Chapter 6

Minkowski-type fractional integral inequalities using Saigo fractional integral

6.1 Introduction

In this chapter we establish some Minkowski type fractional inequality. The celebrated Minkowski inequality is important classical inequality in mathematics. It can be found in many books on real functions, analysis, functional analysis or $L_p$-spaces.

The content of this chapter is published in the following papers.

i) New Fractional Inequalities Involving Saigo Fractional Integral Operator, Math. Sci. Lett., 3(3), 2014, 133-139. Due to usefulness in analysis and its applications,
the inequality has received a considerable attention in the past decades and a number of papers have appeared in the literatures which deal with their various generalizations, extensions and applications.

In [7] A. Anber and et al. have studies the fractional integral inequalities (1.3.5) and (1.3.6) using Riemann-Liouville fractional integral. In same paper authors have established the following theorem.
Let \( \alpha > 0 \), and \( f, g \) be two positive function on \([0, \infty[\), such that \( f \) is nondecreasing and \( g \) is non-increasing. Then

\[
J^\alpha f^\gamma(t) g^\delta(t) \leq \frac{\Gamma(\alpha + 1)}{t^\alpha} J^\alpha f^\gamma(t) J^\alpha g^\delta(t),
\]

for any \( t > 0 \), \( \gamma > 0 \), \( \delta > 0 \).

For \( \beta > 0 \),

\[
\frac{t^\beta}{\Gamma(\beta + 1)} J^\alpha (f^\gamma(t) g^\delta(t)) + \frac{t^\alpha}{\Gamma(\alpha + 1)} J^\beta (f^\gamma(t) g^\delta(t)) \leq (J^\alpha f^\gamma(t))(J^\beta g^\delta(t)) + (J^\alpha g^\delta(t))(J^\beta f^\gamma(t)). \tag{6.1.2}
\]

Motivated from [7, 30, 39], our purpose is to establish some new results using Saigo fractional integral.
6.2 Reverse Minkowski fractional integral inequality

In this section, we establish reverse Minkowski fractional integral inequality involving Saigo fractional integer operator (1.2.7).

**Theorem 6.2.1.** Let \( p \geq 1 \) and let \( f, g \) be two positive functions on \([0, \infty)\), such that for all \( x > 0 \), \( I^{\alpha, \beta, \eta}_{0, x}[f^p(x)] < \infty \), \( I^{\alpha, \beta, \eta}_{0, x}[g^q(x)] < \infty \). If \( 0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M \), \( \tau \in (0, x) \), then we have

\[
\left[ I^{\alpha, \beta, \eta}_{0, x}[f^p(x)] \right]^\frac{1}{p} + \left[ I^{\alpha, \beta, \eta}_{0, x}[g^q(x)] \right]^\frac{1}{q} \leq \frac{1 + M(m + 2)}{(m + 1)(M + 1)} \left[ I^{\alpha, \beta, \eta}_{0, x}[(f + g)^p(\tau)] \right]^\frac{1}{p},
\]

(6.2.1)

for any \( \alpha > \max\{0, -\beta\} \), \( \beta < 1 \), \( \beta - 1 < \eta < 0 \).

**Proof :-** Using the condition \( \frac{f(\tau)}{g(\tau)} \leq M \), \( \tau \in (0, x) \), \( x > 0 \), we have

\[
(M + 1)^p f(\tau) \leq M^p (f + g)^p(\tau).
\]

(6.2.2)

Clearly, we can say that the function \( G(x, \tau) \) defined by (1.2.10) which remains positive because for all \( \tau \in (0, x) \), \( x > 0 \) since each term of the (1.2.10) is positive. Multiplying both sides of (6.2.2) by \( G(x, \tau) \), then integrating the resulting identity with respect to \( \tau \) from 0 to \( x \),
we get

\[
(M + 1)^p \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x - \tau)^{\alpha-1} _2F_1(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x}) f^p(\tau) d\tau
\]

\[
\leq \frac{M^p}{\Gamma(\alpha)} \int_0^x (x - \tau)^{\alpha-1} _2F_1(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x}) (f + g)^p(\tau) d\tau,
\]

(6.2.3)

which is equivalent to

\[
I_{0,x}^{\alpha,\beta,\eta}[f^p(x)] \leq \frac{M^p}{(M + 1)^p} \left[ I_{0,x}^{\alpha,\beta,\eta}[(f + g)^p(x)] \right],
\]

(6.2.4)

hence, we can write

\[
\left[ I_{0,x}^{\alpha,\beta,\eta}[f^p(x)] \right]^\frac{1}{p} \leq \frac{M}{(M + 1)} \left[ I_{0,x}^{\alpha,\beta,\eta}[(f + g)^p(x)] \right]^\frac{1}{p}.
\]

(6.2.5)

On other hand, using condition \( m \leq \frac{f(\tau)}{g(\tau)} \), we obtain

\[
(1 + \frac{1}{m}) g(\tau) \leq \frac{1}{m} (f(\tau) + g(\tau)),
\]

(6.2.6)

which implies

\[
(1 + \frac{1}{m})^p g^p(\tau) \leq \left( \frac{1}{m} \right)^p (f(\tau) + g(\tau))^p.
\]

(6.2.7)

Now, multiplying both sides of (6.2.7) by \( G(x, \tau) \), \( (\tau \in (0, x), x > 0) \),

where \( G(x, \tau) \) is defined by (1.2.10). Then integrating the resulting identity with respect to \( \tau \) from 0 to \( x \), we have

\[
\left[ I_{0,x}^{\alpha,\beta,\eta}[g^p(x)] \right]^\frac{1}{p} \leq \frac{1}{(m + 1)} \left[ I_{0,x}^{\alpha,\beta,\eta}[(f + g)^p(x)] \right]^\frac{1}{p}.
\]

(6.2.8)
By adding inequality (6.2.5) and (6.2.8) we get required inequality (6.2.8).

Now we give our second result is as follows.

**Theorem 6.2.2.** Let $p \geq 1$ and $f, g$ be two positive functions on $[0, \infty)$, such that for all $x > 0$, $I_{0,x}^{\alpha,\beta,\eta} [f^p(x)] < \infty$, $I_{0,x}^{\alpha,\beta,\eta} [g^q(x)] < \infty$. If $0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M$, $\tau \in (0, t)$, then we have

$$
\left[ I_{0,x}^{\alpha,\beta,\eta} [f^p(x)] \right]^\frac{1}{p} + \left[ I_{0,x}^{\alpha,\beta,\eta} [g^q(x)] \right]^\frac{1}{p} \geq \left( \frac{M + 1}{M} (m + 1) - 2 \right) \times
\left[ I_{0,x}^{\alpha,\beta,\eta} [f^p(x)] \right]^\frac{1}{p} \left[ I_{0,x}^{\alpha,\beta,\eta} [g^q(x)] \right]^\frac{1}{p}.
$$

(6.2.9)

**Proof :-** Multiplying the inequalities (6.2.5) and (6.2.8), we obtain

$$
\frac{(M + 1)(m + 1)}{M} \left[ I_{0,x}^{\alpha,\beta,\eta} [f^p(x)] \right]^\frac{1}{p} \left[ I_{0,x}^{\alpha,\beta,\eta} [g^q(x)] \right]^\frac{1}{p} \\
\leq \left( \left[ I_{0,x}^{\alpha,\beta,\eta} [(f(x) + g(x))^p(x)] \right] \right)^\frac{1}{p}. \quad (6.2.10)
$$

Applying Minkowski inequalities to the right hand side of (6.2.10), we have

$$
\left( \left[ I_{0,x}^{\alpha,\beta,\eta} [(f(x) + g(x))^p(x)] \right] \right)^\frac{1}{p} \leq \left( \left[ I_{0,x}^{\alpha,\beta,\eta} [f^p(x)] \right] + \left[ I_{0,x}^{\alpha,\beta,\eta} [g^q(x)] \right] \right)^\frac{1}{p}, \quad (6.2.11)
$$
which implies that
\[
\left[ I_{0,x}^{\alpha,\beta,\eta}[(f(x) + g(x))^p] \right]^\frac{2}{p} \leq \left[ I_{0,x}^{\alpha,\beta,\eta}[f^p(x)] \right]^\frac{2}{p} + \left[ I_{0,x}^{\alpha,\beta,\eta}[g^q(x)] \right]^\frac{2}{p} \\
+ 2 \left[ I_{0,x}^{\alpha,\beta,\eta}[f^p(x)] \right]^\frac{1}{p} \left[ I_{0,x}^{\alpha,\beta,\eta}[g^q(x)] \right]^\frac{1}{p}.
\]
(6.2.12)

Using (6.2.10) and (6.2.12), we obtain (6.2.9). This completes proof of theorem.

\[\square\]

### 6.3 Fractional integral inequalities involving Saigo fractional integral operator

In this section, we establish some new integral inequalities involving the Saigo fractional integral operator.

**Theorem 6.3.1.** Let \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \) and \( f, g \) be two positive functions on \([0, \infty)\), such that \( I_{0,x}^{\alpha,\beta,\eta}[f(x)] < \infty \), \( I_{0,x}^{\alpha,\beta,\eta}[g(x)] < \infty \). If \( 0 < m \leq \frac{f(\tau)}{g(\tau)} \leq M < \infty \), \( \tau \in [0, x] \) we have
\[
\left[ I_{0,x}^{\alpha,\beta,\eta}[f(x)] \right]^\frac{1}{p} \left[ I_{0,x}^{\alpha,\beta,\eta}[g(x)] \right]^\frac{1}{q} \leq \left( \frac{M}{m} \right)^\frac{1}{pq} \left[ I_{0,x}^{\alpha,\beta,\eta}[[f(x)]^\frac{1}{p}[g(x)]^\frac{1}{q}] \right],
\]
(6.3.1)

hold. For all \( x > 0, \alpha > \max\{0, -\beta\}, \beta < 1, \beta - 1 < \eta < 0 \).
Proof :- Since \( \frac{f(\tau)}{g(\tau)} \leq M, \tau \in [0, x], x > 0 \), we have

\[ [g(\tau)]^{\frac{1}{p}} \geq M^{-\frac{1}{q}} [f(\tau)]^{\frac{1}{p}}, \tag{6.3.2} \]

and

\[ [f(\tau)]^{\frac{1}{p}} [g(\tau)]^{\frac{1}{q}} \geq M^{\frac{1}{q}} [f(\tau)]^{\frac{1}{p}} \]
\[ \geq M^{\frac{1}{q}} [f(\tau)]^{\frac{1}{p} + \frac{1}{q}} \]
\[ \geq M^{\frac{1}{q}} [f(\tau)]. \tag{6.3.3} \]

Multiplying both sides of (6.3.3) by \( G(x, \tau), \ (\tau \in (0, x), x > 0) \), where \( G(x, \tau) \) which is defined by (1.2.10). Then integrating the resulting identity with respect to \( \tau \) from 0 to \( x \), we have

\[ \frac{x^{-\alpha - \beta}}{\Gamma(\alpha)} \int_0^x (x - \tau)^{\alpha - 1} 2F_1(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x}) f(\tau)^{\frac{1}{p}} g(\tau)^{\frac{1}{q}} d\tau \]
\[ \leq \frac{M^{-\frac{1}{q}} x^{-\alpha - \beta}}{\Gamma(\alpha)} \int_0^x (x - \tau)^{\alpha - 1} 2F_1(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x}) f(\tau) d\tau, \tag{6.3.4} \]

which implies that

\[ I_{0,x}^{\alpha,\beta,\eta} \left( [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} \right) \leq M^{-\frac{1}{q}} I_{0,x}^{\alpha,\beta,\eta} f(x). \tag{6.3.5} \]

Consequently

\[ \left( I_{0,x}^{\alpha,\beta,\eta} \left( [f(x)]^{\frac{1}{p}} [g(x)]^{\frac{1}{q}} \right) \right)^{\frac{1}{p}} \leq M^{\frac{1}{pq}} \left[ I_{0,x}^{\alpha,\beta,\eta} f(x) \right]^{\frac{1}{p}}. \tag{6.3.6} \]
on other hand, since \( mg(\tau) \leq f(\tau), \; \tau \in [0, x), \; x > 0 \), then we have

\[
[f(\tau)]^{\frac{1}{\beta}} \geq m^{\frac{1}{\beta}}[g(\tau)]^{\frac{1}{\beta}},
\]  

(6.3.7)

multiplying equation (6.3.7) by \([g(\tau)]^{\frac{1}{\xi}}\), we have

\[
[f(\tau)]^{\frac{1}{\beta}}[g(\tau)]^{\frac{1}{\xi}} \geq m^{\frac{1}{\beta}}[g(\tau)]^{\frac{1}{\xi}}[g(\tau)]^{\frac{1}{\beta}} = m^{\frac{1}{\beta}}[g(\tau)].
\]

(6.3.8)

Multiplying both sides of (6.3.8) by \( G(x, \tau), \; (\tau \in (0, x), \; x > 0) \), where \( G(x, \tau) \) is defined by (1.2.10). Then integrating the resulting identity with respect to \( \tau \) from 0 to \( x \), we have

\[
\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_{0}^{x} (x - \tau)^{\alpha-1} 2F_{1}(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x}) [f(\tau)^{\frac{1}{\beta}} g(\tau)^{\frac{1}{\xi}}] d\tau
\]

\[
\leq \frac{m^{\frac{1}{\beta}} x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_{0}^{x} (x - \tau)^{\alpha-1} 2F_{1}(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x}) g(\tau) d\tau,
\]

(6.3.9)

we have

\[
I_{0,x}^{\alpha,\beta,\eta} \left[ [f(x)]^{\frac{1}{\beta}} [g(x)]^{\frac{1}{\xi}} \right] \leq m^{\frac{1}{\beta}} I_{0,x}^{\alpha,\beta,\eta} g(x).
\]

(6.3.10)

Hence we can write

\[
\left( I_{0,x}^{\alpha,\beta,\eta} \left[ [f(x)]^{\frac{1}{\beta}} [g(x)]^{\frac{1}{\xi}} \right] \right)^{\frac{1}{q}} \leq m^{\frac{1}{\beta}} I_{0,x}^{\alpha,\beta,\eta} g(x)^{\frac{1}{q}},
\]

(6.3.11)

multiplying equation (6.3.6) and (6.3.11), we get the result (6.3.1). □
**Theorem 6.3.2.** Let $f$ and $g$ be two positive functions on $[0, \infty[$, such that

$$I_{0,x}^{\alpha,\beta,\eta}[f^p(x)] < \infty, \ I_{0,x}^{\alpha,\beta,\eta}[g^q(x)] < \infty, \ x > 0, \ I_{0}^{\alpha,\beta,\eta}[f(x)^p] \leq M < \infty, \ \tau \in [0,x].$$

Then

$$\left[ I_{0,x}^{\alpha,\beta,\eta} f^p(x) \right]^\frac{1}{p} \left[ I_{0,x}^{\alpha,\beta,\eta} g^q(x) \right]^\frac{1}{q} \leq \left( \frac{M}{m} \right)^{\frac{1}{p}} \left[ I_{0,x}^{\alpha,\beta,\eta}(f(x)g(x)) \right], \quad (6.3.12)$$

holds, where $p > 1, \ \frac{1}{p} + \frac{1}{q} = 1$, for all $x > 0, \ \alpha > \max\{0,-\beta\}, \ \beta < 1, \ \beta - 1 < \eta < 0$.

**Proof :-** Replacing $f(\tau)$ and $g(\tau)$ by $f(\tau)^p$ and $g(\tau)^q, \ \tau \in [0,x], \ x > 0$ in Theorem 6.3.1, we obtain (6.3.12). \qed

**Theorem 6.3.3.** Let $f, g$ be two positive functions on $[0, \infty)$, such that $f$ is non-decreasing and $g$ is non-increasing. Then

$$I_{0,x}^{\alpha,\beta,\eta} f^\gamma(x)g^\delta(x) \leq \frac{\Gamma(1 - \beta)\Gamma(1 + \alpha + \eta)}{\Gamma(1 - \beta + \eta)} x^\beta I_{0,x}^{\alpha,\beta,\eta}[f^\gamma(x)] I_{0,x}^{\alpha,\beta,\eta}[g^\delta(x)], \quad (6.3.13)$$

for all $x > 0, \ \alpha > \max\{0,-\beta\}, \ \beta < 1, \ \beta - 1 < \eta < 0, \ \gamma > 0, \ \delta > 0$.

**Proof :-** Let $\tau, \rho \in [0,x], \ x > 0$, for any $\delta > 0, \ \gamma > 0$, we have

$$\left( f^\gamma(\tau) - f^\gamma(\rho) \right) \left( g^\delta(\rho) - g^\delta(\tau) \right) \geq 0. \quad (6.3.14)$$
\[ f^\gamma(\tau)g^\delta(\rho) - f^\gamma(\tau)g^\delta(\tau) - f^\gamma(\rho)(g^\delta(\rho) + f^\gamma(\rho)g^\delta(\tau) \geq 0. \quad (6.3.15) \]

we have

\[ f^\gamma(\tau)g^\delta(\tau) + f^\gamma(\rho)(g^\delta(\rho) \leq f^\gamma(\tau)g^\delta(\rho) + f^\gamma(\rho)g^\delta(\tau). \quad (6.3.16) \]

Now, multiplying both sides of (6.3.16) by \( G(x, \tau), (\tau \in (0, x), x > 0) \), where \( G(x, \tau) \) is defined by (1.2.10). Then integrating the resulting identity with respect to \( \tau \) from 0 to \( x \), we have

\[
\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x - \tau)^{\alpha-1} \frac{f^\gamma(\tau)g^\delta(\tau)}{2F_1(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x})} d\tau + \\
\frac{f^\gamma(\rho)g^\delta(\rho)x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x - \tau)^{\alpha-1} \frac{2F_1(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x})}{1 + f^\gamma(\tau)} d\tau \\
\leq \frac{g^\delta(\rho)x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x - \tau)^{\alpha-1} \frac{2F_1(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x})}{g^\delta(\tau)} d\tau + \\
\frac{f^\gamma(x)x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x - \tau)^{\alpha-1} \frac{2F_1(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x})}{g^\delta(x)} d\tau. \quad (6.3.17) \\
\]

\[
I_{0,x}^{\alpha,\beta,\eta}[f^\gamma(x)g^\delta(x)] + f^\gamma(\rho)(g^\delta(\rho)I_{0,x}^{\alpha,\beta,\eta}[1] \\
\leq g^\delta(\rho)I_{0,x}^{\alpha,\beta,\eta}[f^\gamma(x)] + f^\gamma(\rho)I_{0,x}^{\alpha,\beta,\eta}[g^\delta(x)]. \quad (6.3.18) \\
\]

Again, multiplying both sides of (6.3.18) by \( G(x, \rho), (\rho \in (0, x), x > 0) \), where \( G(x, \rho) \) is defined by (1.2.10). Then integrating the resulting
identity with respect to \( \rho \) from 0 to \( x \), we have
\[
I_{0,x}^{\alpha,\beta,\eta} \left[ f^\gamma(t) g^\delta(x) \right] I_{0,x}^{\alpha,\beta,\eta} [1] + I_{0,x}^{\alpha,\beta,\eta} \left[ f^\gamma(x) g^\delta(x) \right] I_{0,x}^{\alpha,\beta,\eta} [1] \\
\leq I_{0,x}^{\alpha,\beta,\eta} \left[ g^\delta(x) \right] I_{0,x}^{\alpha,\beta,\eta} \left[ f^\gamma(x) \right] + I_{0,x}^{\alpha,\beta,\eta} \left[ f^\gamma(x) \right] I_{0,x}^{\alpha,\beta,\eta} \left[ g^\delta(x) \right],
\]
then we have
\[
2I_{0,x}^{\alpha,\beta,\eta} \left[ f^\gamma(x) g^\delta(x) \right] \leq \frac{1}{\left[ I_{0,x}^{\alpha,\beta,\eta} [1] \right]^{-1}} 2I_{0,x}^{\alpha,\beta,\eta} \left[ f^\gamma(x) \right] I_{0,x}^{\alpha,\beta,\eta} \left[ g^\delta(x) \right].
\]
This gives the required inequality (6.3.13). This completes the proof of theorem.

**Theorem 6.3.4.** Let \( f, g \) be two positive functions on \([0, \infty)\), such that \( f \) is non-decreasing and \( g \) is non-increasing. Then we have
\[
\frac{\Gamma(1 - \beta + \eta)}{\Gamma(1 - \beta) \Gamma(1 + \alpha + \eta) x^\beta} I_{0,x}^{\psi,\phi,\zeta} \left[ f^\gamma(x) g^\delta(x) \right] + \\
\frac{\Gamma(1 - \phi + \zeta)}{\Gamma(1 - \phi) \Gamma(1 + \psi + \zeta) x^\phi} I_{0,x}^{\alpha,\beta,\eta} \left[ f^\gamma(x) g^\delta(x) \right] \\
\leq \left(I_{0,x}^{\alpha,\beta,\eta} \left[ f^\gamma(x) \right]\right) \left( I_{0,x}^{\psi,\phi,\zeta} \left[ g^\delta(x) \right] \right) + \\
\left(I_{0,x}^{\psi,\phi,\zeta} \left[ g^\delta(x) \right] \right) \left( I_{0,x}^{\alpha,\beta,\eta} \left[ f^\gamma(x) \right] \right),
\]
for all \( x > 0, \alpha > \max\{0, -\beta\}, \psi > \max\{0, -\phi\} \beta < 1, \beta - 1 < \eta < 0, \phi < 1, \phi - 1 < \zeta < 0, \gamma > 0 \delta > 0.\)
Proof : - Multiplying both sides of equation (6.3.18) by \( \frac{x^{-\psi-\phi}}{\Gamma(\psi)}(x - \rho)^{\psi-1}2F_1(\psi + \phi, -\zeta; \psi; 1 - \frac{\rho}{x}) \) \((\rho \in (0, x), x > 0)\), which (in view of the argument mentioned above in proof of theorem 6.3.3 ) remains positive. Then integrating the resulting identity with respect to \( \rho \) from 0 to \( x \), we have

\[
I_{0,x}^{\alpha,\beta,\eta}[f^\gamma(x)g^\delta(x)] \frac{x^{-\psi-\phi}}{\Gamma(\psi)} \int_0^x (x - \rho)^{\psi-1}2F_1(\psi + \phi, -\zeta; \psi; 1 - \frac{\rho}{x})[1]d\rho + \\
I_{0,x}^{\alpha,\beta,\eta}[1] \frac{x^{-\psi-\phi}}{\Gamma(\psi)} \int_0^x (x - \rho)^{\psi-1}2F_1(\psi + \phi, -\zeta; \psi; 1 - \frac{\rho}{x})f^\gamma(\rho)g^\delta(\rho)d\rho \\
\leq I_{0,x}^{\alpha,\beta,\eta}[f^\gamma(x)] \frac{x^{-\psi-\phi}}{\Gamma(\psi)} \int_0^x (x - \rho)^{\psi-1}2F_1(\psi + \phi, -\zeta; \psi; 1 - \frac{\rho}{x})g^\delta(\rho)d\rho + \\
I_{0,x}^{\alpha,\beta,\eta}[g^\delta(x)] \frac{x^{-\psi-\phi}}{\Gamma(\psi)} \int_0^x (x - \rho)^{\psi-1}2F_1(\psi + \phi, -\zeta; \psi; 1 - \frac{\rho}{x})f^\gamma(\rho)d\rho, \\
\tag{6.3.20}
\]

which gives the require inequality (6.3.19). This completes proof of theorem.

\[\square\]

Remark 6.3.1. The inequalities (6.3.13) and (6.3.19) are reversed if the functions are

\[
(f^\gamma(\tau) - f^\gamma(\rho)) (g^\delta(\rho) - g^\delta(\tau)) \leq 0.
\]

and if we put for \( \alpha = \psi, \beta = \phi \) and \( \eta = \zeta \), in theorem 6.3.4 reduces to the theorem 6.3.3.
\textbf{Theorem 6.3.5.} Let \( f \geq 0, \ g \geq 0 \) be two functions defined on \([0, \infty)\), such that \( g \) is non-decreasing. If

\[
I_{0,x}^{\alpha, \beta, \eta} f(x) \geq I_{0,x}^{\alpha, \beta, \eta} g(x), \tag{6.3.21}
\]

then for all \( x > 0, \ \alpha > \max\{0, -\beta\}, \ \beta < 1, \ \beta - 1 < \eta < 0, \ \gamma > 0, \ \delta > 0 \) and \( \gamma - \delta > 0 \), we have

\[
I_{0,x}^{\alpha, \beta, \eta} f^{\gamma-\delta}(x) \leq I_{0,x}^{\alpha, \beta, \eta} f^{\eta}(x) g^{-\delta}(x), \tag{6.3.22}
\]

\textbf{Proof :-} From the arithmetic-geometric inequality, for \( \gamma > 0, \ \delta > 0 \), we have

\[
\frac{\gamma}{\gamma - \delta} f^{\gamma-\delta}(\tau) - \frac{\delta}{\gamma - \delta} g^{\gamma-\delta}(\tau) \leq f^{\eta}(\tau) g^{-\delta}(\tau), \ \tau \in (0, x), \ x > 0.
\tag{6.3.23}
\]

Now, multiplying both sides of (6.3.23) by \( G(x, \tau), \ (\tau \in (0, x), \ x > 0) \), where \( G(x, \tau) \) is defined by (1.2.10). Then integrating the resulting identity with respect to \( \tau \) from 0 to \( x \), we have

\[
\frac{\gamma}{\gamma - \delta} \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x - \tau)^{\alpha-1} 2F_1(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x})[f^{\gamma-\delta}(\tau)]d\tau -
\]

\[
\frac{\delta}{\gamma - \delta} \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x - \tau)^{\alpha-1} 2F_1(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x})[g^{\gamma-\delta}(\tau)]d\tau
\]
\[ \leq \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x - \tau)^{\alpha-1} 2F_1(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x}) [f^{\gamma}(\tau)g^{-\delta}(\tau)] d\tau, \]

(6.3.24)

we get

\[ \frac{\gamma}{\gamma - \delta} I_{0,x}^{\alpha,\beta,\eta} [f^{\gamma-\delta}(x)] - \frac{\delta}{\gamma - \delta} I_{0,x}^{\alpha,\beta,\eta} [g^{\gamma-\delta}(x)] \leq I_{0,x}^{\alpha,\beta,\eta} [f^{\gamma}(x)g^{-\delta}(x)], \]

(6.3.25)

which implies that

\[ \frac{\gamma}{\gamma - \delta} I_{0,x}^{\alpha,\beta,\eta} [f^{\gamma-\delta}(x)] \leq I_{0,x}^{\alpha,\beta,\eta} [f^{\gamma}(x)g^{-\delta}(x)] + \frac{\delta}{\gamma - \delta} I_{0,x}^{\alpha,\beta,\eta} [g^{\gamma-\delta}(x)], \]

(6.3.26)

that is

\[ I_{0,x}^{\alpha,\beta,\eta} [f^{\gamma-\delta}(x)] \leq \frac{\gamma - \delta}{\gamma} I_{0,x}^{\alpha,\beta,\eta} [f^{\gamma}(t)g^{-\delta}(x)] \leq \frac{\delta}{\gamma} I_{0,x}^{\alpha,\beta,\eta} [f^{\gamma-\delta}(x)], \]

(6.3.27)

from (6.3.22) we get the required inequality. This completes the proof of theorem.

\[ \square \]

**Theorem 6.3.6.** Suppose that \( f, g \) and \( h \) be positive and continuous functions on \([0, \infty)\), such that

\[ (g(\tau) - g(\rho)) \left( \frac{f(\rho)}{h(\rho)} - \frac{f(\tau)}{h(\tau)} \right) \geq 0; \quad \tau, \rho \in [0, x), \]

(6.3.28)
then for all \( x > 0, \alpha > \max\{0, -\beta\}, \beta < 1, \beta - 1 < \eta < 0 \), we have
\[
\frac{I_{0,x}^{\alpha,\beta,\eta}[f(x)]}{I_{0,x}^{\alpha,\beta,\eta}[h(x)]} \geq \frac{I_{0,x}^{\alpha,\beta,\eta}[(gf)(x)]}{I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)]}.
\] (6.3.29)

**Proof:** Since \( f, g \) and \( h \) be three positive and continuous functions
on \([0, \infty[\), by (6.3.28), we have
\[
g(\tau) \frac{f(\rho)}{h(\rho)} + g(\rho) \frac{f(\tau)}{h(\tau)} - g(\rho) \frac{f(\rho)}{h(\rho)} - g(\tau) \frac{f(\rho)}{h(\tau)} \geq 0.
\] (6.3.30)

Now, multiplying equation (6.3.30) by \( h(\rho)h(\tau) \) on both side, we have,
\[
g(\tau)f(\rho)h(\tau) - g(\tau)f(\tau)h(\rho) - g(\rho)f(\rho)h(\tau) + g(\rho)f(\tau)h(\rho) \geq 0.
\] (6.3.31)

Now, multiplying equation (6.3.31) by \( G(x, \tau), (\tau \in (0, x), x > 0) \),
where \( G(x, \tau) \) is defined by (1.2.10). Then integrating the resulting
identity with respect to \( \tau \) from 0 to \( x \), we have
\[
f(\rho) x^{-\alpha-\beta} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} \binom{2}{1}(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x})[g(\tau)h(\tau)]d\tau -
\]
\[
h(\rho) x^{-\alpha-\beta} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} \binom{2}{1}(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x})[f(\tau)g(\tau)]d\tau +
\]
\[
f(\rho)g(\rho) x^{-\alpha-\beta} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} \binom{2}{1}(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x})[h(\tau)]d\tau
\]
\[
g(\rho)h(\rho) x^{-\alpha-\beta} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} \binom{2}{1}(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x})[f(\tau)]d\tau \geq 0,
\] (6.3.32)
Thus we have
\[ f(\rho)I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)] + g(\rho)h(\rho)I_{0,x}^{\alpha,\beta,\eta}[f(x)] - \\
g(\rho)f(\rho)I_{0,x}^{\alpha,\beta,\eta}[h(x)] - h(\rho)I_{0,x}^{\alpha,\beta,\eta}[(gf)(x)] \geq 0. \] (6.3.33)

Again, multiplying (6.3.33) by \( G(x, \rho), \quad (\rho \in (0, x), \quad x > 0) \), where \( G(x, \rho) \) is defined by (1.2.10). Then integrating the resulting identity with respect to \( \rho \) from 0 to \( x \), we have
\[ I_{0,x}^{\alpha,\beta,\eta}[f(x)]I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)] - I_{0,x}^{\alpha,\beta,\eta}[h(x)]I_{0,x}^{\alpha,\beta,\eta}[(gf)(x)] - \\
I_{0,x}^{\alpha,\beta,\eta}[(gf)(x)]I_{0,x}^{\alpha,\beta,\eta}[h(x)] + I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)]I_{0,x}^{\alpha,\beta,\eta}[f(x)] \geq 0, \] (6.3.34)

which implies that
\[ I_{0,x}^{\alpha,\beta,\eta}[f(x)]I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)] \geq I_{0,x}^{\alpha,\beta,\eta}[h(x)]I_{0,x}^{\alpha,\beta,\eta}[(gf)(x)], \] (6.3.35)

which is required inequality (6.3.29) This completes the proof of theorem. \( \square \)

**Theorem 6.3.7.** Suppose that \( f, g \) and \( h \) be positive and continuous functions on \([0, \infty)\), such that
\[ (g(\tau) - g(\rho)) \left( \frac{f(\rho)}{h(\rho)} - \frac{f(\tau)}{h(\tau)} \right) \geq 0, \quad \tau, \rho \in (0, t) \quad t > 0, \] (6.3.36)
then for all \( x > 0, \alpha > \max\{0, -\beta\}, \psi > \max\{0, -\phi\}, \beta < 1, \beta - 1 < \eta < 0, \phi < 1, \phi - 1 < \zeta < 0, \)

\[
\frac{I_{0,x}^{\alpha,\beta,\eta}[f(x)]I_{0,x}^{\psi,\phi,\zeta}[(gh)(x)] + I_{0,x}^{\psi,\phi,\zeta}[f(x)]I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)]}{I_{0,x}^{\alpha,\beta,\eta}[h(x)]I_{0,x}^{\psi,\phi,\zeta}[(gf)(x)] + I_{0,x}^{\psi,\phi,\zeta}[h(x)]I_{0,x}^{\alpha,\beta,\eta}[(gf)(x)]} \geq 1, \quad (6.3.37)
\]

holds.

**Proof :-** Multiplying equation (6.3.33) by \( \frac{x^{\psi - \phi}}{\Gamma(\psi)}(x - \rho)^{\psi - 1} {}_2F_1(\psi + \phi, -\gamma; \psi; 1 - \frac{\rho}{x}) \) \((\rho \in (0, x), x > 0)\), which (in view of the argument mentioned above in proof of theorem 6.3.3) remains positive. Then integrating the resulting identity with respect to \( \rho \) from 0 to \( x \), we have

\[
I_{0,x}^{\psi,\phi,\zeta}[f(x)]I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)] - I_{0,x}^{\psi,\phi,\zeta}[h(x)]I_{0,x}^{\alpha,\beta,\eta}[(gf)(x)]
- I_{0,x}^{\psi,\phi,\zeta}[(gf)(x)]I_{0,x}^{\alpha,\beta,\eta}[h(x)] + I_{0,x}^{\psi,\phi,\zeta}[(gh)(x)]I_{0,x}^{\alpha,\beta,\eta}[f(x)] \geq 0, \quad (6.3.38)
\]

we get

\[
I_{0,x}^{\psi,\phi,\zeta}[f(x)]I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)] + I_{0,x}^{\psi,\phi,\zeta}[(gh)(x)]I_{0,x}^{\alpha,\beta,\eta}[f(x)]
\geq I_{0,x}^{\psi,\phi,\zeta}[h(x)]I_{0,x}^{\alpha,\beta,\eta}[(gf)(x)] + I_{0,x}^{\psi,\phi,\zeta}[(gf)(x)]I_{0,x}^{\alpha,\beta,\eta}[h(x)], \quad (6.3.39)
\]

which is the required inequality (6.3.37). This completes the proof of theorem. \( \square \)

**Remark 6.3.2.** If we take \( \alpha = \psi, \beta = \phi \) and \( \eta = \zeta \), in theorem 6.3.7 reduces to the theorem 6.3.6.
Theorem 6.3.8. Suppose that $f$ and $h$ are two positive continuous functions such that $f \leq h$ on $[0, \infty)$. If $\frac{f}{h}$ is decreasing and $f$ is increasing on $[0, \infty)$, then for any $p \geq 0$, For all $x > 0$, $\alpha > \max\{0, -\beta\}$, $\beta < 1$, $\beta - 1 < \eta < 0$

$$\frac{h^p(\tau)[f(x)]}{I_{0,x}^{\alpha, \beta, \eta}[h(x)]} \geq \frac{I_{0,x}^{\alpha, \beta, \eta}[f^p(x)]}{I_{0,x}^{\alpha, \beta, \eta}[h^p(x)]}.$$ (6.3.40)

Proof :- We take $g = f^{p-1}$ in theorem 6.3.6.

$$\frac{I_{0,x}^{\alpha, \beta, \eta}[f(x)]}{I_{0,x}^{\alpha, \beta, \eta}[h(x)]} \geq \frac{I_{0,x}^{\alpha, \beta, \eta}[(f^{p-1})(x)]}{I_{0,x}^{\alpha, \beta, \eta}[(h^{p-1})(x)]}.$$ (6.3.41)

Since $f \leq h$ on $[0, \infty)$ then we have

$$h^{p-1}(x) \leq h^p.$$ (6.3.42)

Multiplying equation (6.3.42) by $G(x, \tau)$, ($\tau \in (0, x)$, $x > 0$), where $G(x, \tau)$ is defined by (1.2.10). Then integrating the resulting identity with respect to $\tau$ from 0 to $x$, we have

$$\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x - \tau)^{\alpha-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \int_0^x (x - \tau)^{\alpha-1} 2F_1(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x}) d\tau \left[f^{p-1}h(\tau)\right] d\tau \leq x^{-\alpha-\beta} \Gamma(\alpha) \int_0^x (x - \tau)^{\alpha-1} 2F_1(\alpha + \beta, -\eta; \alpha; 1 - \frac{\tau}{x}) [h^p(\tau)] d\tau.$$ (6.3.43)
implies that
\[ I_{0,x}^{\alpha,\beta,\eta}[hf^{p-1}(x)] \leq I_{0,x}^{\alpha,\beta,\eta}[h^p(x)], \quad (6.3.44) \]
and we have
\[ \frac{h^p(\tau)[(f f^{p-1})(x)]}{h^p(\tau)[(h f^{p-1})(x)]} \geq \frac{h^p(\tau)[f^p(x)]}{h^p(\tau)[h^p(x)]}, \quad (6.3.45) \]
then from equation (6.3.41) and (6.3.45), we required inequality (6.3.40).

This completes proof of the theorem. \( \square \)

**Theorem 6.3.9.** Suppose that \( f \) and \( h \) are two positive continuous functions such that \( f \leq h \) on \([0, \infty)\). If \( \frac{f}{h} \) is decreasing and \( f \) is increasing on \([0, \infty)[\), then for any \( p \geq 1 \), for all \( x > 0 \), \( \alpha > \max\{0, -\beta\} \), \( \psi > \max\{0, -\phi\} \), \( \beta < 1 \), \( \beta - 1 < \eta < 0 \), \( \phi < 1 \), \( \phi - 1 < \zeta < 0 \),

\[ \frac{I_{0,x}^{\alpha,\beta,\eta}[f(x)] I_{0,x}^{\psi,\phi,\zeta}[h^p(x)] + I_{0,x}^{\psi,\phi,\zeta}[f(x)] I_{0,x}^{\alpha,\beta,\eta}[h^p(x)]}{I_{0,x}^{\alpha,\beta,\eta}[h(x)] I_{0,x}^{\psi,\phi,\zeta}[f^p(x)] + I_{0,x}^{\psi,\phi,\zeta}[h(x)] I_{0,x}^{\alpha,\beta,\eta}[f^p(x)]} \geq 1. \quad (6.3.46) \]

**Proof :-** We take \( g = f^{p-1} \) in theorem 6.3.7, then we obtain
\[ \frac{I_{0,x}^{\alpha,\beta,\eta}[f(x)] I_{0,x}^{\psi,\phi,\zeta}[hf^{p-1}(x)] + I_{0,x}^{\psi,\phi,\zeta}[f(x)] I_{0,x}^{\alpha,\beta,\eta}[hf^{p-1}(x)]}{I_{0,x}^{\alpha,\beta,\eta}[h(x)] I_{0,x}^{\psi,\phi,\zeta}[f^p(x)] + I_{0,x}^{\psi,\phi,\zeta}[h(x)] I_{0,x}^{\alpha,\beta,\eta}[f^p(x)]} \geq 1, \quad (6.3.47) \]
then by hypothesis, \( f \leq h \) on \([0, \infty)\), which implies that
\[ hf^{p-1} \leq h^p. \quad (6.3.48) \]
Now, multiplying both sides of (6.3.48) by \( \frac{x^{-\phi-\psi}}{\Gamma(\psi)}(x - \rho)^{\psi-1}2F_1(\psi + \phi, -\zeta; \psi; 1 - \frac{\rho}{x}) \) \((\rho \in (0, x), x > 0)\), which (in view of the argument
mentioned above in proof of theorem 6.3.3) remain positive. Then integrating the resulting identity with respect to \( \rho \) from 0 to \( x \), we have,

\[
I_{0,x}^{\psi,\phi,\zeta}[h f^{p-1}(x)] \leq I_{0,x}^{\psi,\phi,\zeta}[h^p(x)],
\]

(6.3.49)

multiplying on both sides of (6.3.49) by \( I_{0,x}^{\alpha,\beta,\eta}[f(x)] \), we obtain

\[
I_{0,x}^{\alpha,\beta,\eta}[f(x)] I_{0,x}^{\psi,\phi,\zeta}[h f^{p-1}(x)] \leq I_{0,x}^{\alpha,\beta,\eta}[f(x)] I_{0,x}^{\psi,\phi,\zeta}[h^p(x)],
\]

(6.3.50)

hence by (6.3.49) and (6.3.50), we obtain

\[
I_{0,x}^{\alpha,\beta,\eta}[f(x)] I_{0,x}^{\psi,\phi,\zeta}[h f^{p-1}(x)] + I_{0,x}^{\psi,\phi,\zeta}[f(x)] I_{0,x}^{\alpha,\beta,\eta}[h f^{p-1}(x)]
\]

\[
\leq I_{0,x}^{\alpha,\beta,\eta}[f(x)] I_{0,x}^{\psi,\phi,\zeta}[h^p(x)] + I_{0,x}^{\psi,\phi,\zeta}[f(x)] I_{0,x}^{\alpha,\beta,\eta}[h^p(x)].
\]

(6.3.51)

By (6.3.47) and (6.3.51), This completes the proof of theorem.

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By (6.3.47) and (6.3.51), This completes the proof of theorem.

Now we present another inequalities involving Saigo fractional integral operator

**Theorem 6.3.10.** Suppose that \( f, g \) and \( h \) be positive and continuous functions on \([0, \infty)\), such that

\[
(f(\tau) - f(\rho))(g(\tau) - g(\rho))(h(\tau) + h(\rho)) \geq 0; \ \tau, \rho \in (0, x) \quad x > 0,
\]

(6.3.52)
then for all $x > 0$, $\alpha > \max\{0, -\beta\}$, $\beta < 1$, $\beta - 1 < \eta < 0$

\[
I^{\alpha, \beta, \eta}_{0,x}[(fgh)(x)]I^{\alpha, \beta, \eta}_{0,x}[(fg)(x)]I^{\alpha, \beta, \eta}_{0,x}[h(x)] \\
\geq I^{\alpha, \beta, \eta}_{0,x}[g(x)]I^{\alpha, \beta, \eta}_{0,x}[(fh)(x)] + I^{\alpha, \beta, \eta}_{0,x}[f(x)]I^{\alpha, \beta, \eta}_{0,x}[(gh)(x)]. \quad (6.3.53)
\]

Proof :- By the assumption for any $\tau, \rho$, we have

\[
f(\tau)g(\tau)h(\tau) + f(\tau)g(\rho)h(\rho) - f(\tau)g(\rho)h(\tau) - f(\tau)g(\rho)h(\rho) - \\
f(\rho)g(\tau)h(\tau) - f(\rho)g(\tau)h(\rho) + f(\rho)g(\rho)h(\tau) + f(\rho)g(\rho)h(\rho) \geq 0,
\]

(6.3.54)

multiplying equation (6.3.54) by $G(x, \tau)$, $(\tau \in (0, x), x > 0)$, where $G(x, \tau)$ is defined by (1.2.10). Then integrating the resulting identity with respect to $\tau$ from 0 to $x$, we have

\[
I^{\alpha, \beta, \eta}_{0,x}[(fgh)(x)] + h(\rho)I^{\alpha, \beta, \eta}_{0,x}[(fg)(x)] + f(\rho)g(\rho)I^{\alpha, \beta, \eta}_{0,x}[h(x)] + \\
f(\rho)g(\rho)h(\rho)I^{\alpha, \beta, \eta}_{0,x}[1] \geq g(\rho)I^{\alpha, \beta, \eta}_{0,x}[(fh)(x)] + g(\rho)h(\rho)I^{\alpha, \beta, \eta}_{0,x}[f(x)] + \\
f(\rho)I^{\alpha, \beta, \eta}_{0,x}[(gh)(x)] + f(\rho)h(\rho)I^{\alpha, \beta, \eta}_{0,x}[g(x)]. \quad (6.3.55)
\]

Again multiplying (6.3.55) by $G(x, \rho)$, $(\rho \in (0, x), x > 0)$, where $G(x, \rho)$ is defined by (1.2.10). Then integrating the resulting identity with respect to $\rho$ from 0 to $x$, we have

\[
I^{\alpha, \beta, \eta}_{0,x}[1]I^{\alpha, \beta, \eta}_{0,x}[(fgh)(x)] + I^{\alpha, \beta, \eta}_{0,x}[h(x)]I^{\alpha, \beta, \eta}_{0,x}[(fg)(x)] +
\]
\[ I_{0,x}^{\alpha,\beta,\eta}[h(x)] I_{0,x}^{\alpha,\beta,\eta}[(fg)(x)] + I_{0,x}^{\alpha,\beta,\eta}[(fgh)(x)] I_{0,x}^{\alpha,\beta,\eta}[1] \]
\[ \geq I_{0,x}^{\alpha,\beta,\eta}[g(x)] I_{0,x}^{\alpha,\beta,\eta}[(fh)(x)] + I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)] I_{0,x}^{\alpha,\beta,\eta}[f(x)] + \]
\[ I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)] I_{0,x}^{\alpha,\beta,\eta}[f(x)] + I_{0,x}^{\alpha,\beta,\eta}[(fh)(x)] I_{0,x}^{\alpha,\beta,\eta}[g(x)], \quad (6.3.56) \]

which gives the equation (6.3.53). This completes the theorem. \(\square\)

**Theorem 6.3.11.** Suppose that \(f\), \(g\) and \(h\) be positive and continuous functions on \([0, \infty)\), such that

\[(f(\tau) - f(\rho))(g(\tau) + g(\rho))(h(\tau) + h(\rho)) \geq 0; \quad \tau, \rho \in (0, x) \quad x > 0, \quad (6.3.57)\]

then for all \(x > 0\), \(\alpha > \max\{0, -\beta\}, \beta < 1, \beta - 1 < \eta < 0\)

\[ I_{0,x}^{\alpha,\beta,\eta}[f(x)] I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)] + I_{0,x}^{\alpha,\beta,\eta}[(fh)(x)] I_{0,x}^{\alpha,\beta,\eta}[g(x)] \]
\[ \geq I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)] I_{0,x}^{\alpha,\beta,\eta}[f(x)] + I_{0,x}^{\alpha,\beta,\eta}[h(x)] I_{0,x}^{\alpha,\beta,\eta}[(fg)(x)]. \quad (6.3.58) \]

**Proof :-** For any \(\tau, \rho\), we have

\[ f(\tau)g(\tau)h(\tau) + f(\tau)g(\rho)h(\rho) + f(\tau)g(\rho)h(\tau) + f(\tau)g(\rho)h(\rho) \]
\[ \geq f(\rho)g(\tau)h(\tau) + f(\rho)g(\tau)h(\rho) + f(\rho)g(\rho)h(\tau) + f(\rho)g(\rho)h(\rho). \quad (6.3.59) \]
Similar to the proof theorem 6.3.10, we have

\[ I_{0,x}^{\alpha,\beta,\eta}[(fg)(x)] + h(\rho)I_{0,x}^{\alpha,\beta,\eta}[(fg)(x)] + g(\rho)I_{0,x}^{\alpha,\beta,\eta}[(f)(x)] + g(\rho)h(\rho)I_{0,x}^{\alpha,\beta,\eta}[f(x)] \geq f(\rho)g(\rho)I_{0,x}^{\alpha,\beta,\eta}[h(x)] + f(\rho)h(\rho)I_{0,x}^{\alpha,\beta,\eta}[g(x)] + f(\rho)I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)] + f(\rho)g(\rho)h(\rho)I_{0,x}^{\alpha,\beta,\eta}[1]. \] (6.3.60)

Again, similar to proof of theorem 6.3.10, we obtain

\[ I_{0,x}^{\alpha,\beta,\eta}[1]I_{0,x}^{\alpha,\beta,\eta}[(fg)(x)] + I_{0,x}^{\alpha,\beta,\eta}[h(x)]I_{0,x}^{\alpha,\beta,\eta}[(f)(x)] + I_{0,x}^{\alpha,\beta,\eta}[g(x)]I_{0,x}^{\alpha,\beta,\eta}[(f)(x)] + I_{0,x}^{\alpha,\beta,\eta}[(gh)(x)]I_{0,x}^{\alpha,\beta,\eta}[f(x)] \geq I_{0,x}^{\alpha,\beta,\eta}[(fg)(x)]I_{0,x}^{\alpha,\beta,\eta}[h(x)] + I_{0,x}^{\alpha,\beta,\eta}[(f)(x)]I_{0,x}^{\alpha,\beta,\eta}[g(x)] + I_{0,x}^{\alpha,\beta,\eta}[(fg)(x)]I_{0,x}^{\alpha,\beta,\eta}[h(x)] + I_{0,x}^{\alpha,\beta,\eta}[(f)(x)]I_{0,x}^{\alpha,\beta,\eta}[1]. \] (6.3.61)

we get equation (6.3.58), this completes the proof of theorem 6.3.11. □