CHAPTER VI

UNSTEADY-STATE THERMOELASTIC PROBLEM IN A CIRCULAR ANNULAR FIN DUE TO INTERNAL HEAT SOURCE

The main results of this chapter have been published as detailed below:

Unsteady State Thermoelastic Problem in a Circular Annular Fin

Due to Internal Heat Source

CHAPTER VI

6.0. INTRODUCTION

The circular annular fin is found in many fields of thermal engineering such as air conditioning, heat exchangers, and microelectronics. Circular annular fin is used mostly in heat exchange devices to increase the heat transfer rate from a heat source for a given temperature difference or to decrease the temperature difference between the heat source and heat sink for a given heat flow rate. Several solutions to the problem of one-dimensional steady-state condition within an annular fin of constant thickness have been presented [1],[2],[3],[5],[7] and [8]. Navneet Lamba et al. [6] have discussed the temperature and thermal stresses on isotropic circular fin. The typical problem of transient thermal stresses in one-dimensional steady-state condition within an annular fin of constant annular fin have been investigated by Wu [9].

In this chapter, an attempt has been made to generalize the one-dimensional problem considered by Wu [6] and obtained the exact solution of two-dimensional transient heat equation problem with radiation boundary condition subjected to internal heat source and temperature distribution, displacement and stress function for annular fin have been obtained. We developed the analysis for temperature field for heating processes by using Marchi-Zgrablich and Laplace transform technique with boundary condition of radiation type.

6.1 FORMULATION OF THE PROBLEM

We consider circular annular fin (Fig. 6.1) occupying the space
\[ D = \{(x, y, z) \in \mathbb{R}^3 : a \leq r \leq b, 0 \leq z \leq l\} \],
where \( r = \sqrt{x^2 + y^2} \) the material of fin is isotropic homogeneous and all properties are assumed to be constant. The governing equations and boundary condition for the stress field [1, 3, 7, 9] are
A nonzero stress strain-displacement equation
\[ \varepsilon_r = \frac{\partial u}{\partial r}, \varepsilon_\phi = \frac{u}{r} \] (6.1.1)

A single equilibrium equation
\[ \frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\phi}{r} = 0 \] (6.1.2)

Two equation of stress strain equation relation as
\[ \sigma_r = \frac{E}{1-\nu^2} [\varepsilon_r - \nu \varepsilon_\phi - (1 + \nu)\alpha T] \] (6.1.3)
\[ \sigma_\phi = \frac{E}{1-\nu^2} [\varepsilon_r - \nu \varepsilon_\phi + (1 + \nu)\alpha T] \] (6.1.4)

and boundary condition
\[ \sigma_r = 0 \text{ at } r = a, \sigma_\phi = 0 \text{ at } r = b \] (6.1.5)

Combining equation (6.1.1) to (6.1.4), integrating twice with respect to \( r \) and applying the boundary condition, we obtain, the stress strain displacement relation as
\[ \sigma_r = \frac{-\alpha E}{r^2} \int_a^r (T - T_\infty) \eta \, d\eta + \frac{\alpha E}{b^2-a^2} \left( 1 - \frac{a^2}{r^2} \right) \int_a^b (T - T_\infty) \eta \, d\eta \] (6.1.6)
\[ \sigma_\phi = -\alpha E(T - T_\infty) + \frac{\alpha E}{r^2} \int_a^r (T - T_\infty) \eta \, d\eta \]
\[ \quad + \frac{\alpha E}{b^2-a^2} \left( 1 + \frac{a^2}{r^2} \right) \int_a^b (T - T_\infty) \eta \, d\eta \] (6.1.7)

Introduce dimensionless quantities \( \theta, \xi, \tau, R, S_r, S_\phi \) in (6.1.6), (6.1.7)
\[ S_r = \frac{-\alpha E}{r^2} \int_1^\xi \theta \xi d\xi + \frac{1}{\xi^2} \int_1^R \theta \xi d\xi \] (6.1.8)
\[ S_\phi = -\theta + \frac{1}{\xi^2} \int_1^\xi \theta \xi d\xi + \frac{1}{\xi^2} \int_1^R \theta \xi d\xi \] (6.1.9)

unsteady–state conduction with an isotropic circular annular fin with internal heat source must satisfy two dimensional equation which in cylindrical coordinate can be obtained by substituting the radial and tangential stresses in to stress equilibrium equation ,lead as \( a \leq r \leq b, \ 0 \leq z \leq l, t > 0 \),
\[ \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right) - \frac{2h}{l k'} (T - T_\infty) + \frac{\psi(r,z,t,T)}{k'} = \frac{1}{k} \frac{\partial T}{\partial t} , \]
\[ a \leq r \leq b, 0 \leq z \leq l, t > 0 \] (6.1.10)
Substitute dimensionless parameters

\[ \theta = \frac{(T - T_\infty)}{T_a - T_\infty}, \quad \xi = \frac{r}{a}, \quad \zeta = \frac{z}{a}, \quad L = \frac{l}{a}, \quad \tau = \frac{k't}{\rho a}, \quad R = \frac{b}{a}, \quad N^2 = \frac{2h^2}{k'l} \]

\[ \psi(r, z, t, T) = \Phi(r, z, t) + \varepsilon(t)T(r, z, t) \] as in \([4, 6, 8]\),

\[ \chi(\xi, \zeta, \tau) = \Phi(r, z, t)e^{-\int_0^t \varepsilon(y) \, dy} \]

For sake of brevity,

we consider \(\chi(r, z, t) = \frac{\delta(r-r_0)\delta(z-z_0)}{2\pi r_0}e^{-\omega t}\), we obtain

\[ \frac{\partial^2 \theta}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial \theta}{\partial \xi} + \frac{\partial^2 \theta}{\partial \zeta^2} - N^2 \theta + \frac{\chi(\xi, \zeta, \tau)}{k'} = \frac{1}{k} \frac{\partial \theta}{\partial t} \]

subject to initial and boundary conditions

\[ M_\tau(\theta, 1, 0, 0) = 0 \] for \(1 \leq \xi \leq R, 0 \leq \zeta \leq L, \tau = 0 \) \hspace{1cm} (6.1.12)

\[ M_\xi(\theta, 1, k_1, 1) = 0, \quad M_\xi(\theta, 1, k_2, R) = F_1(\zeta, \tau), \quad \text{for} \quad 0 \leq \zeta \leq L, \tau > 0 \] \hspace{1cm} (6.1.13)

\[ M_\zeta(\theta, 1, 0, 0) = 0, \quad M_\zeta(\theta, 1, 0, L) = f(\xi, \tau), \text{for} \quad 1 \leq \xi \leq R, \tau > 0 \] \hspace{1cm} (6.1.14)

being \(M_\theta(f, \kappa, \bar{\kappa}, \bar{\mu}) = (\bar{k}f + \bar{\kappa}\bar{f})\) \(\theta = \$\) \hspace{1cm} (6.1.15)

where the dot denotes differentiation with respect to \(\theta\), \(k_1, k_2\) are radiation constant on the curved surface of the annular disc thus equations (6.1.1) to (6.1.15) constitute the mathematical formulation of the heating problem under consideration.

### 6.2 SOLUTION OF THE PROBLEM

Applying March-Zgrablich integral transform stated in (1.2.30) to the (6.1.11), (6.1.12) and (6.1.14) using (6.1.13), we obtain

\[ \frac{d^2 \bar{\theta}}{d \xi^2} - (\mu_n^2 + N^2) \bar{\theta} + \frac{\bar{\kappa}}{k'} = \frac{1}{k} \frac{d \bar{\theta}}{d t} + Q \] \hspace{1cm} (6.2.1)

where \( Q = -\frac{R}{k_2} S_0(k_1, k_2, \mu_n R)F_1(\zeta, \tau) \)
and the eigen values $\mu_n$ are the solution of the equation

$$J_0(k_1,\mu)Y_0(k_2,\mu R) - J_0(k_2,\mu R)Y_0(k_1,\mu) = 0 \quad (6.2.2)$$

$$M_\tau(\bar{\theta}, 1,0,0) = 0 \quad (6.2.3)$$

$$M_\xi(\bar{\theta}, 1,0,0) = 0 \quad (6.2.4)$$

$$M_\xi(\bar{\theta}, 1,0,L) = \bar{f}(\xi, \tau) \quad (6.2.5)$$

where $\bar{\theta}$ denotes the Marchi-Zgrablich integral transform of $\theta$ and $\tau$ is Marchi-Zgrablich integral transform parameter, $k_1, k_2$ are radiation constants.

Applying Laplace transform to (6.2.1), (6.2.4), (6.2.5), and using (6.2.3), we obtain

$$\frac{d^2\tilde{\varphi}^*}{dt^2} - q^2\tilde{\varphi}^* = k \left( Q^* + \frac{\bar{x}}{k} \right) \quad (6.2.7)$$

where $q^2 = \mu_n^2 + N^2 + \frac{s}{k}$

$$Q^* = -\frac{R}{k_2} S_0(k_1, k_2, \mu_n R) F_1^* (\zeta, \tau)$$

$$M_\xi(\tilde{\theta}^*, 1,0,0) = 0 \quad (6.2.9)$$

$$M_\xi(\tilde{\theta}^*, 1,0,L) = \tilde{f}^*(\xi, s) \quad (6.2.10)$$

where $\tilde{\theta}^*$ denotes the Laplace transform of $\bar{\theta}$ and $s$ is the Laplace transform parameter.

The equation (6.2.7) is second order differential equation whose solution is in the form

$$\tilde{\varphi}^* = Ae^{q\xi} + Be^{-q\xi} + PI \quad (6.2.11)$$

where $PI = \frac{1}{D^2-q^2} \left\{ k \left( Q^* + \frac{\bar{x}}{k} \right) \right\}$, $D \equiv \frac{d}{dt}$ and $A, B$ are constants.

using equation (5.2.9), (5.2.10) in (5.2.11), we obtain

$$0 = (1 + cq)A + (1 - cq) B + [PI]_{\xi=0} + c \left\{ \frac{dPI}{d\xi} \right\}_{\xi=0} \quad (6.2.12)$$
\[ \bar{f}^* = (1 + cq)e^{qL}A + (1 - cq)e^{-qL}B + [PI]_{\xi = L} + c \left[ \frac{dPI}{d\xi} \right]_{\xi = L} \]  
(6.2.13)

Solving (6.2.12) and (6.2.13), we obtain

\[ A = \frac{\left[ -[PI]_{\xi = 0} - c \left[ \frac{dPI}{d\xi} \right]_{\xi = 0} \right] e^{-qL} - \left[ \bar{f}^* - [PI]_{\xi = L} - c \left[ \frac{dPI}{d\xi} \right]_{\xi = L} \right] e^{qL}}{-2(1 + cq)\sin h(qL)} \]

\[ B = \frac{\left[ -[PI]_{\xi = 0} - c \left[ \frac{dPI}{d\xi} \right]_{\xi = 0} \right] e^{qL} - \left[ \bar{f}^* - [PI]_{\xi = L} - c \left[ \frac{dPI}{d\xi} \right]_{\xi = L} \right] e^{-qL}}{2(1 - cq)\sin h(qL)} \]

Substitute the value of A and B in (6.2.11), we obtain

\[ \bar{\theta}^*(n, \zeta, s) = \frac{\left[ \bar{f}^* - [PI]_{\xi = L} - c \left[ \frac{dPI}{d\xi} \right]_{\xi = L} \right] \sin h[q \zeta] - cq \cosh[q \zeta]}{(1 - cq^2)\sinh(qL)} \]
\[ - \frac{\left[ -[PI]_{\xi = 0} - c \left[ \frac{dPI}{d\xi} \right]_{\xi = 0} \right] \sinh[q \zeta] - cq \cosh[q \zeta]}{(1 - cq^2)\sinh(qL)} + PI \]  
(6.2.14)

Applying inverse Laplace transform to the equation (6.2.14), we obtain

\[ \bar{\theta}(n, \zeta, t) = L^{-1} \left[ \frac{\left[ \bar{f}^* - [PI]_{\xi = L} - c \left[ \frac{dPI}{d\xi} \right]_{\xi = L} \right] \sinh[q \zeta] - cq \cosh[q \zeta]}{(1 - cq^2)\sinh(qL)} \right] \]
\[ - L^{-1} \left[ \frac{\left[ -[PI]_{\xi = 0} - c \left[ \frac{dPI}{d\xi} \right]_{\xi = 0} \right] \sinh[q \zeta] - cq \cosh[q \zeta]}{(1 - cq^2)\sinh(qL)} \right] \]
\[ + L^{-1} [PI] \]  
(6.2.15)

To evaluate \( L^{-1} \left[ \bar{f}^*(\xi, s) \tilde{g}^*_1(s) \right] \)  
(6.2.16)

where \( \tilde{g}^*_1(s) = \frac{\sin h[q \xi]}{\sin h(qL)} \)  
(6.2.17)

Using inverse integral (1.2.4) to the equation (6.2.16), we obtain

\[ \tilde{g}_1(t) = \frac{1}{2\pi i} \int_{c-I\infty}^{c+I\infty} e^{st} \frac{\sin h[q \xi]}{\sin h(qL)} ds \]  
(6.2.18)

Now calculate the inverse integral (6.2.18) where c is greater than the real part of all singularities of integrand. The integral is single valued function
of s in region bounded by the closed Bromwich contour of the figure give below

![Bromwich contour diagram]

The line NL is chosen so as to lie all the poles to the right, which are given by \( s = s_m = \frac{im\pi}{L}, m = 1,2, \ldots \)

Choosing the contour so that the curved portion LMN is an arc of the circle \( \Gamma \) with center at origin and radius \( R \) so that it will not pass through zero of \( \sinh(qL) \) The integral over the circular arc tends to zero as \( m \) tends to zero

Now \( \sinh(qL) = 0 \) gives \( q = \frac{im\pi}{L} \) ie \( s = s_m = \frac{im\pi}{L}, m = 1,2, \ldots \)

Residue at \( s_m \)

\[
= \lim_{s \to s_m} \left[ \frac{s - s_m}{\sinh \left( L \sqrt{\frac{\mu_n^2 + N^2 + \frac{s}{\alpha}}{\alpha}} \right)} e^{st} \sinh \left[ \int \frac{\mu_n^2 + N^2 + \frac{s}{\alpha}}{\alpha} \right] \right] \\
= \frac{2k\text{im} \pi}{L^2 \cos \pi m} \left( \frac{1}{i} \right) \sin \left( i\frac{m\pi}{L} \right) e^{-kt \left( \mu_n^2 + N^2 + \left( \frac{m\pi}{L} \right)^2 \right)} \quad (6.2.19)
\]

hence the value of \( \tilde{g}_1(t) \) is given by

\[
\tilde{g}_1(t) = \frac{2k\pi}{L^2} \sum_{m=1}^{\infty} (-1)^{m+1} m \sin(\lambda_m \zeta) e^{-kt(\mu_n^2 + N^2 + \lambda_m^2)} , \quad (6.2.20)
\]

\( \lambda_m = \frac{m\pi}{L} \)

Applying the convolution theorem stated in (1.2.3) to the equation (6.2.16), we obtain
\[
\frac{2k\pi}{L^2} \sum_{m=1}^{\infty} (-1)^{m+1} m \sin(\lambda_m \zeta) \int_0^\tau \bar{f}(\xi, \tau) e^{-k(\mu_n^2 + N^2 + \lambda_m^2)(\tau - \tau')} d\tau'
\]

Similarly, we can find Laplace inverse from (6.2.15), we obtain

\[
\tilde{\theta}(n, \zeta, t) = \frac{2k\pi}{L^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{1+c\lambda_m^2} m [\sin(\lambda_m \zeta) - c\lambda_m \cos(\lambda_m \zeta)]
\]

\[
\times \int_0^\tau \left[ \bar{f} - [PI]_{\xi=L} - c \left[ \frac{d[PI]}{dz} \right]_{\xi=L} \right] e^{-k(\mu_n^2 + N^2 + \lambda_m^2)(\tau - \tau')} d\tau'
\]

\[
= \frac{2k\pi}{L^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{1+c\lambda_m^2} m [\sin(\lambda_m \zeta) - c\lambda_m \cos(\lambda_m \zeta)]
\]

\[
\times \int_0^\tau \left[ - [PI]_{\xi=0} - c \left[ \frac{d[PI]}{dz} \right]_{\xi=0} \right] e^{-k(\mu_n^2 + N^2 + \lambda_m^2)(\tau - \tau')} d\tau'
\]

\[
+ L^{-1}[PI] (6.2.21)
\]

Applying inverse March-Zgrablich integral transform stated in (1.2.31) to the equation (6.2.21), we obtain, the expression for temperature as

\[
\theta(\xi, \zeta, t) = \frac{2k\pi}{L^2} \sum_{n=1}^{\infty} \frac{S_0(k_1, k_2, \mu_n \xi)}{\mu_n \zeta}
\]

\[
\times \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{1+c\lambda_m^2} m [\sin(\lambda_m \zeta) - c\lambda_m \cos(\lambda_m \zeta)]
\]

\[
\times \int_0^\tau \left[ \bar{f} - [PI]_{\xi=L} - c \left[ \frac{d[PI]}{dz} \right]_{\xi=L} \right] e^{-k(\mu_n^2 + N^2 + \lambda_m^2)(\tau - \tau')} d\tau'
\]

\[
- \frac{2k\pi}{L^2} \sum_{n=1}^{\infty} \frac{S_0(k_1, k_2, \mu_n \xi)}{\mu_n \zeta}
\]

\[
\times \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{1+c\lambda_m^2} m [\sin(\lambda_m \zeta) - c\lambda_m \cos(\lambda_m \zeta)]
\]

\[
\times \int_0^\tau \left[ - [PI]_{\xi=0} - c \left[ \frac{d[PI]}{dz} \right]_{\xi=0} \right] e^{-k(\mu_n^2 + N^2 + \lambda_m^2)(\tau - \tau')} d\tau'
\]

\[
+ \sum_{n=1}^{\infty} \frac{S_0(k_1, k_2, \mu_n \xi)}{\mu_n \zeta} L^{-1}[PI] (6.2.22)
\]

where \(\mu_m\) are the positive roots of

\(J_0(k_1, \mu)Y_0(k_2, \mu R) - J_0(k_2, \mu R)Y_0(k_1, \mu) = 0\), \(\lambda_m = \frac{mn}{L}\)
6.3 DETERMINATION THERMAL STRESSES

Using (6.2.22) in (6.1.8), (6.1.9), we obtain the radial and tangential stresses are

\[ S_r = -\frac{2k\pi}{L^2} \frac{1}{\xi^2} \]

\[ \times \int_1^\xi \sum_{m,n=1}^{\infty} \frac{(1)^{m+1}S_0(k_1,k_2,\mu_n,\xi)}{c_n(1+c\lambda_m^2)} m[\sin(\lambda_m \zeta) - c\lambda_m \cos(\lambda_m \zeta)] \]

\[ \times \int_0^\tau \left[ -[\Pi]_{\zeta=0} - c \left[ \frac{d\Pi}{dz} \right]_{\zeta=0} \right] e^{-k(\mu_n^2+N^2+\lambda_m^2)(\tau-\tau')} d\tau' d\xi \]

\[ + \frac{2k\pi}{L^2} \frac{1}{\xi^2} \int_1^\xi \sum_{m,n=1}^{\infty} \frac{(1)^{m+1}S_0(k_1,k_2,\mu_n,\xi)}{c_n(1+c\lambda_m^2)} m[\sin(\lambda_m \zeta) - c\lambda_m \cos(\lambda_m \zeta)] \]

\[ \times \int_0^\tau \left[ -[\Pi]_{\zeta=0} - c \left[ \frac{d\Pi}{dz} \right]_{\zeta=0} \right] e^{-k(\mu_n^2+N^2+\lambda_m^2)(\tau-\tau')} d\tau' d\xi \]

\[ - \frac{2k\pi}{L^2} \frac{1}{\xi^2} \sum_{m,n=1}^{\infty} \frac{(1)^{m+1}S_0(k_1,k_2,\mu_n,\xi)}{c_n(1+c\lambda_m^2)} m[\sin(\lambda_m \zeta) - c\lambda_m \cos(\lambda_m \zeta)] \]

\[ \times \int_0^\tau \left[ -[\Pi]_{\zeta=0} - c \left[ \frac{d\Pi}{dz} \right]_{\zeta=0} \right] e^{-k(\mu_n^2+N^2+\lambda_m^2)(\tau-\tau')} d\tau' d\xi \]

\[ + \frac{2k\pi}{L^2} \frac{1}{\xi^2} \sum_{m,n=1}^{\infty} \frac{(1)^{m+1}S_0(k_1,k_2,\mu_n,\xi)}{c_n(1+c\lambda_m^2)} m[\sin(\lambda_m \zeta) - c\lambda_m \cos(\lambda_m \zeta)] \]

\[ \times \int_0^\tau \left[ -[\Pi]_{\zeta=0} - c \left[ \frac{d\Pi}{dz} \right]_{\zeta=0} \right] e^{-k(\mu_n^2+N^2+\lambda_m^2)(\tau-\tau')} d\tau' d\xi \]

\[ + \frac{1}{\xi^2} \sum_{m,n=1}^{\infty} \frac{(1)^{m+1}S_0(k_1,k_2,\mu_n,\xi)}{c_n(1+c\lambda_m^2)} \]

\[ L^{-1}[\Pi] d\xi \]

\[ S_\phi = -\frac{2k\pi}{L^2} \sum_{m=1}^{\infty} \frac{S_0(k_1,k_2,\mu_n,\xi)}{c_n(1+c\lambda_m^2)} m[\sin(\lambda_m \zeta) - c\lambda_m \cos(\lambda_m \zeta)] \]

\[ \times \int_0^\tau \left[ -[\Pi]_{\zeta=0} - c \left[ \frac{d\Pi}{dz} \right]_{\zeta=0} \right] e^{-k(\mu_n^2+N^2+\lambda_m^2)(\tau-\tau')} d\tau' \]

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\[ + \frac{2k\pi}{L^2} \sum_{n=1}^{\infty} \frac{S_0(k_1k_2\mu_n\xi)}{c_n} \]
\[ \times \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{1+c\lambda_m^2} m[Sin(\lambda_m\zeta) - c\lambda_m Cos(\lambda_m\zeta)] \]
\[ \times \int_0^\tau \left[ -[PI]_{\zeta=0} - c \left[ \frac{dp_1}{d\zeta} \right]_{\zeta=0} \right] e^{-k(\mu_n^2+N^2+\lambda_m^2)(\tau-\tau')} d\tau' \]
\[ - \sum_{n=1}^{\infty} \frac{S_0(k_1k_2\mu_n\xi)}{c_n} L^{-1}[PI] \]
\[ + \frac{2k\pi}{L^2} \int_1^\xi \sum_{n=1}^{\infty} \frac{(-1)^{m+1}e^{-\mu_n^2+N^2+\lambda_m^2}}{c_n(1+c\lambda_m^2)} m[Sin(\lambda_m\zeta) - c\lambda_m Cos(\lambda_m\zeta)] \]
\[ \times \int_0^\tau \left[ \tilde{f} - [PI]_{\zeta=L} - c \left[ \frac{dp_1}{dz} \right]_{\zeta=L} \right] e^{-k(\mu_n^2+N^2+\lambda_m^2)(\tau-\tau')} d\tau' d\xi \]
\[ - \frac{2k\pi}{L^2} \int_1^\xi \sum_{n=1}^{\infty} \frac{(-1)^{m+1}e^{-\mu_n^2+N^2+\lambda_m^2}}{c_n(1+c\lambda_m^2)} m[Sin(\lambda_m\zeta) - c\lambda_m Cos(\lambda_m\zeta)] \]
\[ \times \int_0^\tau \left[ -[PI]_{\zeta=0} - c \left[ \frac{dp_1}{d\zeta} \right]_{\zeta=0} \right] e^{-k(\mu_n^2+N^2+\lambda_m^2)(\tau-\tau')} d\tau' d\xi \]
\[ + \frac{1}{\xi^2} \int_1^\xi \sum_{n=1}^{\infty} \frac{\xi S_0(k_1k_2\mu_n\xi)}{c_n} L^{-1}[PI] d\xi \]
\[ + \frac{2k\pi}{L^2} \int_1^{\xi^2-1} \int_1^R \sum_{n=1}^{\infty} \frac{(-1)^{m+1}e^{-\mu_n^2+N^2+\lambda_m^2}}{c_n(1+c\lambda_m^2)} m[Sin(\lambda_m\zeta) - c\lambda_m Cos(\lambda_m\zeta)] \]
\[ \times \int_0^\tau \left[ \tilde{f} - [PI]_{\zeta=L} - c \left[ \frac{dp_1}{dz} \right]_{\zeta=L} \right] e^{-k(\mu_n^2+N^2+\lambda_m^2)(\tau-\tau')} d\tau' d\xi \]
\[ - \frac{2k\pi}{L^2} \int_1^{\xi^2-1} \int_1^R \sum_{n=1}^{\infty} \frac{(-1)^{m+1}e^{-\mu_n^2+N^2+\lambda_m^2}}{c_n(1+c\lambda_m^2)} m[Sin(\lambda_m\zeta) - c\lambda_m Cos(\lambda_m\zeta)] \]
\[ \times \int_0^\tau \left[ -[PI]_{\zeta=0} - c \left[ \frac{dp_1}{d\zeta} \right]_{\zeta=0} \right] e^{-k(\mu_n^2+N^2+\lambda_m^2)(\tau-\tau')} d\tau' d\xi \]
\[ + \frac{1}{\xi^2} \int_1^{\xi^2-1} \sum_{n=1}^{\infty} \frac{\xi S_0(k_1k_2\mu_n\xi)}{c_n} L^{-1}[PI] d\xi \]
6.4 SPECIAL CASE

Setting,

\[ f(\xi, \tau) = (1 - e^{-\tau}) \frac{\delta(\xi - \xi_0)}{2\pi\xi} \zeta \]  \hspace{1cm} (6.4.1)

where \( \delta \) Dirac-delta function.

Applying the finite Marchi-Zgrablich integral transform to (6.4.1), we obtain

\[ \tilde{f}(n, \tau) = \int_{1}^{\infty} \xi(1 - e^{-\tau}) \frac{\delta(\xi - \xi_0)}{2\pi\xi} \zeta_0(k_1, k_2, \mu_n, \xi) d\xi \]

and \( \chi(\xi, \zeta, \tau) = \frac{\delta(\xi - \xi_0)}{2\pi\xi} g(\tau) \delta(\zeta), Q=0 \),

where \( g(\tau) = \text{constant} \)  \hspace{1cm} (6.4.2)

\[ [\Pi]_{\xi=L} + c \left[ \frac{d\Pi}{d\xi} \right]_{\xi=L} = A1, \text{cons.} \]

\[ [\Pi]_{\xi=0} + c \left[ \frac{d\Pi}{d\xi} \right]_{\xi=0} = 0 \]

\( L^{-1}[\Pi] = \text{fun of } \xi = B1\xi \), we obtain

\[ \theta(\xi, \zeta, t) = \frac{2k\pi}{L^2} \sum_{n=1}^{\infty} \frac{S_0(k_1, k_2, \mu_n, \xi)S_0(k_1, k_2, \mu_n, \xi_0)}{C_n} \]

\[ \times \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(1+c\lambda_m^2)} m[Sin(\lambda_m \zeta) - c\lambda_m Cos(\lambda_m \xi)] \]

\[ \times \int_{0}^{\infty} \left[ (1 - e^{-\tau}) - A1 \right] e^{-k(\mu_n^2 + N^2 + \lambda_m^2)(\tau - \tau')} d\tau' \]

\[ + \sum_{n=1}^{\infty} \frac{S_0(k_1, k_2, \mu_n, \xi)S_0(k_1, k_2, \mu_n, \xi_0)}{C_n} B1\xi \]  \hspace{1cm} (6.4.3)

\[ S_r = -\frac{2k\pi}{L^2} \int_{1}^{\xi} \sum_{n, n=1}^{\infty} \frac{(-1)^{m+1}S_0(k_1, k_2, \mu_n, \xi)S_0(k_1, k_2, \mu_n, \xi_0)}{C_n(1+c\lambda_m^2)} \]

\[ \times m[Sin(\lambda_m \zeta) - c\lambda_m Cos(\lambda_m \xi)] \]

\[ \times \int_{0}^{\infty} \left[ (1 - e^{-\tau}) - A1 \right] e^{-k(\mu_n^2 + N^2 + \lambda_m^2)(\tau - \tau')} d\tau' \]

\[ -\frac{1}{\xi^2} \int_{1}^{\xi} \sum_{n=1}^{\infty} \frac{\xi S_0(k_1, k_2, \mu_n, \xi)S_0(k_1, k_2, \mu_n, \xi_0)}{C_n} \] \hspace{1cm} B1\xi d\xi
\[ S_\varphi = \frac{2k\pi}{L^2} \sum_{n=1}^{\infty} \frac{S_0(k_1,k_2,\mu_n,\xi)S_0(k_1,k_2,\mu_n,\xi_0)}{c_n} \times m[\sin(\lambda_m \xi) - c\lambda_m \cos(\lambda_m \xi)] \]

\[ \times \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(1+c\lambda_m \xi)} \times \sum_{n=1}^{\infty} \frac{(-1)^{m+1}}{(1+c\lambda_m \xi)} \times m[\sin(\lambda_m \xi) - c\lambda_m \cos(\lambda_m \xi)] \]

\[ \times \int_0^{\tau} [(1 - e^{-\tau'}) - A1] e^{-k(\mu_n^2 + N^2 + \lambda_m^2)(\tau - \tau')} d\tau' d\xi \]

\[ + \frac{1}{\xi^2} \int_1^{\xi} \sum_{n=1}^{\infty} \frac{\xi S_0(k_1,k_2,\mu_n,\xi)S_0(k_1,k_2,\mu_n,\xi_0)}{c_n} B1\xi d\xi \]

\[ \times m[\sin(\lambda_m \xi) - c\lambda_m \cos(\lambda_m \xi)] \]

\[ \times \int_0^{\tau} [(1 - e^{-\tau'}) - A1] e^{-k(\mu_n^2 + N^2 + \lambda_m^2)(\tau - \tau')} d\tau' d\xi \]

\[ + \frac{2k\pi}{L^2} \frac{\xi^2-1}{\xi^2} \int_1^{\xi} \sum_{n=1}^{\infty} \frac{(-1)^{m+1}}{(1+c\lambda_m \xi)} \times \sum_{n=1}^{\infty} \frac{(-1)^{m+1}}{(1+c\lambda_m \xi)} \times m[\sin(\lambda_m \xi) - c\lambda_m \cos(\lambda_m \xi)] \]

\[ \times \int_0^{\tau} [(1 - e^{-\tau'}) - A1] e^{-k(\mu_n^2 + N^2 + \lambda_m^2)(\tau - \tau')} d\tau' d\xi \]

\[ + \frac{1}{\xi^2} \int_1^{\xi} \sum_{n=1}^{\infty} \frac{\xi S_0(k_1,k_2,\mu_n,\xi)S_0(k_1,k_2,\mu_n,\xi_0)}{c_n} B1\xi d\xi \]

\[ + \frac{2k\pi}{L^2} \frac{\xi^2-1}{\xi^2} \int_1^{\xi} \sum_{n=1}^{\infty} \frac{(-1)^{m+1}}{(1+c\lambda_m \xi)} \times \sum_{n=1}^{\infty} \frac{(-1)^{m+1}}{(1+c\lambda_m \xi)} \times m[\sin(\lambda_m \xi) - c\lambda_m \cos(\lambda_m \xi)] \]

\[ \times \int_0^{\tau} [(1 - e^{-\tau'}) - A1] e^{-k(\mu_n^2 + N^2 + \lambda_m^2)(\tau - \tau')} d\tau' d\xi \]

\[ + \frac{1}{\xi^2} \int_1^{\xi} \sum_{n=1}^{\infty} \frac{\xi S_0(k_1,k_2,\mu_n,\xi)S_0(k_1,k_2,\mu_n,\xi_0)}{c_n} B1\xi d\xi \]  

\[ \text{(6.4.4)} \]

\[ \text{6.5 NUMERICAL RESULT} \]

The numerical calculation have been carried out for low carbon steel (AISI) with parameter $k_1 = k_2 = 1, c=1$, radius $R=4$ cm, $\xi_0 = 1.2$ cm, $h=1$ cm $L=1,N=2$, $\omega = 1$ $\tau = 1$ $A1=0.124,B1=0.24$, thermal Conductivity $k' = 26$ Btu/hr ft $^\circ F$, thermal diffusivity $k=0.48ft^2/hr$ and $\mu_m=1.10821,2.13634, 3.17041, 4.21067, 5.2536, 6.2979, 3.7.34305,$
8.38869, 9.43467, 21.9954 are the positive root of transcendental equation of \( J_0(k_1, \mu)Y_0(k_2, \mu R) - J_0(k_2, \mu R)Y_0(k_1, \mu) = 0 \) and \( \lambda_n = 3.142, 6.284, 9.426, 12.568, 15.71, 18.853, 21.994, 25.136, 28.278, 31.42 \) are the positive root of Transcendental equation of \( \lambda_m = \frac{\pi m}{l} \), we obtain

\[
\theta(\xi, \zeta, t) = 2.4 \sum_{n=1}^{\infty} \frac{S_0(1,1,\mu_n \xi)S_0(1,1,\mu_n 1.2)}{\int_1^{\tau} [\xi S_0(k_1,k_2,\mu_n \xi)]^2 d\xi} \times \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(1+\pi m^2)} m[Sin(\pi m \xi) - c \pi m Cos(\pi m \xi)] \\
\times \int_0^{\tau} [(1 - e^{-\tau'}) - 0.124] e^{-k(\mu_n^2 + 4 + \pi m^2)(\tau - \tau')} d\tau' \\
+ \sum_{n=1}^{\infty} \frac{S_0(1,1,\mu_n \xi)S_0(1,1,\mu_n 1.2)}{\int_1^{\tau} [\xi S_0(k_1,k_2,\mu_n \xi)]^2 d\xi} \times 0.24 \xi d\xi
\]

\[
S_r = -2.4 \frac{1}{\xi_2} \sum_{n=1}^{\infty} \frac{\xi(-1)^{n+1} S_0(1,1,\mu_n \xi)S_0(1,1,\mu_n 1.2)}{\int_1^{\tau} [\xi S_0(k_1,k_2,\mu_n \xi)]^2 d\xi} \times \sum_{n=1}^{\infty} \frac{\xi S_0(1,1,\mu_n \xi)S_0(1,1,\mu_n 1.2)}{\int_1^{\tau} [\xi S_0(k_1,k_2,\mu_n \xi)]^2 d\xi} \times 0.2 \xi d\xi \\
\times m[Sin(\pi m \xi) - c \pi m Cos(\pi m \xi)] \\
\times \int_0^{\tau} [(1 - e^{-\tau'}) - 0.124] e^{-k(\mu_n^2 + 4 + \pi m^2)(\tau - \tau')} d\tau' d\xi \\
+ 2.4 \frac{1}{\xi_2} \int_1^{\tau} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \xi S_0(k_1,k_2,\mu_n \xi)S_0(k_1,k_2,\mu_n \xi_0)}{G_n(1+c\lambda_m^2)} \\
\times m[Sin(\lambda_m \xi) - c \lambda_m Cos(\lambda_m \xi)] \\
\times \int_0^{\tau} [(1 - e^{-\tau'}) - 0.124] e^{-k(\mu_n^2 + N^2 + \lambda_m^2)(\tau - \tau')} d\tau' d\xi \\
+ \frac{1}{\xi_2} \int_1^{\tau} \sum_{n=1}^{\infty} \frac{\xi S_0(1,1,\mu_n \xi)S_0(1,1,\mu_n 1.2)}{\int_1^{\tau} [\xi S_0(k_1,k_2,\mu_n \xi)]^2 d\xi} \times 0.2 \xi d\xi
\]

(6.5.1)

6.5 CONCLUSION

In this chapter, we generalized the idea proposed by Wu S.S. [9] for two dimensional non-homogeneous radiation boundary value problem of circular annular fin occupying the space

\[
D = \{(x,y,z) \in \mathbb{R}^3 : a \leq r \leq b, 0 \leq z \leq l \},
\]

where \( r = \sqrt{x^2 + y^2} \) the material of fin is isotropic homogeneous and all properties are assumed to
be constant with heat source and temperature distribution, displacement and stress function for annular fin have been obtained. We developed the analysis for temperature field for heating processes by using Marchi-Zgrablich and Laplace transforms technique with boundary condition of radiation type. Although some authors are finding the temperature distribution, displacement and stress function for annular fin with first and second kind boundary condition, they can not apply internal heat source. The series solution is converges since the thickness of annular fin is very small. Any particular case of special interest may be derived by assigning suitable value of the parameter and function in the series expansion. The result can be applied to the design of useful structures or machines in engineering applications.

REFERENCES


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**Nomenclature**

- \( a, b \): Inner and outer radii of the fin
- \( c \): Specific heat of material of the fin
- \( c_1, c_2 \): Constants
- \( E \): Young’s modulus of material of the fin
- \( h \): Heat transfer coefficient
- \( k \): Thermal conductivity of material of the fin
- \( N \): Dimensionless parameter
- \( q_b \): Heat flux from the base of the fin
- \( S_r, S_\theta \): Dimensionless radial and tangential stresses
- \( T \): Temperature of the fin
- \( T_\infty \): Ambient temperature
- \( U \): Radial displacement
- \( \varepsilon_r, \varepsilon_\theta \): Radial and tangential strains
- \( \sigma_r, \sigma_\theta \): Radial and tangential stresses
- \( r, \phi \): Polar coordinates
- \( t \): Time
- \( R \): Dimensionless outer radius
- \( S_r, S_\theta \): Dimensionless tangential stresses
- \( \theta \): Dimensionless temperature of the fin
- \( \tau \): Dimensionless time
- \( L \): Dimensionless thickness
Graphical Analysis

Figure 6.1: shows characteristic of annular fin with radius r=a, r=b

Fig.6.2: shows that variation of temperature $\theta(r, z, t)$ Vs $r$. It is clear that temperature suddenly decreases at time $t=0.1$ sec., $t=0.2$ sec. up to zero at $r=1.5$ and slightly increase and again go to decrease due to internal heat.
Fig. 6.3: shows that variation of temperature $\theta(r, z, t)$ Vs r. It is clear that temperature suddenly decreases at value of $z=0.1$, $z=0.2$ up to zero at $r=2$ in the interval $[1.5, 2]$ due to internal heat source.

Fig. 6.4 shows that variation of thermal Stress $S_r$ Vs $z$. It is clear that thermal Stress slightly decreases at time $t=1$ sec., $t=5$ sec. and then increase due to internal heat source.
Fig. 6.5 shows that variation of thermal Stress $S_\varphi$ Vs z. It is clear that thermal Stress slightly decreases at time $t=1$ sec., $t=5$ sec. and then suddenly increase due to internal heat source.