Chapter 5

Classes related to Multiplier Transformations

In this chapter\(^1\), we define the classes \(T^\lambda_j(m, l, A, B)\), \(R^\lambda_j(m, l, A, B)\), and \(K^\lambda_j(m, l, A, B, C, D)\) using the Janowski class and the multiplier transformation \(I(m, \lambda, l)f\) and certain properties of neighborhoods and partial sums for functions belonging to these classes are studied. For different choices of the parameters we get the results obtained by Altintas and Owa [3] and Aouf [7]. Moreover, we obtain sharp lower bounds for \(\Re \left\{ \frac{f(z)}{f_n(z)} \right\}\), \(\Re \left\{ \frac{f_n(z)}{f(z)} \right\}\), \(\Re \left\{ \frac{f'(z)}{f_n'(z)} \right\}\) and \(\Re \left\{ \frac{f_n'(z)}{f'(z)} \right\}\).

\(^1\)Reference [60] is based on this Chapter.
5.1 Neighborhood results

Let $T(j)$ denote the class of functions of the form

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k, \quad (a_k \geq 0; \ j \in \mathbb{N}) \quad (5.1.1)$$

which are analytic in the open unit disc $\mathcal{E} = \{z : |z| < 1\}$.

**Definition 5.1.1.** For any function $f \in T(j)$, $z \in \mathcal{E}$ and $\delta \geq 0$, we define the $(n, \delta)$-neighborhood of $f$ as

$$N_{j,\delta}(f) = \left\{ g \in T(j) : g(z) = z - \sum_{k=j+1}^{\infty} b_k z^k, \ \sum_{k=j+1}^{\infty} k|a_k - b_k| \leq \delta \right\}. \quad (5.1.2)$$

In particular, for the identity function $e(z) = z$, we have

$$N_{j,\delta}(e) = \left\{ g \in T(j) : g(z) = z - \sum_{k=j+1}^{\infty} b_k z^k, \ \sum_{k=j+1}^{\infty} k|b_k| \leq \delta \right\}. \quad (5.1.3)$$

In this section, we discuss certain properties of $(n, \delta)$-neighborhoods of the following subclasses of the class $T(j)$ of normalized analytic functions in $\mathcal{E}$ with negative coefficients.

**Definition 5.1.2.** For $f \in T(j)$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, the multiplier transformation $I(m, \lambda, l)f [12]$ is defined by

$$I(m, \lambda, l)f(z) = z - \sum_{k=j+1}^{\infty} \left( \frac{\lambda(k - 1) + l + 1}{l + 1} \right)^m a_k z^k, \quad z \in \mathcal{E}. \quad (5.1.4)$$
**Definition 5.1.3.** A $f \in T(j)$ is in the class $T_j^\lambda(m, l, A, B)$ if and only if

$$\frac{z[I(m, \lambda, l)f(z)']}{I(m, \lambda, l)f(z)} < \frac{1 + Az}{1 + Bz}, \quad (m \in \mathbb{N}_0)$$

for $-1 \leq A < B \leq 1$ and $z \in \mathcal{E}$.

We note that $T_j^0(1, 0, 2\alpha - 1, 1) \equiv S_j^*(\alpha)$ introduced by Chatterjea [13] and $T_j^1(1, 0, 2\alpha - 1, 1) \equiv C_j(\alpha)$ studied by Srivastava [80].

Now we prove the following coefficient inequality.

**Theorem 5.1.1.** A function $f \in T(j)$ is in the class $T_j^\lambda(m, l, A, B)$ if and only if

$$\sum_{k=j+1}^{\infty} \left( \frac{\lambda(k - 1) + l + 1}{l + 1} \right)^m [k(B + 1) - (A + 1)]a_k \leq B - A, \quad (5.1.4)$$

for $m \in \mathbb{N}_0$, $-1 \leq A < B \leq 1$.

**Proof.** Since $f \in T_j^\lambda(m, l, A, B)$, we have

$$\frac{z[I(m, \lambda, l)f(z)']}{I(m, \lambda, l)f(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)}$$
By Schwarz’s Lemma, it follows that

\[ \left| \sum_{k=j+1}^{\infty} \left( \frac{\lambda(k-1) + l + 1}{l+1} \right)^m (k-1)a_kz^k \right| \leq \frac{1}{(B-A) + \sum_{k=j+2}^{\infty} \left( \frac{\lambda(k-1) + l + 1}{l+1} \right)^m (A-Bk)a_kz^k} \]

\[ \sum_{k=j+1}^{\infty} \left( \frac{\lambda(k-1) + l + 1}{l+1} \right)^m [k(B+1) - (A+1)]a_k \leq B - A, \]

for \( m \in \mathbb{N}_0, -1 \leq A < B \leq 1. \) \( \square \)

Now, we discuss about the neighborhood results for the class \( \mathcal{T}_j^\lambda(m, l, A, B). \)

**Theorem 5.1.2.** \( \mathcal{T}_j^\lambda(m, l, A, B) \subset \mathcal{N}_{j, \delta}(e), \) where,

\[ \delta = \frac{(j+1)(B-A)}{\left( \frac{\lambda j + l + 1}{l+1} \right)^m \left[ (j+1)(B+1) - (A+1) \right]} . \tag{5.1.5} \]
Proof. Let \( f \in \mathcal{T}_j^\lambda(m, l, A, B) \). By the Theorem 5.1.1, we have
\[
\left( \frac{\lambda j + l + 1}{l + 1} \right)^m [(j + 1)(B + 1) - (A + 1)] \sum_{k=j+1}^{\infty} a_k \leq B - A,
\]
which implies that,
\[
\sum_{k=j+1}^{\infty} a_k \leq \frac{B - A}{\left( \frac{\lambda j + l + 1}{l + 1} \right)^m [(j + 1)(B + 1) - (A + 1)]}. \quad (5.1.6)
\]
Using (5.1.4) and (5.1.6), we have,
\[
\left( \frac{\lambda j + l + 1}{l + 1} \right)^m (B + 1) \sum_{k=j+1}^{\infty} k a_k \leq B - A + \left( \frac{\lambda j + l + 1}{l + 1} \right)^m (A + 1) \sum_{k=j+1}^{\infty} a_k,
\]
\[
\leq B - A + \left( \frac{\lambda j + l + 1}{l + 1} \right)^m (A + 1) \sum_{k=j+1}^{\infty} a_k, \quad (5.1.6)
\]
\[
\leq \frac{(j + 1)(B - A)}{\left( \frac{\lambda j + l + 1}{l + 1} \right)^m [(j + 1)(B + 1) - (A + 1)]}.
\]
That is,
\[
\sum_{k=j+1}^{\infty} k a_k \leq \frac{(j + 1)(B - A)}{\left( \frac{\lambda j + l + 1}{l + 1} \right)^m [(j + 1)(B + 1) - (A + 1)]} = \delta.
\]
Thus, by the definition given by (5.1.3), \( f \in \mathcal{N}_{j, \delta}(e) \), which completes the proof.

Putting \( j = 1 \), in the above Theorem, we have the following.
**Corollary 5.1.3.** \( T^\lambda_1(m, l, A, B) \subset \mathcal{N}_{1,\delta}(e) \), where,

\[
\delta = \frac{2(B - A)}{\left( \frac{\lambda + l + 1}{l + 1} \right)^m [2(B + 1) - (A + 1)]}.
\]

Letting \( A = 2\alpha - 1 \), \( B = 1 \), \( m = 1 \), \( \lambda = 0 \) and \( l = 0 \) in the above Theorem, we have

**Corollary 5.1.4.** \( \mathcal{S}^*_j(\alpha) \subset \mathcal{N}_{j,\delta}(e) \), where,

\[
\delta = \frac{(j + 1)(1 - \alpha)}{j + 1 - \alpha}.
\]

Generalizing the classes defined by Yaguchi, Aouf, Sarangi and Urale-gaddi, we define the subclass \( \mathcal{R}_j^\lambda(m, l, A, B) \) of \( \mathcal{T}(j) \) and obtain neighborhood for functions belonging to this class.

**Definition 5.1.4.** A function \( f \in \mathcal{T}(j) \) is said to be in the class \( \mathcal{R}_j^\lambda(m, l, A, B) \) if it satisfies

\[
(I(m, \lambda, l)f(z))' \prec \frac{1 + Az}{1 + Bz}, \quad z \in \mathcal{E},
\]

for \( m \in \mathbb{N}_0 \) and \(-1 \leq B < A \leq 1\).

Different parametric values yield the following classes.

i. \( \mathcal{R}_1^1(n, 2\alpha - 1, 1) \equiv \mathcal{R}_1(m, 0, \alpha) \) defined by Yaguchi and Aouf [84].

ii. \( \mathcal{R}_j^0(1, 0, 2\alpha - 1, 1) \equiv \mathcal{R}_j(\alpha) \) introduced by Sarangi and Urale-gaddi [68].
iii. $\mathcal{R}_j^1(m, 0, 2\alpha - 1, 1) \equiv \mathcal{R}_j(m, \alpha)$ defined by Aouf [7].

**Theorem 5.1.5.** A function $f \in \mathcal{T}(j)$ is in the class $\mathcal{R}_j^\lambda(m, l, A, B)$ if and only if

$$\sum_{k=j+1}^{\infty} \left( \frac{\lambda(k-1)+l+1}{l+1} \right)^m k(B+1)a_k \leq B - A. \quad (5.1.7)$$

**Proof.** Proof is on similar lines of that of the Theorem 5.1.1. $\square$

**Theorem 5.1.6.** $\mathcal{R}_j^\lambda(m, l, A, B) \subset \mathcal{N}_{j, \delta}(e)$, where,

$$\delta = \frac{B - A}{\left( \frac{\lambda j + l + 1}{l + 1} \right)^m (B + 1)}. \quad (5.1.8)$$

**Proof.** Let $f \in \mathcal{T}_j^\lambda(m, l, A, B)$. By the Theorem 5.1.6, we have,

$$\left( \frac{\lambda j + l + 1}{l + 1} \right)^m (B + 1) \sum_{k=j+1}^{\infty} ka_k \leq B - A.$$

That is,

$$\sum_{k=j+1}^{\infty} ka_k \leq \frac{B - A}{\left( \frac{\lambda j + l + 1}{l + 1} \right)^m (B + 1)} = \delta.$$

Thus, by the definition given by (5.1.3), $f \in \mathcal{N}_{j, \delta}(e)$.

This completes the proof. $\square$

Putting $j = 1$, in the above Theorem, we have the following.

**Corollary 5.1.7.** $\mathcal{R}_1^\lambda(m, l, A, B) \subset \mathcal{N}_{1, \delta}(e)$, where,

$$\delta = \frac{B - A}{\left( \frac{\lambda + l + 1}{l + 1} \right)^m (B + 1)}.$$
Letting $A = 2\alpha - 1$, $B = 1$, $m = 1$, $\lambda = 0$ and $l = 0$ in the above Theorem, we have

**Corollary 5.1.8.** \[4\] $R_j^*(\alpha) \subset N_{j,\delta}(e)$, where, $\delta = 1 - \alpha$.

Letting $A = 2\alpha - 1$, $B = 1$, $\lambda = 1$ and $l = 0$ in the above Theorem, we have,

**Corollary 5.1.9.** \[7\] $T_j(n,\alpha) \subset N_{j,\delta}(e)$, where,

$$\delta = \frac{1 - \alpha}{(j + 1)^m}.$$ 

**Definition 5.1.5.** A function $f \in T(j)$ is said to be in the class $K_j^\lambda(m, l, A, B, C, D)$ if it satisfies

$$\left| \frac{f(z)}{g(z)} - 1 \right| \leq B - A, \quad z \in \mathcal{E}, \quad (5.1.9)$$

for some $-1 \leq A < B \leq 1$ and $g \in T_j^\lambda(m, l, C, D)$ \((-1 \leq C < D \leq 1)\).

**Theorem 5.1.10.** $N_{j,\delta}(g) \subset K_j^\lambda(m, l, A, B, C, D)$, where, $g \in T_j^\lambda(m, l, C, D)$ and

$$A = B - \frac{\left(\frac{\lambda j + l + 1}{l + 1}\right)^m [(j + 1)(D + 1) - (C + 1)]\delta}{(j + 1) \left[\left(\frac{\lambda j + l + 1}{l + 1}\right)^m [(j + 1)(D + 1) - (C + 1)] - (D - C)\right]}.$$ 

**Proof.** Let $g \in N_{n,\delta}$. Then, we have

$$\sum_{k=j+1}^{\infty} k|a_k - b_k| \leq \delta, \quad (5.1.10)$$
\[
\sum_{k=j+1}^{\infty} b_k \leq \left( \frac{\lambda j + l + 1}{l + 1} \right)^m \frac{D - C}{[(j + 1)(D + 1) - (C + 1)]},
\]
so that,

\[
\left| \frac{f(z)}{g(z)} - 1 \right| \leq \sum_{k=j+1}^{\infty} |a_k - b_k| = \frac{\sum_{k=j+1}^{\infty} d_k}{1 - \sum_{k=j+1}^{\infty} b_k} \leq \delta \cdot \left( \frac{\lambda j + l + 1}{l + 1} \right)^m \frac{[(j + 1)(D + 1) - (C + 1)]}{[(j + 1)(D + 1) - (C + 1)] - (D - C)} = \frac{\delta \left( \frac{\lambda j + l + 1}{l + 1} \right)^m}{(j + 1) \left[ (j + 1)(D + 1) - (C + 1)] - (D - C) \right]}
\]

\[
= B - A.
\]

Thus, by the definition \( f \in K_j^{\lambda}(m, l, A, B, C, D) \), \(-1 \leq A < B \leq 1\). \(\square\)

Putting \( j = 1 \), in the above Theorem, we have the following.

**Corollary 5.1.11.** \( N_{1, \delta}(g) \subset K_1^{\lambda}(m, l, A, B, C, D) \), where, \( g \in T_1^{\lambda}(m, l, C, D) \) and

\[
A = B - \frac{\left( \frac{\lambda + l + 1}{l + 1} \right)^m \left[ 2(D + 1) - (C + 1) \right] \delta}{2 \left[ \left( \frac{\lambda + l + 1}{l + 1} \right)^m \left[ 2(D + 1) - (C + 1) \right] - (D - C) \right]}.
\]
Letting $A = 2\alpha - 1$, $B = 1$, $C = 2\beta - 1$, $D = 1$, $m = 1$, $\lambda = 0$ and $l = 0$ in the above Theorem, we have

**Corollary 5.1.12.** \[ 4 \] $N_{j,\delta}(g) \subset S_j(\alpha, \beta)$, where,

$$\alpha = 1 - \frac{(j + 1 - \beta)\delta}{j(j + 1)}.$$

### 5.2 Partial Sums

In this section, we consider partial sums of functions in the class $T_{j\lambda}(m, l, A, B)$ and obtain sharp lower bounds for the ratios of real part of $f$ to $f_n$ and $f'$ to $f'_n$. Now, applying methods used by Silverman [75] and Silvia [78], we will investigate the ratio of a function $f$ of the form (5.1.1) to its sequence of partial sums

$$f_n(z) = z - \sum_{k=j+1}^{n} a_k z^k,$$

when the coefficients are sufficiently small to satisfy conditions (5.1.4). More precisely, we will determine sharp lower bounds for

$$\Re \left\{ \frac{f(z)}{f_n(z)} \right\}, \Re \left\{ \frac{f_n(z)}{f(z)} \right\}, \Re \left\{ \frac{f'(z)}{f'_n(z)} \right\} \text{ and } \Re \left\{ \frac{f'_n(z)}{f'(z)} \right\}.$$

In the sequel, we will make use of the result that

$$\Re \left\{ \frac{1 + \omega(z)}{1 - \omega(z)} \right\} > 0, \quad z \in \mathcal{E}$$
if and only if \( \omega(z) = \sum_{k=j}^{\infty} c_k z^k \) satisfies the inequality \( |\omega(z)| < |z| \).

**Theorem 5.2.1.** If \( f \in T_j^\lambda(m, l, A, B) \), then

\[
\Re \left\{ \frac{f(z)}{f_n(z)} \right\} > 1 - \frac{1}{c_{n+1}}, \quad z \in \mathcal{E}
\]

where,

\[
c_k = \frac{\left( \frac{\lambda(k - 1) + l + 1}{l + 1} \right)^m [k(B + 1) - (A + 1)]}{B - A}. \quad (5.2.2)
\]

The estimate in (5.2.1) is sharp for every \( n \) with extremal function

\[
f(z) = z - \frac{z^{n+1}}{c_{n+1}}, \quad z \in \mathcal{E}, \quad k \geq 1. \quad (5.2.3)
\]

**Proof.** The function \( f \in T_j^\lambda(m, l, A, B) \), if and only if

\[
\sum_{k=j+1}^{\infty} c_k a_k \leq 1.
\]

It is easy to verify that \( c_{k+1} > c_k > 1 \).

Therefore, we have

\[
\sum_{k=j+1}^{n} a_k + c_{n+1} \sum_{k=n+1}^{\infty} a_k \leq \sum_{k=j+1}^{\infty} c_k a_k \leq 1. \quad (5.2.4)
\]

By setting,

\[
\omega(z) = \frac{c_{n+1}}{c_{n+1}} \left\{ \frac{f(z)}{f_n(z)} - \left( 1 - \frac{1}{c_{n+1}} \right) \right\} = 1 - \frac{\sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 - \sum_{k=j+1}^{n} a_k z^{k-1}}
\]
and applying (5.2.4), we find that

\[
\left| \frac{1 - \omega(z)}{1 + \omega(z)} \right| \leq \frac{c_{n+1} \sum_{k=n+1}^{\infty} a_k}{2 - 2 \sum_{k=j+1}^{n} a_k - c_{n+1} \sum_{k=n+1}^{\infty} a_k} \leq 1, \quad z \in \mathcal{E},
\]

which readily yields the assertion (5.2.1).

To see that

\[ f(z) = z - \frac{z^{n+1}}{c_{n+1}} \]

gives sharp results, we observe that

\[
\frac{f(z)}{f_n(z)} = 1 - \frac{z^n}{c_{n+1}}.
\]

Letting \( z \to 1^- \), we have

\[
\frac{f(z)}{f_n(z)} = 1 - \frac{1}{c_{n+1}},
\]

which shows that the bound in (5.2.1) is the best possible for each \( n \in \mathbb{N} \).

\[ \square \]

**Theorem 5.2.2.** If \( f \in T_{j}^{\lambda}(m, l, A, B) \), then

\[
\Re \left\{ \frac{f_n(z)}{f(z)} \right\} > \frac{c_{n+1}}{1 + c_{n+1}}, \quad z \in \mathcal{E}, \quad (5.2.5)
\]

where, \( c_n \) is defined in (5.2.2).

The result is sharp for every \( n \) with extremal function given by (5.2.3).
Proof. By setting,

\[
\omega(z) = (1 + c_{n+1}) \left\{ \frac{f_n(z)}{f(z)} - \frac{c_{n+1}}{1+c_{n+1}} \right\} = \\
1 - \sum_{k=j+1}^{n} a_k z^{k-1} + c_{n+1} \sum_{k=n+1}^{\infty} a_k z^{k-1} \\
= \frac{1 - \sum_{k=j+1}^{\infty} a_k z^{k-1}}{1 - \sum_{k=j+1}^{\infty} a_k z^{k-1}}
\]

and using (5.2.4) we find that

\[
\left| \frac{1 - \omega(z)}{1 + \omega(z)} \right| \leq \frac{(1 + c_{n+1}) \sum_{k=n+1}^{\infty} a_k}{2 - 2 \sum_{k=j+1}^{n} a_k - (1 - c_{n+1}) \sum_{k=n+1}^{\infty} a_k} \leq 1, \quad z \in \mathcal{E}.
\]

which gives (5.2.5). The bound in (5.2.5) is sharp for all \( n \in \mathbb{N} \) with the external function given by (5.2.3). \qed

**Theorem 5.2.3.** If \( f \in T_j^\lambda(m, l, A, B) \), then

\[
\Re \left\{ \frac{f''(z)}{f_n'(z)} \right\} > 1 - \frac{n+1}{c_{n+1}}, \quad z \in \mathcal{E}
\]

(5.2.6)

where, \( c_n \) is defined in (5.2.2).
Proof. By setting,
\[
\omega(z) = c_{n+1} \left\{ \frac{f'(z)}{f_n'(z)} - \left(1 - \frac{n+1}{c_{n+1}}\right) \right\} \\
1 - \frac{c_{n+1}}{n+1} \sum_{k=n+1}^{\infty} ka_k z^{k-1} - \sum_{k=j+1}^{\infty} ka_k z^{k-1} \\
= \frac{1 - \sum_{k=j+1}^{n} ka_k z^{k-1}}{1 - \sum_{k=j+1}^{n} ka_k z^{k-1}}.
\]

We have,
\[
\left| \frac{1 - \omega(z)}{1 + \omega(z)} \right| \leq \frac{\frac{c_{n+1}}{n+1} \sum_{k=n+1}^{\infty} ka_k}{2 - 2 \sum_{k=j+1}^{n} ka_k - \frac{c_{n+1}}{n+1} \sum_{k=n+1}^{\infty} ka_k} \leq 1,
\]

if and only if
\[
2 \frac{c_{n+1}}{n+1} \sum_{k=n+1}^{\infty} ka_k \leq 2 - 2 \sum_{k=j+1}^{n} ka_k,
\]

which is equivalent to
\[
\sum_{k=j+1}^{n} ka_k + \frac{c_{n+1}}{n+1} \sum_{k=n+1}^{\infty} ka_k \leq 1
\]

thus we obtain (5.2.6).

The result is sharp for \( f \) given by (5.2.3).
Theorem 5.2.4. If $f \in T^\lambda_j(m, l, A, B)$, then
\[
\Re \left\{ \frac{f_n'(z)}{f'(z)} \right\} > \frac{c_{n+1}}{n + 1 + c_{n+1}}, \quad z \in \mathcal{E}, \quad (5.2.7)
\]
where, $c_n$ is defined in (5.2.2).

The result is sharp for every $n$ with extremal function given by (5.2.3).

Proof. By setting,
\[
\omega(z) = [(n + 1) + c_{n+1}] \left\{ \frac{f_n'(z)}{f'(z)} - \frac{c_{n+1}}{n + 1 + c_{n+1}} \right\} \\
\quad = 1 + \left( 1 + \frac{c_{n+1}}{n+1} \right) \sum_{k=n+1}^{\infty} k a_k z^{k-1} \\
\quad = 1 + \left( 1 + \frac{c_{n+1}}{n+1} \right) \sum_{k=n+1}^{\infty} k a_k z^{k-1}.
\]

We deduce that
\[
\left| \frac{1 - \omega(z)}{1 + \omega(z)} \right| \leq \frac{(1 + \frac{c_{n+1}}{n+1}) \sum_{k=n+1}^{\infty} k a_k}{2 - 2 \sum_{k=j+1}^{n} k a_k - (1 + \frac{c_{n+1}}{n+1}) \sum_{k=n+1}^{\infty} k a_k} \leq 1,
\]
if and only if
\[
2(1 + \frac{c_{n+1}}{n+1}) \sum_{k=n+1}^{\infty} k a_k \leq 2 - 2 \sum_{k=j+1}^{n} k a_k,
\]
which is equivalent to
\[
\sum_{k=j+1}^{n} k a_k + (1 + \frac{c_{n+1}}{n+1}) \sum_{k=n+1}^{\infty} k a_k \leq 1,
\]
which gives (5.2.7).

The bound in (5.2.7) is sharp for all \( n \in \mathbb{N} \) with the extremal function given by (5.2.3). \( \square \)

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