then by Theorem 3.2.1 we have
\[
\sum_{n=2}^\infty n|a_n| \leq \frac{2(1 - \alpha)}{(3 - \alpha) B_{\lambda}(m, 2)}.
\]
Thus  \( |f'(z)| \leq 1 + \frac{2(1 - \alpha)}{(3 - \alpha) B_{\lambda}(m, 2)} r. \)

Similarly we get  \( |f'(z)| \geq 1 - \frac{2(1 - \alpha)}{(3 - \alpha) B_{\lambda}(m, 2)} r. \)

This completes the result.

\[\mathbf{\square}\]

**Theorem 3.2.8.** Let \( f \) of the form (1.1.1) be in the class \( \mathcal{VS}_p(\lambda, m, \alpha) \),
with  \( \arg a_n = \theta_n \) where  \( [\theta_n + (n-1)\beta] = \pi (\mod 2\pi) \).
Define  \( f_1(z) = z \) and  \( f_n(z) = z + \frac{1 - \alpha}{(2n - 1 - \alpha) B_{\lambda}(m, n)} e^{i\theta_n} z^n, \quad n \geq 2, \quad z \in \mathcal{U}. \)

Then  \( f \in \mathcal{VS}_p(\lambda, m, \alpha) \) if and only if  \( f \) can be expressed in the form
\[
f(z) = \sum_{n=1}^\infty \mu_n f_n(z) \quad \text{where} \quad \mu_n \geq 0 \quad \text{and} \quad \sum_{n=1}^\infty \mu_n = 1.
\]

**Proof.** If  \( f(z) = \sum_{n=1}^\infty \mu_n f_n(z) \) with  \( \sum_{n=1}^\infty \mu_n = 1 \) and  \( \mu_n \geq 0 \) then
\[
\sum_{n=2}^\infty \frac{(2n - 1 - \alpha) B_{\lambda}(m, n)}{(2n - 1 - \alpha) B_{\lambda}(m, n)} \mu_n
= \sum_{n=2}^\infty \mu_n (1 - \alpha) = (1 - \mu_1)(1 - \alpha) \leq 1 - \alpha.
\]
Hence \( f \in \mathcal{VS}_p(\lambda, m, \alpha) \).

Conversely, let the function \( f \) defined by (1.1.1) be in the class \( \mathcal{VS}_p(\lambda, m, \alpha) \), since
\[
|a_n| \leq \frac{1 - \alpha}{(2n - 1 - \alpha)B_\lambda(m, n)}, \quad n = 2, 3, \ldots
\]
We may set
\[
\mu_n = \frac{(2n - 1 - \alpha)B_\lambda(m, n)|a_n|}{1 - \alpha}, \quad n \geq 2 \quad \text{and}
\]
\[
\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n. \quad \text{Then } f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z), \quad \text{this completes the proof.}
\]

**Definition 3.2.1.** A function \( f \in \mathcal{V} \) of the form (1.1.1) is in \( \mathcal{VS}_p(\lambda, m, \alpha, \beta) \) if \( f \) satisfies the analytic criteria
\[
\Re \left\{ \frac{z(D_\lambda^m f(z))'}{D_\lambda^m f(z)} \right\} \geq \alpha \left| \frac{z(D_\lambda^m f(z))'}{D_\lambda^m f(z)} - 1 \right| + \beta, \quad (3.2.7)
\]
where \( \alpha \geq 0, \ \beta \geq 0 \) and \( z \in \mathcal{U} \) and \( D_\lambda^m f \) is defined in [Definition 1.2.23].

The proof of the following is similar to that of Theorem 3.2.1 and will be omitted.

**Theorem 3.2.9.** A function \( f \) of the form (1.1.1), is in \( \mathcal{VS}_p(\lambda, m, \alpha, \beta) \) if and only if
\[
\sum_{n=2}^{\infty} E_n B_\lambda(m, n)|a_n| \leq 1 - \beta \quad (3.2.8)
\]
where \( E_n = n(\alpha + 1) - (\alpha + \beta) \).
Similarly we can also prove the above results for function \( f \in \mathcal{V} \) in the class \( \mathcal{VS}_p(\lambda, m, \alpha, \beta) \).

**Theorem 3.2.10.** Let \( f \) of the form (1.1.1) be in the class \( \mathcal{VS}_p(\lambda, m, \alpha, \beta) \).

Then,
\[
\frac{1 - \beta}{E_2B_\lambda(m,2)} r^2 \leq |f(z)| \leq \frac{1 - \beta}{E_2B_\lambda(m,2)} r^2
\]

and
\[
1 - \frac{2(1 - \beta)}{E_2B_\lambda(m,2)} r \leq |f'(z)| \leq 1 + \frac{2(1 - \beta)}{E_2B_\lambda(m,2)} r.
\]

**Theorem 3.2.11.** Let \( f \) of the form (1.1.1) be in the class \( \mathcal{VS}_p(\lambda, m, \alpha, \beta) \), with \( \arg a_n = \theta_n \) where \( \lfloor \theta_n + (n - 1)\beta \rfloor = \pi(\mod 2\pi) \). Define \( f_1(z) = z \) and \( f_n(z) = z + \frac{1 - \beta}{E_nB_\lambda(m,n)} e^{i\theta_n} z^n, \quad n \geq 2, \ z \in \mathcal{U} \).

Then \( f \in \mathcal{VS}_p(\lambda, m, \alpha, \beta) \) if and only if \( f \) expressed in the form
\[
f(z) = \sum_{n=2}^{\infty} \mu_n f_n(z) \quad \text{where} \quad \mu_n \geq 0 \quad \text{and} \quad \sum_{n=2}^{\infty} \mu_n = 1.
\]

**Remark 3.2.12.**

i. By taking \( n = 0, \beta = 0, \lambda = 1 \) these results reduce to the results obtained for the functions \( f \) of the form \( f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \), in the class \( \mathcal{T}S_p(\alpha) \) [76].

ii. If \( \alpha = 0, n = 0, \lambda = 1 \). The above results coincide with the results obtained in [74].

iii. For \( \lambda = 1 \) we get the results of [80].

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