Chapter 3

Fractional integral inequalities for Chebeyshev functional

3.1 Introduction

In this chapter we establish some fractional inequalities for Chebey-shev functional using Hadamard fractional integral.

The content of this chapter is published in the following papers.


Consider the functional (1.1.7), where $f$ and $g$ are two integrable functions which are synchronous on $[a, b]$, (i.e satisfies the equation (1.2.12)) given in [19]. Many researchers have studied (1.1.7) and related inequalities, see [4, 13, 46, 67, 69, 87].

In [13], authors have established the inequality for Chebyshev functional by using Riemann-Liouville fractional integral. Moreover author proved the generalized form for extended Chebyshev functional. Some authors have studied fractional calculus, see [58, 77, 98, 107]. Recently many authors have studied fractional integral inequalities using Riemann-Liouville, Caputo derivative and fractional $q$-integral, [3, 4, 5, 6, 7, 13, 14, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 56, 69, 75, 82, 101, 103, 105, 111, 114, 119]. In literature few results have been obtained on some fractional integral inequalities using Hadamard fractional integral [20, 21, 22, 23, 24, 27, 81, 108, 109].
3.2 Inequalities for Chebyshev functional

Now, we prove the first inequality for Chebyshev functional using Hadamard fractional integral

**Theorem 3.2.1.** Let \(f\) and \(g\) be two synchronous function on \([0, \infty[\).

Then for all \(t > 0, \alpha > 0\), we have

\[
H D_{1,t}^{-\alpha} (fg)(t) \geq \frac{\Gamma(\alpha + 1)}{(\ln t)^{\alpha}} (H D_{1,t}^{-\alpha} f(t))(H D_{1,t}^{-\alpha} g(t)).
\]  

(3.2.1)

**Proof** :- Since \(f\) and \(g\) are synchronous on \([0, \infty[\), for all \(\tau \geq 0, \rho \geq 0\), we have

\[
(f(\tau) - f(\rho))(g(\tau) - g(\rho)) \geq 0,
\]

(3.2.2)

From (3.2.2)

\[
f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau),
\]

(3.2.3)

Multiplying both sides of (3.2.3) by \(\frac{(\ln(\tau^{\alpha}))^{\alpha - 1}}{\tau^{\alpha}}\), which is positive because \(\tau \in (0, t), t > 0\). Then integrating the resulting identity with respect
to \( \tau \) over \((1,t)\), we have
\[
\frac{1}{\Gamma(\alpha)} \int_{1}^{t} \ln\left(\frac{t}{\tau}\right)^{\alpha-1} f(\tau) g(\tau) \frac{d\tau}{\tau} + \frac{1}{\Gamma(\alpha)} \int_{1}^{t} \ln\left(\frac{t}{\tau}\right)^{\alpha-1} f(\rho) g(\rho) \frac{d\tau}{\tau} \\
\geq \frac{1}{\Gamma(\alpha)} \int_{1}^{t} \ln\left(\frac{t}{\tau}\right)^{\alpha-1} f(\tau) g(\tau) \frac{d\tau}{\tau} + \frac{1}{\Gamma(\alpha)} \int_{1}^{t} \ln\left(\frac{t}{\tau}\right)^{\alpha-1} f(\rho) g(\tau) \frac{d\tau}{\tau},
\tag{3.2.4}
\]
consequently
\[
H D_{1,t}^{\alpha} (f g)(t) + f(\rho) g(\rho) \frac{1}{\Gamma(\alpha)} \int_{1}^{t} \ln\left(\frac{t}{\tau}\right)^{\alpha-1} \frac{d\tau}{\tau} \\
\geq \frac{g(\rho)}{\Gamma(\alpha)} \int_{1}^{t} \ln\left(\frac{t}{\tau}\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau} + \frac{f(\rho)}{\Gamma(\alpha)} \int_{1}^{t} \ln\left(\frac{t}{\tau}\right)^{\alpha-1} g(\tau) \frac{d\tau}{\tau}, \tag{3.2.5}
\]
we get
\[
H D_{1,t}^{-\alpha} (f g)(t) + f(\rho) g(\rho) H D_{1,t}^{-\alpha} (1) \geq g(\rho) H D_{1,t}^{-\alpha} f(t) + f(\rho) H D_{1,t}^{-\alpha} g(t). \tag{3.2.6}
\]

Multiplying both sides of (3.2.6) by \(\frac{\ln\left(\frac{t}{\rho}\right)}{\rho \Gamma(\alpha)}\), which is positive because \(\rho \in (0,t), \, t > 0\). Then integrating the resulting identity with respect to \(\rho\) over \((1,t)\), we obtain
\[
\frac{H D_{1,t}^{-\alpha} (f g)(t)}{\Gamma(\alpha)} \int_{1}^{t} \ln\left(\frac{t}{\rho}\right)^{\alpha-1} \frac{d\rho}{\rho} + \frac{H D_{1,t}^{-\alpha} (1)}{\Gamma(\alpha)} \int_{1}^{t} \ln\left(\frac{t}{\tau}\right)^{\alpha-1} f(\rho) g(\rho) \frac{d\rho}{\rho} \\
\geq \frac{H D_{1,t}^{-\alpha} f(t)}{\Gamma(\alpha)} \int_{1}^{t} \ln\left(\frac{t}{\rho}\right)^{\alpha-1} g(\rho) \frac{d\rho}{\rho} + \frac{H D_{1,t}^{-\alpha} g(t)}{\Gamma(\alpha)} \int_{1}^{t} \ln\left(\frac{t}{\rho}\right)^{\alpha-1} f(\rho) \frac{d\rho}{\rho}, \tag{3.2.7}
\]
hence

\[ \text{\( H D_{1,t}^{-\alpha} (fg)(t) + H D_{1,t}^{-\alpha} (1) H D_{1,t}^{-\alpha} (fg)(t) \)} \]

\[ \geq \text{\( H D_{1,t}^{-\alpha} (f(t)H D_{1,t}^{-\alpha} (g(t)) + H D_{1,t}^{-\alpha} (g(t)H D_{1,t}^{-\alpha} (f(t), \quad (3.2.8) \)} \]

we get

\[ \text{\( H D_{1,t}^{-\alpha} (1) [2H D_{1,t}^{-\alpha} (fg)(t)] \geq 2H D_{1,t}^{-\alpha} (f(t)H D_{1,t}^{-\alpha} (g(t)). \quad (3.2.9) \)} \]

This completes the proof of theorem 3.2.1. \( \square \)

Now, we prove our second result.

**Theorem 3.2.2.** Let \( f \) and \( g \) be two synchronous functions on \([0, \infty[\),

then for all \( t > 0, \alpha > 0, \beta > 0 \) we have

\[ \frac{(\ln t)^\beta}{\Gamma(\beta + 1)} D_{1,t}^{-\alpha} (fg)(t) + \frac{(\ln t)^\alpha}{\Gamma(\alpha + 1)} D_{1,t}^{-\beta} (fg)(t) \]

\[ \geq \text{\( H D_{1,t}^{-\alpha} (f(t)H D_{1,t}^{-\beta} (g(t)) + H D_{1,t}^{-\alpha} (g(t)H D_{1,t}^{-\beta} (f(t). \quad (3.2.10) \)} \]

**Proof :-** To prove above theorem multiplying equation (3.2.6) by \( \frac{(\ln t)^\beta}{\Gamma(\beta)} \).
which is positive because $\rho \in (0, t)$, $t > 0$. Then integrating the resulting identity with respective $\rho$ over 1 to $t$, we obtain

$$
\frac{h D_{1,t}^{-\alpha} (fg)(t)}{\Gamma(\beta)} \int_1^t \left( \ln \left( \frac{t}{\rho} \right) \right)^{\beta-1} \frac{d\rho}{\rho} + \frac{h D_{1,t}^{-\alpha}(1)}{\Gamma(\beta)} \int_1^t \left( \ln \left( \frac{t}{\rho} \right) \right)^{\beta-1} f(\rho) g(\rho) \frac{d\rho}{\rho} \\
\geq \frac{h D_{1,t}^{-\alpha} f(t)}{\Gamma(\beta)} \int_t^1 \left( \ln \left( \frac{t}{\rho} \right) \right)^{\beta-1} g(\rho) \frac{d\rho}{\rho} + \frac{h D_{1,t}^{-\alpha} g(t)}{\Gamma(\beta)} \int_t^1 \left( \ln \left( \frac{t}{\rho} \right) \right)^{\beta-1} f(\rho) \frac{d\rho}{\rho},
$$

(3.2.11)

and this completes the proof of theorem 3.2.2.

\[\square\]

**Remark 3.2.1.** Applying theorem 3.2.2 for $\alpha = \beta$, we obtain theorem 3.2.1.

**Theorem 3.2.3.** Let $(f_i)_{i=1,2,...,n}$ be positive increasing function on $[0, \infty[$, then for all $t > 0$, $\alpha > 0$, we have

$$
H D_{1,t}^{-\alpha} \left( \prod_{i=1}^{n} f_i \right)(t) \geq \left[ H D_{1,t}^{-\alpha}(1) \right]^{1-n} \prod_{i=1}^{n} H D_{1,t}^{-\alpha} f_i(t).
$$

(3.2.12)

**Proof :-** We prove this theorem by induction. Clearly, for $n = 1$, $H D_{1,t}^{-\alpha} f_1(t) \geq H D_{1,t}^{-\alpha} f_1(t)$, for all $t > 0$, $\alpha > 0$. for $n = 2$, applying equation (3.2.1), we obtain

$$
H D_{1,t}^{-\alpha} (f_1 f_2) \geq \left[ H D_{1,t}^{-\alpha}(1) \right]^{-1} H D_{1,t}^{-\alpha} f_1(t) H D_{1,t}^{-\alpha} f_2(t).
$$

(3.2.13)
Then continuing for \( n > 2 \)

\[
H D_{1,t}^{-\alpha} \left( \prod_{i=1}^{n-1} f_i(t) \right) \geq \left[ H D_{1,t}^{-\alpha}(1) \right]^{-2-n} \prod_{i=1}^{n-1} H D_{1,t}^{-\alpha} f_i(t) \quad \text{for all } t > 0, \quad \alpha > 0.
\]

(3.2.14)

Now, since \((f_i)_{i=1,2,...,n}\) are positive increasing function, then \( (\prod_{i=1}^{n-1} f_i)(t) \) is an increasing function, applying theorem 3.2.1 to the function \( \prod_{i=1}^{n-1} f_i = g, \ f_n = f \), we obtain

\[
H D_{1,t}^{-\alpha} \prod_{i=1}^{n} (f_i)(t) \geq \left( H D_{1,t}^{-\alpha} \prod_{i=1}^{n-1} f_i f_n \right)(t) \geq_H D_{1,t}^{-\alpha}(g f)(t)
\]

\[
\geq \left[ H D_{1,t}^{-\alpha}(1) \right]^{-1} D_{1,t}^{-\alpha} g(t) H D_{1,t}^{-\alpha} f(t)
\]

\[
\geq \left[ H D_{1,t}^{-\alpha}(1) \right]^{-1} D_{1,t}^{-\alpha} (\prod_{i=1}^{n-1} f_i)(t) H D_{1,t}^{-\alpha} f_n
\]

\[
\geq \left[ H D_{1,t}^{-\alpha}(1) \right]^{-1} \left[ H D_{1,t}^{-\alpha}(1) \right]^{2-n} (\prod_{i=1}^{n-1} H D_{1,t}^{-\alpha} f_i)(t) H D_{1,t}^{-\alpha} f_n
\]

\[
\geq \left[ H D_{1,t}^{-\alpha}(1) \right]^{1-n} \prod_{i=1}^{n} H D_{1,t}^{-\alpha} f_i(t). \tag{3.2.15}
\]

This completes the proof of theorem 3.2.3.

\[\square\]

### 3.3 Inequalities for extended Chebyshev functional

Now, we prove lemma which will be useful in proving further results.
Lemma 3.3.1. Let $f$ and $g$ be two synchronous functions on $[0, \infty[$, and $x, y : [0, \infty) \to [0, \infty)$. Then for all $t > 0$, $\alpha > 0$, we have

\[
H D_{1,t}^\alpha x(t) H D_{1,t}^\alpha (yfg)(t) + H D_{1,t}^\alpha y(t) H D_{1,t}^\alpha (xfg)(t) \geq \\
H D_{1,t}^\alpha (xf)(t) H D_{1,t}^\alpha (yhg)(t) + H D_{1,t}^\alpha (yf)(t) H D_{1,t}^\alpha (xg)(t).
\]

(3.3.1)

Proof :- Since $f$ and $g$ are synchronous on $[0, \infty[,$ for all $\tau \geq 0$, $\rho \geq 0$, we have

\[
(f(\tau) - f(\rho))(g(\tau) - g(\rho)) \geq 0.
\]

(3.3.2)

From (3.3.2), we have

\[
f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau).
\]

(3.3.3)

Now, multiplying both sides of (3.3.3) by $\frac{(\ln(\frac{t}{\tau}))^{\alpha-1}x(\tau)}{\tau^{\Gamma(\alpha)}}$, $\tau \in (0, t)$, $t > 0$. Then the integrating the resulting identity with respect to $\tau$ from 1 to $t$ we have

\[
\frac{1}{\Gamma(\alpha)} \int_1^t \ln(\frac{t}{\tau})^{\alpha-1}x(\tau)f(\tau)g(\tau) \frac{d\tau}{\tau} + \frac{1}{\Gamma(\alpha)} \int_1^t \ln(\frac{t}{\tau})^{\alpha-1}x(\tau)f(\rho)g(\rho) \frac{d\tau}{\tau} \geq \\
\frac{1}{\Gamma(\alpha)} \int_1^t \ln(\frac{t}{\tau})^{\alpha-1}x(\tau)f(\tau)g(\rho) \frac{d\tau}{\tau} + \frac{1}{\Gamma(\alpha)} \int_1^t \ln(\frac{t}{\tau})^{\alpha-1}x(\tau)f(\rho)g(\tau) \frac{d\tau}{\tau},
\]

(3.3.4)

consequently

\[
H D_{1,t}^\alpha (xfg)(t) + f(\rho)g(\rho)H D_{1,t}^\alpha (x)(t)
\]
\[ \geq g(\rho) H D_{1,t}^{-\alpha}(xf)(t) + f(\rho) H D_{1,t}^{-\alpha}(xg)(t). \] (3.3.5)

Multiplying both sides of (3.3.5) by \( \frac{(\ln(t/\rho))^{\alpha-1} y(\rho)}{\rho^{\alpha}} \), \( \rho \in (0, t) \), \( t > 0 \). Then integrating the resulting identity with respect to \(\rho\) from 1 to \(t\), we obtain

\[
H D_{1,t}^{-\alpha}(xfg)(t) \frac{1}{\Gamma(\alpha)} \int_1^t \ln\left(\frac{t}{\rho}\right)^{\alpha-1} y(\rho) d\rho \\
+ H D_{1,t}^{-\alpha}(x)(t) \frac{1}{\Gamma(\alpha)} \int_1^t \ln\left(\frac{t}{\rho}\right)^{\alpha-1} y(\rho) f(\rho) g(\rho) d\rho \\
\geq H D_{1,t}^{-\alpha}(xf)(t) \frac{1}{\Gamma(\alpha)} \int_1^t \ln\left(\frac{t}{\rho}\right)^{\alpha-1} y(\rho) g(\rho) d\rho \\
+ H D_{1,t}^{-\alpha}(xg)(t) \frac{1}{\Gamma(\alpha)} \int_1^t \ln\left(\frac{t}{\rho}\right)^{\alpha-1} y(\rho) f(\rho) d\rho. \quad (3.3.6)
\]

This completes the proof of inequality 3.3.1. \(\square\)

Now, we give our main result.

**Theorem 3.3.2.** Let \(f\) and \(g\) be two synchronous function on \([0, \infty[\), and \(r, p, q : [0, \infty) \to [0, \infty)\). Then for all \(t > 0\), \(\alpha > 0\), we have

\[
2H D_{1,t}^{-\alpha} r(t) \left[ H D_{1,t}^{-\alpha} p(t) H D_{1,t}^{-\alpha}(qfg)(t) + H D_{1,t}^{-\alpha} q(t) H D_{1,t}^{-\alpha}(pfg)(t) \right] + \\
2H D_{1,t}^{-\alpha} p(t) H D_{1,t}^{-\alpha} q(t) H D_{1,t}^{-\alpha}(rfg)(t) \geq \\
H D_{1,t}^{-\alpha} r(t) \left[ H D_{1,t}^{-\alpha} (pf)(t) H D_{1,t}^{-\alpha}(qg)(t) + H D_{1,t}^{-\alpha} (qf)(t) H D_{1,t}^{-\alpha}(pg)(t) \right] + \\
H D_{1,t}^{-\alpha} p(t) \left[ H D_{1,t}^{-\alpha} (rf)(t) H D_{1,t}^{-\alpha}(qg)(t) + H D_{1,t}^{-\alpha} (qf)(t) H D_{1,t}^{-\alpha}(rg)(t) \right] +
\]
\[ H D_{1,t}^{-\alpha} q(t) \left[ H D_{1,t}^{-\alpha} (rf)(t) H D_{1,t}^{-\alpha} (pg)(t) + H D_{1,t}^{-\alpha} (pf)(t) H D_{1,t}^{-\alpha} (rg)(t) \right] \]

(3.3.7)

**Proof:** Now, to prove the theorem, put \( x = p, \ y = q \), in lemma 3.3.1, we have

\[ H D_{1,t}^{-\alpha} p(t) H D_{1,t}^{-\alpha} (qf)(t) + H D_{1,t}^{-\alpha} q(t) H D_{1,t}^{-\alpha} (pf)(t) \geq \]

\[ H D_{1,t}^{-\alpha} (pf)(t) H D_{1,t}^{-\alpha} (qg)(t) + H D_{1,t}^{-\alpha} (qf)(t) H D_{1,t}^{-\alpha} (pg)(t). \]  

(3.3.8)

Now, multiplying both sides of (3.3.8) by \( H D_{1,t}^{-\alpha} r(t) \), we have

\[ H D_{1,t}^{-\alpha} r(t) \left[ H D_{1,t}^{-\alpha} p(t) H D_{1,t}^{-\alpha} (qf)(t) + H D_{1,t}^{-\alpha} q(t) H D_{1,t}^{-\alpha} (pf)(t) \right] \geq \]

\[ H D_{1,t}^{-\alpha} r(t) \left[ H D_{1,t}^{-\alpha} (pf)(t) H D_{1,t}^{-\alpha} (qg)(t) + H D_{1,t}^{-\alpha} (qf)(t) H D_{1,t}^{-\alpha} (pg)(t) \right], \]  

(3.3.9)

similarly put \( x = r, \ y = q \), in lemma 3.3.1, we have

\[ H D_{1,t}^{-\alpha} r(t) H D_{1,t}^{-\alpha} (qf)(t) + H D_{1,t}^{-\alpha} q(t) H D_{1,t}^{-\alpha} (rf)(t) \geq \]

\[ H D_{1,t}^{-\alpha} (rf)(t) H D_{1,t}^{-\alpha} (qg)(t) + H D_{1,t}^{-\alpha} (qf)(t) H D_{1,t}^{-\alpha} (rg)(t), \]  

(3.3.10)

multiplying both sides of (3.3.10) by \( H D_{1,t}^{-\alpha} p(t) \), we have

\[ H D_{1,t}^{-\alpha} p(t) \left[ H D_{1,t}^{-\alpha} r(t) H D_{1,t}^{-\alpha} (qf)(t) + H D_{1,t}^{-\alpha} q(t) H D_{1,t}^{-\alpha} (rf)(t) \right] \geq \]
\[ H D_{1,t}^{-\alpha} p(t) \left[ H D_{1,t}^{-\alpha} (rf)(t) H D_{1,t}^{-\alpha} (qg)(t) + H D_{1,t}^{-\alpha} (qf)(t) H D_{1,t}^{-\alpha} (rg)(t) \right]. \] (3.3.11)

Similarly, we have
\[ H D_{1,t}^{-\alpha} q(t) \left[ H D_{1,t}^{-\alpha} pf(t) H D_{1,t}^{-\alpha} (pfg)(t) + H D_{1,t}^{-\alpha} p(t) H D_{1,t}^{-\alpha} (rfg)(t) \right] \geq \]
\[ H D_{1,t}^{-\alpha} (rf)(t) H D_{1,t}^{-\alpha} (pg)(t) + H D_{1,t}^{-\alpha} (pf)(t) H D_{1,t}^{-\alpha} (rg)(t). \] (3.3.12)

Adding the inequalities (3.3.9), (3.3.11) and (3.3.12), we get required inequality (3.3.7).

\textbf{Lemma 3.3.3.} Let \( f \) and \( g \) be two synchronous functions on \([0, \infty[\), and \( x, y : [0, \infty[ \rightarrow [0, \infty[\). Then for all \( t > 0, \alpha > 0 \), we have
\[
H D_{1,t}^{-\alpha} x(t) H D_{1,t}^{-\beta} (yfg)(t) + H D_{1,t}^{-\beta} y(t) H D_{1,t}^{-\alpha} (xfg)(t) \geq
H D_{1,t}^{-\alpha} (xf)(t) H D_{1,t}^{-\beta} (yg)(t) + H D_{1,t}^{-\beta} (yf)(t) H D_{1,t}^{-\alpha} (xg)(t). \] (3.3.13)

\textbf{Proof} :- Now, multiplying both sides of (3.3.5) by \( \frac{(\ln(t))^\beta y(\rho)}{\rho \Gamma(\beta)} \), \( \rho \in (0, t), t > 0 \), we obtain
\[
\frac{(\ln(t))^\beta y(\rho)}{\rho \Gamma(\beta)} H D_{1,t}^{-\alpha} (xf)(t) + \frac{(\ln(t))^\beta y(\rho)}{\rho \Gamma(\beta)} f(\rho)g(\rho) H D_{1,t}^{-\alpha} x(t) \geq
\frac{(\ln(t))^\beta y(\rho)}{\rho \Gamma(\beta)} g(\rho) H D_{1,t}^{-\alpha} (xf)(t) + \frac{(\ln(t))^\beta y(\rho)}{\rho \Gamma(\beta)} f(\rho) H D_{1,t}^{-\alpha} (xg)(t). \] (3.3.14)
Now integrating (3.3.14) from 1 to \( t \), we obtain

\[
H D_{1,t}^{-\alpha}(x f g)(t) \frac{1}{\Gamma(\beta)} \int_1^t \ln \left( \frac{t}{\rho} \right)^{\beta-1} y(\rho) \frac{d\rho}{\rho} +
H D_{1,t}^{-\alpha}(x)(t) \frac{1}{\Gamma(\beta)} \int_1^t \ln \left( \frac{t}{\rho} \right)^{\beta-1} y(\rho) f(\rho) g(\rho) \frac{d\rho}{\rho}
\geq H D_{1,t}^{-\alpha}(x f)(t) \frac{1}{\Gamma(\beta)} \int_1^t \ln \left( \frac{t}{\rho} \right)^{\beta-1} y(\rho) g(\rho) \frac{d\rho}{\rho}
+ H D_{1,t}^{-\alpha}(x g)(t) \frac{1}{\Gamma(\beta)} \int_1^t \ln \left( \frac{t}{\rho} \right)^{\beta-1} y(\rho) f(\rho) \frac{d\rho}{\rho}.
\] (3.3.15)

This completes the proof. \( \square \)

**Theorem 3.3.4.** Let \( f \) and \( g \) be two synchronous functions on \([0, \infty[, \) and \( r, p, q : [0, \infty) \to [0, \infty). \) Then for all \( t > 0, \alpha > 0, \) we have

\[
H D_{1,t}^{-\alpha} r(t) \{ H D_{1,t}^{-\alpha} q(t) H D_{1,t}^{-\beta} (p f g)(t) + 2 H D_{1,t}^{-\alpha} p(t) H D_{1,t}^{-\beta} (q f g)(t)
+ H D_{1,t}^{-\beta} q(t) H D_{1,t}^{-\alpha} (p f g)(t) \}
+ \left[ H D_{1,t}^{-\alpha} p(t) H D_{1,t}^{-\beta} q(t) + H D_{1,t}^{-\beta} p(t) H D_{1,t}^{-\alpha} q(t) \right] H D_{1,t}^{-\alpha} (r f g)(t) \geq
H D_{1,t}^{-\alpha} r(t) \left[ H D_{1,t}^{-\alpha} (p f)(t) H D_{1,t}^{-\beta} (q g)(t) + H D_{1,t}^{-\beta} (q f)(t) H D_{1,t}^{-\alpha} (p g)(t) \right] +
H D_{1,t}^{-\alpha} p(t) \left[ H D_{1,t}^{-\alpha} (r f)(t) H D_{1,t}^{-\beta} (q g)(t) + H D_{1,t}^{-\beta} (q f)(t) H D_{1,t}^{-\alpha} (r g)(t) \right] +
H D_{1,t}^{-\alpha} q(t) \left[ H D_{1,t}^{-\alpha} (r f)(t) H D_{1,t}^{-\beta} (p g)(t) + H D_{1,t}^{-\beta} (p f)(t) H D_{1,t}^{-\alpha} (r g)(t) \right].
\] (3.3.16)
**Proof**: Now, to prove the theorem, put \( x = p, \ y = q, \) in lemma 3.3.3, we have

\[
H D_{1,t}^{-\alpha} p(t) H D_{1,t}^{-\beta} (q f g)(t) + H D_{1,t}^{-\beta} q(t) H D_{1,t}^{-\alpha} (p f g)(t) \geq \\
H D_{1,t}^{-\alpha} (p f)(t) H D_{1,t}^{-\beta} (q g)(t) + H D_{1,t}^{-\beta} (q f)(t) H D_{1,t}^{-\alpha} (p g)(t). \tag{3.3.17}
\]

Now, multiplying both sides of (3.3.17) by \( H D_{1,t}^{-\alpha} r(t) \), we have

\[
H D_{1,t}^{-\alpha} r(t) \left[ H D_{1,t}^{-\alpha} p(t) H D_{1,t}^{-\beta} (q f g)(t) + H D_{1,t}^{-\beta} q(t) H D_{1,t}^{-\alpha} (p f g)(t) \right] \geq \\
H D_{1,t}^{-\alpha} r(t) \left[ H D_{1,t}^{-\alpha} (p f)(t) H D_{1,t}^{-\beta} (q g)(t) + H D_{1,t}^{-\beta} (q f)(t) H D_{1,t}^{-\alpha} (p g)(t) \right]. \tag{3.3.18}
\]

Similarly, put \( x = r, \ y = q, \) in lemma 3.3.3, we have

\[
H D_{1,t}^{-\alpha} r(t) H D_{1,t}^{-\beta} (q f g)(t) + H D_{1,t}^{-\beta} q(t) H D_{1,t}^{-\alpha} (r f g)(t) \geq \\
H D_{1,t}^{-\alpha} (r f)(t) H D_{1,t}^{-\beta} (q g)(t) + H D_{1,t}^{-\beta} (q f)(t) H D_{1,t}^{-\alpha} (r g)(t), \tag{3.3.19}
\]

multiplying both sides of (3.3.19) by \( H D_{1,t}^{-\alpha} p(t) \), we have

\[
H D_{1,t}^{-\alpha} p(t) \left[ H D_{1,t}^{-\alpha} r(t) H D_{1,t}^{-\beta} (q f g)(t) + H D_{1,t}^{-\beta} q(t) H D_{1,t}^{-\alpha} (r f g)(t) \right] \geq \\
H D_{1,t}^{-\alpha} p(t) \left[ H D_{1,t}^{-\alpha} (r f)(t) H D_{1,t}^{-\beta} (q g)(t) + H D_{1,t}^{-\beta} (q f)(t) H D_{1,t}^{-\alpha} (r g)(t) \right]. \tag{3.3.20}
\]
Similarly, we have
\[
H D_{1,t}^{-\alpha} q(t) \left[ H D_{1,t}^{-\alpha} r(t) H D_{1,t}^{-\beta} (p f g)(t) + H D_{1,t}^{-\beta} p(t) H D_{1,t}^{-\alpha} (r f g)(t) \right] \\
H D_{1,t}^{-\alpha} q(t) \left[ H D_{1,t}^{-\alpha} (r f)(t) H D_{1,t}^{-\beta} (p g)(t) + H D_{1,t}^{-\beta} (p f)(t) H D_{1,t}^{-\alpha} (r g)(t) \right].
\]
(3.3.21)

Adding the inequalities (3.3.18), (3.3.20) and (3.3.21), we get the inequality (3.3.16). This complete the proof. \[\square\]

**Remark 3.3.1.** If \( f, g, r, p \) and \( q \) satisfies the following condition,

1. The function \( f \) and \( g \) is asynchronous on \([0, \infty)\).
2. The function \( r, p, q \) are negative on \([0, \infty)\).
3. Two of the function \( r, p, q \) are positive and the third is negative on \([0, \infty)\).

then the inequality 3.3.7 and 3.3.16 are reversed.

Also, for \( \alpha = \beta \), in theorem 3.3.4 , we obtain theorem 3.3.2.