Chapter 6

The Minimum Covering Energy of a Graph

6.1 Introduction

The energy of a graph can be traced back to the year 1930. In 1930, the German scholar Erich Hückel put forward a method for finding approximate solution of the Schrödinger equation of a class of organic molecules, the so-called unsaturated conjugated hydrocarbons [48, 49]. This approach is nowadays referred to as the Hückel molecular orbital (HMO) theory [20, 32].

Within HMO theory, the total energy of $\pi$-electrons is equal to the sum of the energies of all $\pi$-electrons in the considered molecule. It can be calculated from the eigenvalues of the underlying molecular graph [20, 39, 43]. Motivated by HMO total $\pi$-electron energy, Gutman [34] conceived the energy of a graph, defined as

Reference [1] is based on this chapter
the sum of the absolute values of all graph eigenvalues. This definition is by no means restricted to molecular graphs, and enabled one to obtain a remarkable number and variety of novel mathematical results. For further information on the theory of graph energy refer to [12, 23, 24, 38, 41].

In spectral graph theory, the eigenvalues of several matrices like adjacency matrix, Laplacian matrix [44], distance matrix [50], incidence matrix [55] and Randić matrix [11] have been studied and they attracted the attention of many mathematicians. In this Chapter we introduce a new matrix called minimum covering matrix of a graph and study its eigenvalues.

All the graphs considered in this chapter are finite, simple and undirected. Let $G$ be a graph of order $n$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E$. A subset $C$ of $V$ is called a covering set of $G$ if every edge of $G$ is incident to at least one vertex of $C$. Any covering set with minimum cardinality is called a minimum covering set. Let $C$ be a minimum covering set of a graph $G$. The minimum covering matrix of $G$ is the $n \times n$ matrix $A_c(G) = (a_{ij})_{n \times n}$, where

$$a_{ij} = \begin{cases} 
1 & \text{if } v_iv_j \in E \\
1 & \text{if } i = j \text{ and } v_i \in C \\
0 & \text{otherwise}.
\end{cases}$$  

(6.1.1)
The characteristic polynomial of $G$ is the characteristic polynomial of $A_c(G)$ and is denoted by

$$f_n(G, \lambda) := \det(\lambda I - A_c(G)) = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n.$$ 

The minimum covering eigenvalues of the graph $G$ are the eigenvalues of $A_c(G)$. Since $A_c(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The minimum covering energy of $G$ is defined as

$$E_c(G) = \sum_{i=1}^{n} |\lambda_i|.$$ (6.1.2)

In this chapter, we are interested in studying mathematical aspects of the minimum covering energy of a graph. It is possible that the minimum covering energy that we are considering in this chapter may have some applications in chemistry as well as in other areas.

The chapter is organized as follows. In Section 6.3, we discuss some basic properties of minimum covering energy and derive an upper bound and lower bound for $E_c(G)$. In Section 6.4, we compute the minimum covering energies of (i) Star graphs (ii) Complete graphs (iii) Complete bipartite graphs (iv) Crown graphs and (v) Cocktail party graphs.
6.2 A Chemical Connection

The formula (6.1.1) by which the minimum covering matrix is defined, can be viewed also as the definition of the ordinary adjacency matrix of a graph with loops. Indeed, $A_c(G)$ is the adjacency matrix of a graph, obtained from $G$ by attaching a loop of weight $+1$ to each of its vertices belonging to the cover $C$. Graphs with loops are the natural representations of heteroconjugated molecules, and have been much studied in chemical graph theory. In particular, rules for constructing their characteristic polynomials were elaborated in due detail [7, 27, 33, 46, 52, 54, 74, 75, 82, 87]. Loops of weight $+1$ are just the graph representation of nitrogen atoms. The HMO theory of graphs with loops (i.e., molecular graphs of heteroconjugated molecules) were also studied in detail, including the total $\pi$-electron energy [35, 36, 42]. All these results can be directly applied to the presently introduced minimal covering eigenvalues and minimal covering energy. For instance, based on a result from [42], the minimal covering energy, as defined by (6.1.2), can be represented by a Coulson-type integral formula:

$$E_c(G) = \frac{1}{\pi} \int_{-\infty}^{+\infty} 1 \left[ n - \frac{ix f'_n(G, ix)}{f_n(G, ix)} \right] dx,$$  \hspace{1cm} (6.2.1)

where $i = \sqrt{-1}$. This formula holds for any covering set $C$, that is, for any value of $|C|$.
6.3 Some Basic Properties of Minimum Covering Energy

We first compute the minimum covering energy of two graphs, depicted in Figure 6.3.1 and Figure 6.3.2.

Example 6.3.1. Let $G$ be a path on 4 vertices $v_1, v_2, v_3, v_4$ with minimum covering set $C = \{v_1, v_3\}$.

\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

The characteristic polynomial of $A_c(P_4)$ is $\lambda^4 - 2\lambda^3 - 2\lambda^2 + 3\lambda + 1$, minimum covering eigenvalues are $\frac{1}{2} \left(1 - \sqrt{7 + 2\sqrt{5}}\right), \frac{1}{2} \left(1 + \sqrt{7 + 2\sqrt{5}}\right), \frac{1}{2} \left(1 - \sqrt{7 - 2\sqrt{5}}\right), \frac{1}{2} \left(1 + \sqrt{7 - 2\sqrt{5}}\right)$ and the minimum covering energy is $E_c(P_4) = \sqrt{7 + 2\sqrt{5}} + \sqrt{7 - 2\sqrt{5}}$. 
If we take another minimum covering set, namely \( C^* = \{v_2, v_3\} \), then

\[
A_{c^*}(P_4) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]

The Characteristic polynomial of \( A_{c^*}(P_4) \) is \( \lambda^4 - 2\lambda^3 + 2\lambda^2 + 2\lambda + 1 \), the minimum covering eigenvalues are \( 1, -1, 1 - \sqrt{2}, 1 + \sqrt{2} \) and this time the minimum covering energy is equal to \( 2 + 2\sqrt{2} \).

Example 6.3.1 illustrate the fact that the minimum covering energy of a graph \( G \) depends on the choice of the minimum covering set. Therefor the minimum covering energy is not a graph invariant.

**Example 6.3.2.** Let \( G \) be a cycle on 4 vertices \( v_1, v_2, v_3, v_4 \) with minimum covering set \( C = \{v_1, v_3\} \).

![Figure 6.3.2](image-url)
Then

\[ A_c(G) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}. \]

The characteristic polynomial of \( A_c(G) \) is \( \lambda^4 - 2\lambda^3 - 3\lambda^2 + 4\lambda \), minimum covering eigenvalues are 0, 1, \( \frac{1}{2} + \frac{1}{2}\sqrt{17}, \frac{1}{2} - \frac{1}{2}\sqrt{17} \) and the minimum covering energy is \( E_c(G) = 1 + \sqrt{17} \).

**Theorem 6.3.3.** Let \( G \) be a graph with vertex set \( V \), edge set \( E \) and the minimum covering set \( C \). Let \( f_n(G, \lambda) := \det(\lambda I - A_c(G)) = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n \) be the characteristic polynomial of \( G \). Then

(i) \( c_0 = 1 \),

(ii) \( c_1 = -|C| \),

(iii) \( c_2 = \left( \frac{|C|}{2} \right) - |E| \) and

(iv) \( c_3 = |C||E| - \sum_{v \in C} d(v) - \left( \frac{|C|}{3} \right) - 2\Delta \), where \( \Delta \) is the number of triangles in \( G \).

**Proof.** (i) Directly from the definition of \( f_n(G, \lambda) \), it follows that \( c_0 = 1 \).

(ii) Since the sum of diagonal elements of \( A_c(G) \) is equal to \( |C| \), sum of determinants of all \( 1 \times 1 \) principal submatrices of \( A_c(G) \) is the trace of \( A_c(G) \) which evidently equal to \( |C| \). Thus \((-1)^1c_1 = |C|\).
(iii) \((-1)^2 c_2\) is equal to the sum of determinants of all the 2 \(\times\) 2 principal submatrices of \(A_c(G)\), that is

\[
C_2 = \sum_{1 \leq i<j \leq n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} = \sum_{1 \leq i<j \leq n} (a_{ii}a_{jj} - a_{ij}a_{ji})
\]

\[
= \sum_{1 \leq i<j \leq n} a_{ii}a_{jj} - \sum_{1 \leq i<j \leq n} a_{ij}^2 = \left(\frac{|C|}{2}\right) - |E|.
\]

(iv) We have

\[
c_3 = (-1)^3 \sum_{1 \leq i<j<k \leq n} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix}
\]

\[
= - \sum_{1 \leq i<j<k \leq n} a_{ii}[a_{jj}a_{kk} - a_{kj}a_{jk}] - a_{ij}[a_{ji}a_{kk} - a_{ki}a_{jk}] + a_{ik}[a_{ji}a_{kj} - a_{ki}a_{jj}]
\]

\[
= - \sum_{1 \leq i<j<k \leq n} a_{ii}a_{jj}a_{kk} + \sum_{1 \leq i<j<k \leq n} [a_{ii}a_{jk} + a_{jj}a_{ik} + a_{kk}a_{ij}]
\]

\[
- \sum_{1 \leq i<j<k \leq n} a_{ij}a_{jk}a_{ki} - \sum_{1 \leq i<j<k \leq n} a_{ik}a_{kj}a_{ji}
\]

\[
= - \left(\frac{|C|}{3}\right) + \sum_{1 \leq i<j<k \leq n} [a_{ii}a_{jk} + a_{jj}a_{ik} + a_{kk}a_{ij}] - 2\Delta
\]

\[
= - \left(\frac{|C|}{3}\right) + \left(\sum_{i=1}^{n} a_{ii}\right) \left(\sum_{1 \leq j<k \leq n} a_{jk}\right) - \sum_{i=1}^{n} a_{ii} \sum_{k=1}^{n} a_{ik} - 2\Delta.
\]

Thus

\[
c_3 = |C| |E| - \sum_{v \in C} d(v) - \left(\frac{|C|}{3}\right) - 2\Delta.
\]
Theorem 6.3.4. If \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are eigenvalues of \( A_c(G) \), then

\[
\sum_{i=1}^{n} \lambda_i^2 = 2|E| + |C|.
\]

Proof. The sum of squares of the eigenvalues of \( A_c(G) \) is just the trace of \( A_c(G)^2 \).

Therefore

\[
\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}a_{ji} = 2\sum_{i<j} (a_{ij})^2 + \sum_{i=1}^{n} (a_{ii})^2 = 2|E| + |C|.
\]

\( \square \)

Bound for \( E_c(G) \), similar to McClelland’s inequalities [8, 77] for graph energy are given in the following two theorems.

Theorem 6.3.5. (Upper bound) Let \( G \) be a graph with \( n \) vertices, \( m \) edges and \( C \) be a minimum covering set of \( G \). Then

\[
E_c(G) \leq \sqrt{n(2m + |C|)}.
\]

Proof. Let \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \) be the eigenvalues of \( A_c(G) \). Now by
Cauchy-Schwarz inequality we have
\[
\left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right).
\]

Putting \( a_i = 1 \) and \( b_i = |\lambda_i| \) in the above inequality we obtain
\[
E_c(G)^2 = \left( \sum_{i=1}^{n} |\lambda_i| \right)^2 \leq n \left( \sum_{i=1}^{n} |\lambda_i|^2 \right) = n \sum_{i=1}^{n} \lambda_i^2 = n(2m + |C|).
\]

This completes the proof. \(\square\)

**Theorem 6.3.6.** (Lower bound) Let \( G \) be a graph with \( n \) vertices and \( m \) edges and \( C \) be a minimum covering set of \( G \). If \( D = |\text{det}A_c(G)| \) then
\[
E_c(G) \geq \sqrt{2m + |C| + n(n-1)D^2/n}.
\]

**Proof.** We have
\[
[E_c(G)]^2 = \left( \sum_{i=1}^{n} |\lambda_i| \right)^2 = \left( \sum_{i=1}^{n} |\lambda_i| \right) \left( \sum_{j=1}^{n} |\lambda_j| \right)
= \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i \neq j} |\lambda_i||\lambda_j|.
\]

Now by inequality between the arithmetic mean and geometric mean, we have
\[
\frac{1}{n(n-1)} \sum_{i \neq j} |\lambda_i||\lambda_j| \geq \left( \prod_{i \neq j} |\lambda_i||\lambda_j| \right)^{1/n(n-1)}.
\]
Thus

\[ [E_c(G)]^2 \geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1) \left( \prod_{i \neq j} |\lambda_i||\lambda_j| \right) \frac{1}{n(n-1)} \]

\[ \geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1) \left( \prod_{i=1}^{n} |\lambda_i|^{2(n-1)} \right) \frac{1}{n(n-1)} \]

\[ = \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1) \left| \prod_{i=1}^{n} \lambda_i \right|^{\frac{2}{n}} \]

\[ = 2m + |C| + n(n-1)D_n^2. \]

Hence the result. \( \square \)

R. B. Bapat and S. Pati showed that if the graph energy is a rational number, then it is an even integer [9] (see also [81]). The analogous result for minimum covering energy is:

**Theorem 6.3.7.** (Parity theorem) Let \( G \) be a graph with a minimum covering set \( C \). If the minimum covering energy \( E_c(G) \) of \( G \) is a rational number, then

\[ E_c(G) \equiv |C| \pmod{2}. \]

**Proof.** Let \( \lambda_1, \lambda_2, \ldots, \lambda_r \) be positive, and the rest of the minimum covering eigenvalues non-positive. Thus

\[ E_c(G) = \sum_{i=1}^{n} |\lambda_i| = (\lambda_1 + \lambda_2 + \cdots + \lambda_r) - (\lambda_{r+1} + \cdots + \lambda_n). \]
implying

\[ E_c(G) = 2(\lambda_1 + \lambda_2 + \cdots + \lambda_r) - |C|. \]

Since \( \lambda_1, \lambda_2, \ldots, \lambda_r \) are algebraic integers, so is their sum. Hence \((\lambda_1 + \lambda_2 + \cdots + \lambda_r)\) must be an integer if \( E_c(G) \) is rational. Hence the theorem. \( \square \)

6.4 Minimum Covering Energies of Some Families of Graphs

In this section, we compute the minimum covering energy of some well known families of graphs.

Theorem 6.4.1. The minimum covering energy of a star graph \( K_{1,n-1} \) is \( \sqrt{4n - 3} \) for \( n \geq 2 \).

Proof. Let \( K_{1,n-1} \) be a star graph with vertex set \( V = \{v_0, v_1, v_2, \ldots, v_{n-1}\} \), center \( v_0 \) and the minimum covering set \( C = \{v_0\} \). Then

\[
A_c(K_{1,n-1}) = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}_{n \times n}
\]
Then its characteristic polynomial is

\[ f_n(k_{1,n-1}, \lambda) = \begin{vmatrix}
\lambda - 1 & -1 & -1 & \cdots & -1 & -1 \\
-1 & \lambda & 0 & \cdots & 0 & 0 \\
-1 & 0 & \lambda & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & 0 & 0 & \cdots & \lambda & 0 \\
-1 & 0 & 0 & \cdots & 0 & \lambda
\end{vmatrix} \]

\[
= (-1)^{n+2} \begin{vmatrix}
-1 & -1 & \cdots & -1 & -1 \\
\lambda & 0 & \cdots & 0 & 0 \\
0 & \lambda & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \lambda & 0
\end{vmatrix} + (-1)^{2n} \lambda \begin{vmatrix}
\lambda - 1 & -1 & -1 & \cdots & -1 \\
-1 & \lambda & 0 & \cdots & 0 \\
-1 & 0 & \lambda & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & 0 & 0 & \cdots & \lambda
\end{vmatrix}.
\]

From this we get the recurrence relation

\[
f_n(K_{1,n-1}, \lambda) = -\lambda^{n-2} + \lambda f_{n-1}(K_{1,n-2}, \lambda).
\]

(6.4.1)

Changing \( n \) to \( n - 1 \) in (6.4.1), we obtain

\[
f_n(K_{1,n-2}, \lambda) = -\lambda^{n-3} + \lambda f_{n-2}(K_{1,n-3}, \lambda).
\]

(6.4.2)
Employing (6.4.2) in (6.4.1), we deduce that

\[ f_n(K_{1,n-1}, \lambda) = -2\lambda^{n-2} + \lambda^2 f_{n-2}(K_{1,n-3}, \lambda). \]

Continuing this process, we find that

\[
\begin{align*}
    f_n(K_{1,n-1}, \lambda) &= -(n-2)\lambda^{n-2} + \lambda^{n-2} f_2(K_{1,1}, \lambda) \\
    &= -(n-2)\lambda^{n-2} + \lambda^{n-2}(\lambda^2 - \lambda - 1) \\
    &= \lambda^{n-2}[(\lambda^2 - \lambda - (n-1)].
\end{align*}
\]

Therefore, minimum covering eigenvalues are \( \frac{1}{2} \left( 1 + \sqrt{4n-3} \right), \frac{1}{2} \left( 1 - \sqrt{4n-3} \right) \) and \( 0(n-2 \text{ times}) \).

Hence the minimum covering energy is \( E_c(K_{1,n-1}) = \sqrt{4n-3}. \)

\[ \square \]

**Corollary 6.4.2.** Each positive integer \( 2p - 1 \geq 3 \) is the minimum covering energy of a star graph.

**Proof.** From the above theorem, minimum covering eigenvalues of \( K_{1, p^2-p} \) are \( \left[ \frac{1+(2p-1)}{2}, \frac{1-(2p-1)}{2}, 0, 0, \ldots, 0 \right] \). Hence the minimum covering energy \( E_c(K_{1, p^2-1}) = 2p - 1. \)

\[ \square \]
If the minimum covering eigenvalues are ordered by \( \lambda_1 > \lambda_2 > \cdots > \lambda_r \) and their multiplicities are \( m_1, m_2, \ldots, m_r \) respectively, then we shall write

\[
MC\ Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}
\]

OR

\[
MC\ Spec(G) = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_r^{m_r}).
\]

**Theorem 6.4.3.** The minimum covering energy of a complete graph \( K_n \) is 

\[ \sqrt{(n + 3)(n - 1)} \], for \( n \geq 2 \).

**Proof.** Let \( K_n \) be the complete graph with vertex set \( V = \{v_1, v_2, \ldots, v_n\} \), and the minimum covering set \( C = \{v_1, v_2, \ldots, v_{n-1}\} \). Then

\[
A_c(K_n) = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & 1 & \cdots & 1 & 0
\end{pmatrix}_{n \times n}.
\]
Then its characteristic polynomial is

\[ f_n(K_n, \lambda) = \begin{vmatrix}
\lambda - 1 & -1 & -1 & \cdots & -1 & -1 \\
-1 & \lambda - 1 & -1 & \cdots & -1 & -1 \\
-1 & -1 & \lambda - 1 & \cdots & -1 & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -1 & -1 & \cdots & \lambda - 1 & -1 \\
-1 & -1 & -1 & \cdots & -1 & \lambda \\
\end{vmatrix}_{n \times n} \]
\[
\begin{vmatrix}
\lambda - 1 & -1 & \cdots & -1 & -1 \\
-1 & \lambda - 1 & \cdots & -1 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -1 & \cdots & \lambda - 1 & -1 \\
-1 & -1 & \cdots & -1 & -1 \\
\end{vmatrix}
+ (\lambda + 1)
\begin{vmatrix}
\lambda - 1 & -1 & \cdots & -1 & -1 \\
-1 & \lambda - 1 & \cdots & -1 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -1 & \cdots & \lambda - 1 & -1 \\
-1 & -1 & \cdots & -1 & \lambda - 1 \\
\end{vmatrix}
\]

\[
= -\lambda^{n-1} + (\lambda + 1)[\lambda^{n-2}(\lambda - (n - 1))] \\
= \lambda^{n-2}[\lambda^2 - (n - 1)\lambda - (n - 1)].
\]

Hence

\[
MC\ Spec(K_n) = 
\begin{pmatrix}
0 & \frac{n-1}{2} + \frac{\sqrt{(n+3)(n-1)}}{2} & \frac{n-1}{2} - \frac{\sqrt{(n+3)(n-1)}}{2} \\
n - 2 & 1 & 1
\end{pmatrix},
\]

and the minimum covering energy of a complete graph is \( E_c(K_n) \)

\[
= \sqrt{(n + 3)(n - 1)}. \]
Theorem 6.4.4. The minimum covering energy of a complete bipartite graph \( K_{m,n} \) is \((m - 1) + \sqrt{4mn + 1}\). In particular if \( n = m + 1 \) the energy is an integer \(3m\).

Proof. For a complete bipartite graph \( K_{m,n} \) \((m \leq n)\) with vertex set \( V = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}\), \( C = \{u_1, u_2, \ldots, u_m\} \) is a minimum covering set. We have

\[
A_c(K_{m,n}) = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\
0 & 0 & 1 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}_{(m+n)\times(m+n)}.
\]
Its characteristic polynomial is

\[
f_{m+n}(K_{m,n}; \lambda) = \begin{vmatrix}
\lambda - 1 & 0 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\
0 & \lambda - 1 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\
0 & 0 & \lambda - 1 & \cdots & 0 & -1 & -1 & \cdots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda - 1 & -1 & -1 & \cdots & -1 \\
-1 & -1 & -1 & \cdots & -1 & \lambda & 0 & 0 & \cdots & 0 \\
-1 & -1 & -1 & \cdots & -1 & 0 & \lambda & 0 & \cdots & 0 \\
-1 & -1 & -1 & \cdots & -1 & 0 & 0 & \lambda & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \cdots & -1 & 0 & 0 & 0 & \cdots & \lambda \\
\end{vmatrix}
\]

\[
= \begin{vmatrix}
(\lambda - 1)I_m & -J_{m\times n}^T \\
-J_{n\times m} & \lambda I_n \\
\end{vmatrix},
\]

where \( J_{n\times m} \) is a matrix with all entries are equal to 1. We have

\[
\begin{vmatrix}
(\lambda - 1)I_m & -J_{m\times n}^T \\
-J_{n\times m} & \lambda I_n \\
\end{vmatrix} = |(\lambda - 1)I_m| |\lambda I_n - (-J) \frac{I_m}{\lambda - 1} (-J^T)|
\]

\[
= (\lambda - 1)^{m-n} |\lambda (\lambda - 1) I_n - J J^T|
\]

\[
= (\lambda - 1)^{m-n} P_{J J^T}[\lambda (\lambda - 1)]
\]

\[
= (\lambda - 1)^{m-n} P_{mJ_n}[\lambda (\lambda - 1)],
\]

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where $P_{mJ_n}(\lambda)$ is the characteristic polynomial of the matrix $mJ_n$. Therefore

\[
f_{m+n}(K_{m,n}; \lambda) = (\lambda - 1)^{m-n}[\lambda(\lambda - 1) - mn][\lambda(\lambda - 1)]^{n-1} = (\lambda - 1)^{m-n}[\lambda^2 - \lambda - mn]\lambda^{n-1}(\lambda - 1)^{n-1} = (\lambda - 1)^{m-1}\lambda^{n-1}[\lambda^2 - \lambda - mn].\]

So

\[
MC\ Spec(K_{m,n}) = \begin{pmatrix}
0 & 1 & \frac{1}{2} + \frac{\sqrt{4mn+1}}{2} & \frac{1}{2} - \frac{\sqrt{4mn+1}}{2} \\
-1 & m-1 & 1 & 1
\end{pmatrix},
\]

and the minimum covering energy of a complete graph is $E_c(K_{m,n}) = (m - 1) + \sqrt{4mn + 1}$. In particular if $n = m + 1$ the minimum covering energy $E_c(K_{m,m+1}) = m - 1 + 2m + 1 = 3m$. \hfill \Box

**Definition 6.4.5.** The Crown graph $S^0_n$ for an integer $n \geq 3$ is the graph with vertex set \{\(u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\)\} and edge set \{\(u_iv_i : 1 \leq i, j \leq n, i \neq j\)\}. $S^0_n$ is therefore equivalent to the complete bipartite graph $K_{n,n}$ with horizontal edges removed.
Theorem 6.4.6. The minimum covering energy of a Crown graph $S^0_n$ for $n \geq 3$ is $(n - 1)\sqrt{5} + \sqrt{4n - 3}$.

Proof. For a Crown graph $S^0_n$ with vertex set $V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ we choose $C = \{u_1, u_2, \ldots, u_n\}$ as a minimum covering set. Note that

$$A_c(S^0_n) = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\
0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 & \cdots & 1 \\
0 & 0 & 1 & \cdots & 0 & 1 & 1 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 1 & 1 & 1 & \cdots & 0 \\
0 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}_{2n \times 2n}.$$
Its characteristic polynomial is

\[ f_{2n}(S^0_n, \lambda) = \begin{vmatrix}
\lambda - 1 & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\
0 & \lambda - 1 & 0 & \cdots & 0 & -1 & 0 & -1 & \cdots & -1 \\
0 & 0 & \lambda - 1 & \cdots & 0 & -1 & -1 & 0 & \cdots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda - 1 & -1 & -1 & -1 & \cdots & 0 \\
0 & -1 & -1 & \cdots & -1 & \lambda & 0 & 0 & \cdots & 0 \\
-1 & 0 & -1 & \cdots & -1 & 0 & \lambda & 0 & \cdots & 0 \\
-1 & -1 & 0 & \cdots & -1 & 0 & 0 & \lambda & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & \lambda \\
\end{vmatrix}_{2n \times 2n} \]

\[ = \begin{vmatrix}
(\lambda - 1)I_n & -K_n^T \\
-K_n & \lambda I_n \\
\end{vmatrix}. \]
Observe that

\[
f_{2n}(S_n^0, \lambda) = \begin{vmatrix} (\lambda - 1)I_n & -K^T_n \\ -K_n & \lambda I_n \end{vmatrix}
= |(\lambda - 1)I_n||\lambda I_n - (-K_n) \frac{I_n}{\lambda - 1}(-K^T_n)|
= (\lambda - 1)^n|\lambda I_n - \frac{K_nK^T_n}{\lambda - 1}|
= |\lambda(\lambda - 1)I_n - K^2_n|
= P_{K^2_n}[\lambda(\lambda - 1)],
\]

where \(P_{K^2_n}(\lambda)\) is the characteristic polynomial of the matrix \(K^2_n\). Therefore

\[
f_{2n}(S_n^0, \lambda) = [\lambda(\lambda - 1) - 1]^{n-1}[\lambda(\lambda - 1) - (n - 1)^2]
= [\lambda^2 - \lambda - 1]^{n-1}[\lambda^2 - \lambda - (n - 1)^2].
\]

Hence

\[
MC \ Spec(S_n^0) = \begin{pmatrix}
\frac{1}{2} + \frac{\sqrt{5}}{2} & \frac{1}{2} - \frac{\sqrt{5}}{2} & \frac{1}{2} + \frac{\sqrt{4n^2 - 8n + 5}}{2} & \frac{1}{2} - \frac{\sqrt{4n^2 - 8n + 5}}{2} \\
1 & -1 & 1 & -1 \\
\end{pmatrix},
\]

and the minimum covering energy of a complete graph is \(E_c(S_n^0)\)

\[
= (n - 1)\sqrt{5} + \sqrt{4n^2 - 8n + 5}.
\]
Definition 6.4.7. The cocktail party graph, denoted by $K_{n \times 2}$, is graph having vertex set $V = \bigcup_{i=1}^{n} \{u_i, v_i\}$ and edge set $E = \{u_iu_j, v_iv_j, u_iv_j, v_iu_j : 1 \leq i < j \leq n\}$. This graph is also called as complete $n$-partite graph.

Theorem 6.4.8. The minimum covering energy of a Cocktail party graph $K_{n \times 2}$ is $(2n - 3) + \sqrt{4n^2 + 4n - 7}$.

Proof. Let $K_{n \times 2}$ be a Cocktail party graph with a vertex set $V = \bigcup_{i=1}^{n} \{u_i, v_i\}$ and the minimum covering set $C = \bigcup_{i=1}^{n-1} \{u_i, v_i\}$. Then

$$A_c(K_{n \times 2}) = \begin{pmatrix}
1 & 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & \cdots & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & \cdots & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & 1 & \cdots & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & \cdots & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0
\end{pmatrix}_{2n \times 2n}.$$
Its characteristic polynomial is

\[ f_{2n}(K_{n \times 2, \lambda}) = \begin{vmatrix} 
\lambda - 1 & 0 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\
0 & \lambda - 1 & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\
-1 & -1 & \lambda - 1 & 0 & \cdots & -1 & -1 & -1 & -1 \\
-1 & -1 & 0 & \lambda - 1 & \cdots & -1 & -1 & -1 & -1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
-1 & -1 & -1 & -1 & \cdots & \lambda - 1 & 0 & -1 & -1 \\
-1 & -1 & -1 & -1 & \cdots & 0 & \lambda - 1 & -1 & -1 \\
-1 & -1 & -1 & -1 & \cdots & -1 & -1 & \lambda & 0 \\
-1 & -1 & -1 & -1 & \cdots & -1 & -1 & 0 & \lambda \\
\end{vmatrix}_{2n \times 2n} \]

\[ = \begin{vmatrix} M & P^T \\
P & \lambda I_2 \end{vmatrix}_{2n \times 2n}, \]
where

\[ M = \begin{pmatrix} \lambda - 1 & 0 & -1 & -1 & \cdots & -1 & -1 \\ 0 & \lambda - 1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \lambda - 1 & 0 & \cdots & -1 & -1 \\ -1 & -1 & 0 & \lambda - 1 & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & \cdots & \lambda - 1 & 0 \\ -1 & -1 & -1 & -1 & \cdots & 0 & \lambda - 1 \end{pmatrix} \quad (2n-2) \times (2n-2) \]

and

\[ P = \begin{pmatrix} -1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & -1 & -1 & \cdots & -1 & -1 \end{pmatrix} \quad 2 \times (2n-2). \]

Note that

\[
 f_{2n}(K_{n \times 2, \lambda}) = \begin{vmatrix} M & P^T \\ P & \lambda I_2 \end{vmatrix} = |M| |\lambda I_2 - PM^{-1}P^T| \\

= (\lambda - 2n + 3)(\lambda + 1)^{n-2}(\lambda - 1)^{n-1}|\lambda I_2 - PM^{-1}P^T| \\

= (\lambda - 2n + 3)(\lambda + 1)^{n-2}(\lambda - 1)^{n-1} \begin{vmatrix} \lambda - \frac{(2n-2)}{\lambda - 2n+3} & -\frac{(2n-2)}{\lambda - 2n+3} \\ -\frac{(2n-2)}{\lambda - 2n+3} & \lambda - \frac{(2n-2)}{\lambda - 2n+3} \end{vmatrix} \\

= \lambda(\lambda + 1)^{n-2}(\lambda - 1)^{n-1}[\lambda^2 - (2n - 3)\lambda - 4(n - 1)].
\]
Therefore

\[
MC \ Spec(K_{n \times 2}) = \begin{pmatrix}
0 & -1 & 1 & \frac{2n-3}{2} + \frac{\sqrt{4n^2 + 4n - 7}}{2} & \frac{2n-3}{2} - \frac{\sqrt{4n^2 + 4n - 7}}{2} \\
1 & n - 2 & n - 1 & 1 & 1
\end{pmatrix},
\]

and the minimum covering energy of a complete graph is \( E_c(K_{n \times 2}) \)

\[= (2n - 3) + \sqrt{4n^2 + 4n - 7}. \]