Fig. A  Schematic diagram of the parallel plate channel flow under transverse magnetic field with porous beds.
Introduction

The problem of flow of viscous incompressible fluids in channel bounded by porous beds has generated a great deal of interest among researchers, because of its applications in numerous scientific and industrial fields such as Lubrication of porous bearings, Ground water hydrology, Petroleum industries, Industrial filtration and Agricultural engineering, etc. The common approach in such studies has been to make use of Darcy’s law in the porous medium (generally with low permeabilities) and the Navier – Stokes equations in clean fluid regions together with certain boundary conditions and interfacial conditions linking the different flow regions.

The overwhelming majority of existing studies pertain to fluid flow in porous media using Darcy’s model \((4,12,13,17,19,20)\). The main limitations of the Darcy’s model are that: it is not appropriate for fast flows, it does not satisfy the no-slip condition on a solid boundary and it does not account for the spatial variation of the porous matrix. Also Darcy’s law is found to give satisfactory results when the porous medium is densely packed. However, when a porous medium consists of a spare distribution of particles, Darcy’s law becomes inadequate and the condition of no-slip at a
solid boundary and conditions between a porous medium and clean fluid can not be completely satisfied. An alternate and improvised model was proposed by Brinkman (3) which is applicable to highly permeable media and can account for all boundary conditions at a solid surface or at a fluid interface.

Vajravelu et al (17) studied the unsteady flow of two immiscible conducting fluids between two permeable beds of different permeabilities. Nirmal C. Sacheti (7) studied the influence of bounding porous media of different permeabilities on the fully developed steady Poiseuille flow of a viscous incompressible fluid through a channel. Rajasekhara (8) investigated the Couette flow over a naturally permeable bed with an impermeable moving upper plate. Sai (10) discussed the unsteady behaviour of Poiseuille flow of viscous and incompressible in plane channel formed by two walls of which the upper one is solid and the lower one is a porous bed of infinite thickness.

The flow of two immiscible conducting fluid bounded by permeable beds is of importance in view of its applications in Petroleum and Chemical Engineering. Hartmann pioneered the study of the Magnetohydrodynamic flow of a conducting fluid in a parallel plate channel under the influence of a transverse magnetic field. It is well known that under a magnetic field the flow tend to exhibit laminar nature, since the turbulence invariably is damped by eddy currents. Extensive work has been done in the last few decades in magnetohydrodynamic flow through a channel with porous beds (15, 20, 14).

This chapter aims in investigating the flow of two immiscible fluids bounded by porous beds making use of the finite element analysis with quadratic elements. The geometry of the interface of the immiscible fluids as well as the interface between each
fluid and the adjacent porous bed are assumed to be known for the purpose of analysis. The clean fluid regions are governed by Navier-Stokes equations while the Brinkman model is used for the fluid through porous beds. The Ritz finite element analysis with line elements is used to obtain quadratic polynomial approximation solutions for the governing equations. The flow region is divided into four zones and the global matrix solution is obtained making use of interelement continuity and boundary conditions. Two cases are discussed viz. firstly, when the flow is under the influence of a transverse magnetic field and secondly the non-magnetic case. The velocity and shear stresses are evaluated at different levels parallel to the boundary walls and their behaviour is discussed computationally for variations in the governing parameters. The advantage of finite element analysis over the routine analytical investigation is that the former predicts the intricacies of the flow behaviour with in the clean fluid region as well as the porous bed region at different horizontal levels where as the later gives the overall behaviour of the flow.
Consider the flow of an incompressible, immiscible, conducting fluid in a horizontal channel bounded by permeable beds. The flow takes place due to the imposed pressure gradient on each immiscible fluid under the influence of a uniform transverse magnetic field. The entire flow configuration is divided into four zones. Zone – I \((-h \leq y \leq 0)\) corresponds to a region of clean fluid with viscosity \(\mu_1\), bounded below by porous bed. Zone – III with \((-s+h) \leq y \leq 1)\) with permeability \(k_1\). \(y = -h\) is the nominal surface separating the clean fluid and porous bed, the flow is due to an imposed pressure gradient \(\frac{dp_1}{dx}\). Zone – II \((0 \leq y \leq h)\) contains the clean fluid with viscosity \(\mu_2\), bounded above by the porous bed. Zone – IV correspond to a porous bed \((h \leq y \leq s + h)\), whose permeability is \(k_2\) and separated by the nominal surface \(y = h\). In the absence of an extraneous forces, the flow is unidirectional.

The following are the equations governing the flow in each zone.

**Zone – I**

\[
\frac{dV_1}{dx} = 0
\]

\[
\frac{d^2V_1}{dy^2} - \frac{\sigma_1 \mu_1^2 H_0^2}{\mu_1} V_1 = \frac{1}{\mu_1} \frac{dp_1}{dx}
\]  \(\text{(2.1)}\)

**Zone – II**

\[
\frac{dV_2}{dx} = 0
\]

\[
\frac{d^2V_2}{dy^2} - \frac{\sigma_1 \mu_2^2 H_0^2}{\mu_2} V_2 = \frac{1}{\mu_2} \frac{dp_2}{dx}
\]  \(\text{(2.2)}\)
\[
\frac{dV_{p1}}{dx} = 0
\]
\[
\frac{d^2V_{p1}}{dy^2} - \frac{\sigma_1\mu_1^2 H_0^2}{\mu_1} V_{p1} - \frac{1}{k_1} V_{p1} = \frac{1}{\mu_1} \frac{d\rho_1}{dx}
\] (2.3)

\[
\frac{dV_{p2}}{dx} = 0
\]
\[
\frac{d^2V_{p2}}{dy^2} - \frac{\sigma_2\mu_2^2 H_0^2}{\mu_2} V_{p2} - \frac{1}{k_2} V_{p2} = \frac{1}{\mu_2} \frac{d\rho_1}{dx}
\] (2.4)

Where \( V_1, V_2, V_{p1} \) and \( V_{p2} \) are the velocities of the fluid in zones I, II, III & IV respectively, and \( \mu_c \) is the magnetic permeability of the medium, \( \sigma_1 \) and \( \sigma_2 \) are the electrical conductivities of the fluids and \( H_0 \) is the intensity of the imposed magnetic field.

The interfacial continuity conditions are as follows:

\[
V_1 = V_{p1} : y = -h
\]

\[
V_2 = V_{p2} : y = h
\] (2.5)

\[
V_1 = V_2 & \quad \mu_1 \frac{dV_1}{dy} = \mu_2 \frac{dV_2}{dy} : y = 0
\]

The boundary conditions on the impermeable wall are given by

\[
V_{p1} = 0 \quad ; \quad y = -(h+s)
\]

\[
V_{p2} = 0 \quad ; \quad y = (h+s)
\] (2.6)
We define the following non-dimensional variable:

\[ V_1' = \frac{V_1}{V_{av}}, \quad V_2' = \frac{V_2}{V_{av}}, \quad \rho_1' = \frac{\rho_1}{\rho_1 V_{av}}, \quad \rho_2' = \frac{\rho_2}{\rho_2 V_{av}}, \quad \tau' = \frac{x}{h}, \quad \mu' = \frac{\mu_1}{\mu} \]  

\[ x' = \frac{x}{h}, \quad y' = \frac{y}{h} \]

Where \( V_{av} \) is the characteristic average velocity in the fluid region

\( \rho_1 \) is the density of the lower immiscible fluid

\( \rho_2 \) is the density of the upper immiscible fluid.

Making use of these nondimensional variables in equations (2.1)-(2.6) (dropping asterisks) we obtain

\[
\frac{d^2 V_1}{dy^2} - M_1 V_1 = P_1
\]  

(2.8)

\[
\frac{d^2 V_2}{dy^2} - M_2 V_2 = P_2
\]  

(2.9)

\[
\frac{d^2 \rho_1}{dy^2} - M_{11} \rho_1 = P_1
\]  

(2.10)

\[
\frac{d^2 \rho_2}{dy^2} - M_{31} \rho_2 = P_2
\]  

(2.11)

where \( P_1 = R_1 \frac{dp_1}{dx} \)

\( P_2 = R_2 \frac{dp_2}{dx} \)

\( R_1 = \frac{\mu_1 V_{av}}{\rho_1} \) is the Reynolds number in Zone 1
\[ R_2 = \frac{h \rho_i V_{\infty}}{\mu_2} \] is the Reynolds number in Zone II.

\[ D_1 = \frac{k^2}{k_I} \] is the Darcy parameter in Zone III.

\[ D_2 = \frac{k^2}{k_2} \] is the Darcy parameter in Zone IV.

\[ M_1 = \mu, H_0, h \sqrt{\frac{\sigma_1}{\mu_1}} \] is the Hartmann number in Zone I.

\[ M_2 = \mu, H_0, h \sqrt{\frac{\sigma_2}{\mu_2}} \] is the Hartmann number in Zone II.

\[ M_{1D} = M_1 + D_1 \]

\[ M_{2D} = M_2 + D_2 \]

The interfacial and boundary conditions in the nondimensional form are given by:

\[ V_1 = V_{p1} \quad : \quad y = -1 \]  \hspace{1cm} (2.12)

\[ V_1 = V_{p2} \quad : \quad y = 1 \]

\[ V_1 = V_{p2} \quad : \quad y = 0 \]

\[ V_{\eta_1} = 0 \quad : \quad y = -(1 + \tau) \]  \hspace{1cm} (2.14)

\[ V_{\eta_2} = 0 \quad : \quad y = (1 + \tau) \]
3. Finite Element Analysis

The variational formulation corresponding to zone I is given by

\[ 0 = \int_{y_1}^{y_2} \left( \frac{dW_1(y)}{dy} \frac{dV_1(y)}{dy} + M_1V_1(y)W_1(y) + P_1W_1(y) \right) dy - W_1(y_{1+})Q_{a_1} - W_1(y_{n+})Q_{a_n} \]  (3.1)

where \( Q_{a_1} = \left( \frac{dV_1}{dy} \right)_{y_{1+}} \), \( Q_{a_n} = \left( \frac{dV_1}{dy} \right)_{y_{n+}} \).

The variational formulation w.r.t. zone II is given by

\[ 0 = \int_{y_1}^{y_2} \left( \frac{dW_2(y)}{dy} \frac{dV_2(y)}{dy} + M_2V_2(y)W_2(y) + P_2W_2(y) \right) dy - W_2(y_{1+})Q_{a_2} - W_2(y_{n+})Q_{a_n} \]  (3.2)

where \( Q_{a_2} = \left( \frac{dV_2}{dy} \right)_{y_{1+}} \), \( Q_{a_n} = \left( \frac{dV_2}{dy} \right)_{y_{n+}} \).

The variational formulation w.r.t. zone III is given by

\[ 0 = \int_{y_1}^{y_2} \left( \frac{dW_3(y)}{dy} \frac{dV_3(y)}{dy} + M_3V_3(y)W_3(y) + P_3W_3(y) \right) dy - W_3(y_{1+})Q_{a_3} - W_3(y_{n+})Q_{a_n} \]  (3.3)

where \( Q_{a_3} = \left( \frac{dV_3}{dy} \right)_{y_{1+}} \), \( Q_{a_n} = \left( \frac{dV_3}{dy} \right)_{y_{n+}} \).

and w.r.t. zone IV it is given by

\[ 0 = \int_{y_1}^{y_2} \left( \frac{dW_4(y)}{dy} \frac{dV_4(y)}{dy} + M_4V_4(y)W_4(y) + P_4W_4(y) \right) dy - W_4(y_{1+})Q_{a_4} - W_4(y_{n+})Q_{a_n} \]  (3.4)

where \( Q_{a_4} = \left( \frac{dV_4}{dy} \right)_{y_{1+}} \), \( Q_{a_n} = \left( \frac{dV_4}{dy} \right)_{y_{n+}} \).
Quadratic polynomial approximations

We use quadratic polynomial approximation to obtain the finite element solution.

For computational purpose, we divide each zone into two quadratic elements.

We now find the global matrix for the velocity in Zone - 1.

Supposing \( u_i^{(k)} \) are the local nodal values of velocity with reference to the typical element \( e_k (y_{2k-1}, y_{2k+1}) \), under quadratic polynomial approximation

\[
V_i^{(k)} = \sum_{j=1}^5 u_i^{(k)} \psi_j^{(k)}
\]

where \( \psi_j^{(k)} \) are shape functions given in appendix.

Substituting \( V_i^{(k)} \) in (3.1) and integrating over the element, we obtain a local stiffness matrix corresponding to \( e_k \). Assembling these local matrices over the two elements and making use of inter element continuity conditions, the global matrix for \( V_i \) in terms of the five global modal values \( U_j \) \((j = 1, \ldots, 5)\) is given by

\[
\begin{bmatrix}
7 & -8 & 1 & 0 & 0 \\
-8 & 16 & -8 & 0 & 0 \\
1 & -8 & 14 & -8 & 1 \\
0 & 0 & -8 & 16 & -8 \\
0 & 0 & 1 & -8 & 7 \\
\end{bmatrix} + \frac{M_i}{60} \begin{bmatrix}
4 & 2 & -1 & 0 & 0 \\
2 & 16 & 2 & 0 & 0 \\
-1 & 2 & 8 & 2 & -1 \\
0 & 0 & 2 & 16 & 2 \\
0 & 0 & -1 & 2 & 4 \\
\end{bmatrix} \begin{bmatrix}
U_1 \\
U_2 \\
U_3 \\
U_4 \\
U_5 \\
\end{bmatrix} = \begin{bmatrix}
P_1 \\
P_2 \\
P_3 \\
P_4 \\
P_5 \\
\end{bmatrix}
\]

where \( Q_{11} = -\left( \frac{dV_i}{dy} \right)_{y=0} \), \( Q_{12} = \left( \frac{dV_i}{dy} \right)_{y=0} \),

In Zone-II, proceeding as above the matrix equation in terms of global nodes \( U_1^2, U_2^2, U_3^2, U_4^2, U_5^2 \) is given by
The element matrix equation in terms of global nodes $U^1_1, U^1_2, U^1_3, U^1_4, U^1_5$ with reference to zone-III is given by

$$
\begin{bmatrix}
7-8 & 1 & 0 & 0 \\
-8 & 16-8 & 0 & 0 \\
1-8 & 14-8 & 1 \\
0 & 0 & 8 & 16-8 \\
0 & 0 & 1 & -8 & 7
\end{bmatrix}
+ \frac{M_3}{60} \begin{bmatrix}
4 & 2 & -1 & 0 & 0 \\
2 & 16 & 2 & 0 & 0 \\
-1 & 2 & 8 & 2 & -1 \\
0 & 0 & 2 & 16 & 2 \\
0 & 0 & -1 & 2 & 4
\end{bmatrix}
\begin{bmatrix}
U_1^1 \\
U_2^1 \\
U_3^1 \\
U_4^1 \\
U_5^1
\end{bmatrix}
= \begin{bmatrix}
P_3^1 \\
4 \\
2 \\
4 \\
1
\end{bmatrix}
(3.6)
$$

where $Q_1^{21} = \left( \frac{dV^1_i}{dy} \right)_{i=0}$, $Q_2^{22} = \left( \frac{dV^1_i}{dy} \right)_{i=1}$

The element matrix equation in terms of global nodes $U^4_1, U^4_2, U^4_3, U^4_4, U^4_5$ with reference to zone-IV is given by

$$
\begin{bmatrix}
7-8 & 1 & 0 & 0 \\
-8 & 16-8 & 0 & 0 \\
1-8 & 14-8 & 1 \\
0 & 0 & 8 & 16-8 \\
0 & 0 & 1 & -8 & 7
\end{bmatrix}
+ \frac{M_4}{60} \begin{bmatrix}
4 & 2 & -1 & 0 & 0 \\
2 & 16 & 2 & 0 & 0 \\
-1 & 2 & 8 & 2 & -1 \\
0 & 0 & 2 & 16 & 2 \\
0 & 0 & -1 & 2 & 4
\end{bmatrix}
\begin{bmatrix}
U_1^4 \\
U_2^4 \\
U_3^4 \\
U_4^4 \\
U_5^4
\end{bmatrix}
= \begin{bmatrix}
P_4^1 \\
4 \\
2 \\
4 \\
1
\end{bmatrix}
(3.7)
$$

where $Q_1^{31} = \left( \frac{dV^4_i}{dy} \right)_{i=1, \ldots, 4}$ and $Q_3^{32} = \left( \frac{dV^4_i}{dy} \right)_{i=1, \ldots, 3}$

Making use of inter element continuity conditions at the nodes.
\[ U_2^3 = U_1^4 = U_5^6 ; U_3^2 = U_4^2 = U_9^3 ; U_5^3 = U_4^4 = U_{13} \]

and using the boundary conditions

\[ U_1^3 = U_1 = 0 ; \quad U_5^4 = U_{17} = 0 \]

and the balance of the secondary variables

\[ Q_3^{32} + Q_1^{11} = 0 ; \quad Q_3^{12} + Q_1^{21} = 0 ; \quad Q_3^{22} + Q_1^{41} = 0 \]

we assemble the matrix equations in the four zones (3.5) – (3.8).

The assembled matrix is of order 17 X 17 in terms of the unknown global nodal values of the velocity is represented by \( U_i \) (i = 2,..., 16) and the unknown secondary variables \( Q_i^{11} \& Q_3^{42} \).

This 17 X 17 matrix equation can be partitions in the form

\[
\begin{bmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{bmatrix}
\begin{bmatrix}
\Delta_{i1}^1 \\
\Delta_{i2}^2
\end{bmatrix} =
\begin{bmatrix}
F_{i1}^1 \\
F_{i2}^2
\end{bmatrix}
\]

(3.9)

where \( \Delta_{i1}^1 \), \( \Delta_{i2}^2 \), \( F_{i1}^1 \), \( F_{i2}^2 \) are column matrices given by

\[
\Delta_{i1}^1 = 
\begin{bmatrix}
0 \\
U_2 \\
U_3 \\
U_4 \\
U_5 \\
U_6 \\
U_7 \\
U_8
\end{bmatrix}
\quad \Delta_{i2}^2 = 
\begin{bmatrix}
0 \\
U_9 \\
U_{10} \\
U_{11} \\
U_{12} \\
U_{13} \\
U_{14} \\
U_{15} \\
U_{16} \\
0
\end{bmatrix}
\]

\[
\Delta_{i1}^1 = 
\begin{bmatrix}
0 \\
U_2 \\
U_3 \\
U_4 \\
U_5 \\
U_6 \\
U_7 \\
U_8
\end{bmatrix}
\quad \Delta_{i2}^2 = 
\begin{bmatrix}
0 \\
U_9 \\
U_{10} \\
U_{11} \\
U_{12} \\
U_{13} \\
U_{14} \\
U_{15} \\
U_{16} \\
0
\end{bmatrix}
\]
\[
F_+^1 = \begin{bmatrix}
\frac{P_1}{12} + Q_1^{11} \\
4P_1 \tau \\
\frac{2P_1 \tau}{12} \\
\frac{4P_3 \tau}{12} \\
\frac{P_1}{12} (r + 1) \\
\frac{4P_3}{12} \\
\frac{2P_3}{12} \\
\frac{4P_3}{12} \\
\frac{P_2 (1 + \tau)}{12} + Q_2^{11}
\end{bmatrix}
\]

\[
F_+^2 = \begin{bmatrix}
\frac{1}{12} (p_1 + p_2) \\
4P_2 \\
\frac{2P_2}{12} \\
\frac{4P_2}{12} \\
\frac{P_2 (1 + \tau)}{12} + Q_2^{12}
\end{bmatrix}
\]

\[
D^{11} = \begin{bmatrix}
a_1 & a_2 & a_3 & 0 & 0 & 0 & 0 & 0 \\
a_2 & a_4 & a_3 & 0 & 0 & 0 & 0 & 0 \\
a_3 & a_2 & 2a_1 & a_2 & a_3 & 0 & 0 & 0 \\
0 & 0 & a_2 & a_4 & a_2 & 0 & 0 & 0 \\
0 & 0 & a_1 & a_2 & a_1 + h_1 b_1 & b_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b_2 & b_4 & b_2 \\
0 & 0 & 0 & 0 & 0 & b_2 & b_4 & b_2 \\
0 & 0 & 0 & 0 & 0 & 0 & b_2 & b_4
\end{bmatrix}
\]

\[
D^{12} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_1 & b_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
b_2 & b_2 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
D^{21} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & b_1 & b_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
D^{22} = \begin{bmatrix}
b_1 + c_1 & c_2 & c_3 & 0 & 0 & 0 & 0 & 0 \\
c_2 & c_4 & c_2 & 0 & 0 & 0 & 0 & 0 \\
c_1 & c_2 & 2c_4 & c_2 & c_3 & 0 & 0 & 0 \\
0 & 0 & c_2 & c_4 & c_2 & 0 & 0 & 0 \\
0 & 0 & c_1 & c_3 & c_1 + d_1 & d_2 & d_3 & 0 \\
0 & 0 & 0 & 0 & d_2 & d_4 & d_3 & 0 \\
0 & 0 & 0 & 0 & d_1 & d_2 & 2d_1 & d_3 \\
0 & 0 & 0 & 0 & 0 & 0 & d_3 & d_2
\end{bmatrix}
\]

where

\[
a_i = \frac{14}{3\tau} + \frac{4\tau}{60} M_{il}, \quad a_2 = \frac{-16}{3\tau} + \frac{2\tau}{60} M_{il},
\]

\[
a_1 = \frac{2}{3\tau} + \frac{16\tau}{60} M_{il}, \quad a_4 = \frac{32}{3\tau} + \frac{16\tau}{60} M_{il}
\]

\[
b_i = \frac{14}{3} + \frac{4}{60} M_i, \quad b_3 = \frac{-16}{3} + \frac{2}{60} M_i
\]

\[
b_1 = \frac{2}{3} + \frac{16}{60} M_i, \quad b_3 = \frac{32}{3} + \frac{16}{60} M_i
\]

\[
c_1 = \frac{14}{3} + \frac{4}{60} M_2, \quad c_2 = \frac{-16}{3} + \frac{2}{60} M_2
\]
Solving the matrix equation (3.9) we obtain the solution for $U_i$ ($i = 2, 3, \ldots, 16$)

The solutions in the four zones may be presented as follows:

**$V_{p1}$**

$U_1\psi_1^i + U_2\psi_2^i + U_3\psi_3^i. \quad (1 + r) \leq y \leq (1 + \pi/2)$

$U_1\psi_1^i + U_4\psi_2^i + U_5\psi_3^i. \quad (1 + \pi/2) \leq y \leq 1$

**$V_1$**

$U_1\psi_1^i + U_6\psi_2^i + U_7\psi_3^i. \quad -1 \leq y \leq -1/2$

$U_1\psi_1^i + U_8\psi_2^i + U_9\psi_3^i. \quad -1/2 \leq y \leq 0$

**$V_2$**

$U_6\psi_1^o + U_9\psi_2^o + U_{10}\psi_3^o. \quad 0 \leq y \leq 1/2$

$U_{11}\psi_1^o + U_{12}\psi_2^o + U_{13}\psi_3^o. \quad 1/2 \leq y \leq 1$

**$V_{p2}$**

$U_{11}\psi_1^i + U_{14}\psi_2^i + U_{15}\psi_3^i. \quad 1 \leq y \leq (1 + \pi/2)$

$U_{16}\psi_2^i + U_{17}\psi_3^i. \quad (1 + \pi/2) \leq y \leq (1 + \pi)$
Non Magnetic Case:

In this case the global matrix for velocity in terms of $2m+1$ global nodal values $U^l_j$ ($j = 1,2,3 ; l = 1,2,...,m$) is given by

\[
\begin{bmatrix}
    d^l_{11} & d^l_{12} & d^l_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    d^l_{21} & d^l_{22} & d^l_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    d^l_{31} & d^l_{32} + d^l_{11} & d^l_{33} & d^l_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & d^l_{21} & d^l_{22} & d^l_{23} & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & d^l_{11} & d^l_{12} & d^l_{13} & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & d^l_{11} & d^l_{12} & d^l_{13} & d^l_{11} & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & d^l_{11} & d^l_{12} & d^l_{13} & d^l_{11} \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & d^l_{11} & d^l_{12} & d^l_{13} \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d^l_{11} & d^l_{12} \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d^l_{11}
\end{bmatrix}
\begin{bmatrix}
    U^l_1 \\
    U^l_2 \\
    U^l_3 \\
    U^l_4 \\
    U^l_5 \\
    U^l_6 \\
    U^l_7 \\
    U^l_8 \\
    \vdots \\
    \vdots
\end{bmatrix}
= \begin{bmatrix}
    f^l_1 \\
    f^l_2 + f^l_3 \\
    \vdots \\
    \vdots \\
    \vdots
\end{bmatrix}
+ \begin{bmatrix}
    \vdots \\
    \vdots \\
    \vdots
\end{bmatrix}
\begin{bmatrix}
    0 \\
    0 \\
    \vdots \\
    \vdots \\
    \vdots
\end{bmatrix}
\tag{3.10}
\]

choosing two quadratic elements in each zone the corresponding global matrices in the four zones are assembled making use of interfacial continuity for the velocity and inter equilibrium conditions of the secondary variables as well as the boundary conditions.

This matrix equation can be written in the form

\[
\Delta^l_i = \begin{bmatrix}
    0 \\
    U_2 \\
    U_3 \\
    U_4 \\
    U_5 \\
    U_6 \\
    U_7 \\
    U_8
\end{bmatrix}
\quad \Delta^l_i = \begin{bmatrix}
    U_{u} \\
    U_{u1} \\
    U_{u2} \\
    U_{u3} \\
    U_{u4} \\
    U_{u5} \\
    U_{u6} \\
    U_{u7} \\
    0
\end{bmatrix}
\]

where $\Delta^l_i$ is the matrix of nodal values for the $l$th zone and $U_{u}$ is the vector of nodal values for the secondary variables.
\[
F_U^1 = \begin{bmatrix}
p_1 \tau / 12h + Q_1^{31} \\
4p_1 \tau / 12h \\
2p_1 \tau / 12h \\
4p_1 \tau / 12h \\
p_1 \left( \frac{\tau}{12} + 1 \right) \\
4p_1 / 12 \\
2p_1 / 12 \\
4p_1 / 12
\end{bmatrix}
\]

\[
F_U^2 = \begin{bmatrix}
p_1 / 12 + p_2 / 12 \\
4p_1 / 12 \\
2p_2 / 12 \\
4p_2 / 12 \\
p_2 / 12(1 + \tau) \\
4p_2 \tau / 12 \\
2p_2 \tau / 12 \\
4p_2 \tau / 12 \\
p_2 \tau / 12 + Q_1^{42}
\end{bmatrix}
\]

\[
S^{11} = \begin{bmatrix}
c_1 & c_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_2 & c_3 & c_2 & 0 & 0 & 0 & 0 & 0 \\
c_3 & c_4 & 2c_1 & c_2 & c_1 & 0 & 0 & 0 \\
0 & 0 & c_2 & c_4 & c_2 & 0 & 0 & 0 \\
0 & 0 & c_3 & c_2 & c_1 + 14/3 & -16/3 & 2/3 & 0 \\
0 & 0 & 0 & 0 & -16/3 & -32/3 & 16/3 & 0 \\
0 & 0 & 0 & 0 & 2/3 & -16/3 & 28/3 & -16/3 \\
0 & 0 & 0 & 0 & 0 & -16/3 & 32/3 & 0
\end{bmatrix}
\]

\[
S^{12} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
S^{31} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
S^{32} = \begin{bmatrix}
\frac{18}{3} & \frac{16}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\
\frac{16}{3} & \frac{12}{3} & \frac{16}{3} & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{3} & \frac{28}{3} & \frac{16}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 \\
0 & \frac{16}{3} & \frac{12}{3} & \frac{16}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{14}{3} + c_5 & c_n & c_7 & 0 \\
0 & 0 & 0 & 0 & c_n & c_n & c_7 & 0 \\
0 & 0 & c_7 & c_n & 2c_5 & c_n & c_7 & 0 \\
0 & 0 & 0 & 0 & 0 & c_n & c_n & c_n \\
\end{bmatrix}
\]

Where
\[
c_1 = \frac{14}{3\tau} + \frac{D_4\tau}{60} \quad c_2 = -\frac{16}{3\tau} + \frac{D_2\tau}{60}
\]
\[
c_3 = \frac{2}{3\tau} + \frac{-D_3\tau}{60} \quad c_4 = \frac{32}{3\tau} + \frac{D_116\tau}{60}
\]
\[
c_5 = \frac{14}{3\tau} + \frac{D_4\tau}{60} \quad c_6 = -\frac{16}{3\tau} + \frac{D_2\tau}{60}
\]
\[
c_7 = \frac{2}{3\tau} + \frac{-\tau D_3}{60} \quad c_8 = \frac{32}{3\tau} + \frac{D_116\tau}{60}
\]
The finite element solution in each zone is given by

Solving the matrix equation (3.9), we obtain the solution for

\[ L_{11} = \frac{4}{3} \]
4. Discussions

The flow region comprises of four zones, two of them \((-1 + \tau) \leq -1\) and \(1 \leq y \leq 1 + \tau\) correspond to porous regions and the other two \(-1 \leq y \leq 1\) correspond to the clean fluid region. The velocity in the flow field is investigated for variations in the governing parameters in both magnetic and nonmagnetic cases and then profiles are shown in Figs (1) – (36).

The imposed pressure radiant has been chosen to be positive so that the actual flow is in the direction from right to left along the axis of the channel. Figs 1 - 4 represent the behaviour of the velocity for variation in the Darcy parameter when the Hartmann number \(M_1\) related to zone I is greater than the Hartmann number \(M_2\) related to zone II. It is interesting to know that the permeability of porous beds exhibit a definite influence on the clean fluid region irrespective of the thickness of the porous beds we find from the above figs 1 to 4 that the magnitude \(V\) increases with increase in \(D_1\) \((\leq 2 \times 10^4)\) and decreases for \(D_1\) \((\leq 2 \times 10^4)\). Also in this case we find that irrespective of the thickness of the porous beds the reversal flow appears in the lower porous bed when \(D_1 \& D_2 \approx 10^4\).
Fig. 9(a)

\[ V_1 \text{ with } P_1 \text{ when } \tau = 0.20 \]
\[ D_2 = 10^4, P_2 = 4.0, M_1 = 5.0, M_2 = 3.0 \]

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_1 )</td>
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<td>40</td>
<td>60</td>
<td>80</td>
<td>100</td>
</tr>
</tbody>
</table>

Fig. 9(b)

\[ V_1 \text{ with } P_1 \text{ when } \tau = 0.50 \]

For Legend see Fig. 9(a)
When \( M_1 < M_2 \) figs 5 to 8 show that this reversal flow appears in the non porous region except at the interface of Zone – II with Zone – IV. An increase in \( D_1 \) through lower values order \( 5 \times 10^3 \) increases \(|V|\) and decreases for \( D_2 \geq 5 \times 10^4 \). In the porous beds the behaviour of \( V \) with reference to \( D_1 \) is similar to that in the clean fluid. However, \( V \) exhibits an oscillatory nature with the variation in the permeability of the upper porous bed for thin porous beds. While for porous beds of large thickness \(|V|\) decreases with increase in \( D_2 (\leq 10^4) \) and increases with \( D_2 (\geq 10^4) \).

The behaviour of \( V \) as a function of Reynold’s number very much depends on the thickness of the porous beds. (see figs 9 to 16). \(|V|\) inverses with increase in \( P_1 (\leq 40) \) and later decreases for larger values of \( P_1 \). This phenomenon is noticeable in the clear fluid region as well as in the porous region. \(|V|\) increases with increase in \( P_2 \) in all the four zones. In the case of thick porous beds we observe \(|V|\) increases with increase in \( P_1 \) as well as \( P_2 \) in all the four Zones. When \( D_1 > D_2 \) and \( M_1 > M_2 \) the appearance of reversal flow is confined to Zones I & III for this porous beds where as in the case of the thick porous beds this phenomenon spreads to Zone – II also. In generally we observe that the velocity in the porous beds attained maximum values at the interfaces with the respective clean invisible fluids. In fact, with in these porous beds the magnitude rises from 0 on the impermeable beds to attain maximum value at the interface in the non – porous zones the magnitude of the velocity attains its maximum at the interface of the immiscible fluids and gradually reduces to its value on the respective interfaces with the porous beds.
Fig. 13(a)

$V_1$ with $P_2$ when $\tau = 0.20$
$D_1 = 2 \times 10^5$, $D_2 = 10^4$, $P_1 = 50$, $M_1 = 2.0$ $M_2 = 4.0$

- I
- II
- III
- IV
- V

$P_2$: 20  40  60  80  100

Fig. 13(b)

$V_1$ with $P_1$ when $\tau = 0.50$

For Legend see Fig. 13(a)
Fig. 18(a) 
V_i with M_i when \( \tau = 0.50 \)
For Legend see Fig. 17(a)

Fig. 18(b) 
V_i with M_i when \( \tau = 0.50 \)
For Legend see Fig. 17(a)
Fig. 21(a) and (b) V, with $M_2$ when $\tau = 0.50$

For Legend see Fig. 17(a)

$D_i = 10^{-2}$, $D_2 = 7 \times 10^{-2}$, $P_i = 20$, $P_2 = 30$

$M_2 = 1.0$, II, III, IV, V
The behaviour of $V$ with reference to the magnetic parameters $M_1, M_2$ may be observed from figs 17 to 24. We find that in case of a thin porous bed an increase in $M_1$ ($\leq 4$) increases $V$ in all the Zones and also decreases as $M_1$ increases through higher values $> 4$. However, when the porous bed is thick this change in behaviour of $V$ is noticeable at lower values of $M_1 \sim 2$ with reference to $M_2$ no such behavioural change is found in $V$ and it increases with increase in $M_2$ in all the four zones.

The contrast to the magnetic case in the non magnetic case for thin porous beds irrespective of the values of the governing parameters the reversal flow takes place in the three zones in the ranges $-1.05 \leq y \leq -1$, $-0.5 \leq y \leq -0.1$ and $0.75 \leq y \leq 1$. No such reversal flow appears in the upper porous bed. In case of thick porous beds we find that the reversal flow is confined to the upper porous bed while the actual flow takes place in the rest of the region.

The behaviour of the velocity with reference to the variation the permeability of upper and lower porous beds may be observed from figs 34 to 37. When $D_2 \leq 10^4$ an increase in $D_1$ reduces the velocity in the entire fluid region where as $D_2 \geq 10^4$ increase in $D_1$ reduces the velocity in the lower half of the fluid region while increases in the upper half. Like wise keeping the permeability of lower porous bed $D_1 \sim 10^4$. Lower the permeability of the upper porous bed higher the velocity in the entire fluid region except at the level $y = -1.05$. 
Fig. 25(a)

$V_i$ with $P_1$ when $\tau = 0.20$, $D_1 = 2 \times 10^4$, $D_2 = 10^4$, $P_2 = 40$

\[
\begin{array}{ccccc}
I & II & III & IV & V \\
P_1 & 20 & 40 & 60 & 80 & 100 \\
\end{array}
\]

Fig. 25(b)

$V_i$ with $P_1$ when $\tau = 0.80$, $D_1 = 2 \times 10^4$, $D_2 = 10^4$, $P_2 = 40$

\[
\begin{array}{ccccc}
I & II & III & IV & V \\
P_1 & 20 & 40 & 60 & 80 & 100 \\
\end{array}
\]
Fig. 26(a)

$V_2$ with $P_1$ when $r = 0.20$. $D_1 = 2 \times 10^4$. $D_2 = 10^4$. $P_2 = 40$

<table>
<thead>
<tr>
<th>$P_1$</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
</tr>
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<td>40</td>
<td>60</td>
<td>80</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 26(b)

$V_2$ with $P_1$ when $r = 0.80$. $D_1 = 2 \times 10^4$. $D_2 = 10^4$. $P_2 = 40$

<table>
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<th>II</th>
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<th>IV</th>
<th>V</th>
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<tbody>
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<td>40</td>
<td>60</td>
<td>80</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
Fig. 27(a)

$V_{p_1}$ with $P_1$ when $\tau = 0.20$. $D_1 = 2 \times 10^3$, $D_2 = 10^4$, $P_2 = 40$

| $P_1$ | 20 | 40 | 60 | 80 | 100 |

Fig. 27(b)

$V_{p_1}$ with $P_1$ when $\tau = 0.80$. $D_1 = 2 \times 10^3$, $D_2 = 10^4$, $P_2 = 40$

| $P_1$ | 20 | 40 | 60 | 80 | 100 |
Fig. 28(a)

$V_{P_1}$ with $P_1$ when $\tau = 0.20$. $D_1 = 2 \times 10^6$, $D_2 = 10^4$, $P_2 = 40$

<table>
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<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
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<td>40</td>
<td>60</td>
<td>80</td>
<td>100</td>
</tr>
</tbody>
</table>

Fig. 28(b)

$V_{P_1}$ with $P_1$ when $\tau = 0.80$. $D_1 = 2 \times 10^4$, $D_2 = 10^4$, $P_2 = 40$

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
</tr>
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<tbody>
<tr>
<td>$P_1$</td>
<td>20</td>
<td>40</td>
<td>60</td>
<td>80</td>
<td>100</td>
</tr>
</tbody>
</table>
Fig. 29(a)

$Vp_1$ with $P_2$ when $\tau = 0.20$. $D_1 = 2 \times 10^4$, $D_2 = 10^4$, $P_1 = 40$

<table>
<thead>
<tr>
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<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
</tr>
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<tbody>
<tr>
<td>$P_2$</td>
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<td>60</td>
<td>80</td>
<td>100</td>
</tr>
</tbody>
</table>

Fig. 29(b)

$Vp_1$ with $P_2$ when $\tau = 0.80$. $D_1 = 2 \times 10^4$, $D_2 = 10^4$, $P_1 = 50$

<table>
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</tbody>
</table>
Fig. 34

$V_i$ with $D_2$ when $\tau = 0.20$
For Legend see Fig. 9

Fig. 35

$V_i$ with $D_1$ when $\tau = 0.20$
$D_2 = 5 \times 10^4$, $P_i = 50$, $P_e = 20$

$D_1$

I $10^4$
II $2 \times 10^4$
III $3 \times 10^4$
IV $4 \times 10^4$
The stresses on the lower & upper impermeable boundaries are evaluated and tabulated in 1 and 2. We observe that at both impermeable boundaries, the stresses enhances with increasing Reynolds numbers for fixed pressure gradients. Also on the lower impermeable wall and increase in $D_1$, decreases the stress for the porous beds of thickness $\tau \leq 0.4$ and enhances same for $\tau > 0.4$. On the upper impermeable boundary, the magnitude of the stress decreases with increasing $D_1$ or $D_2$. The stress on the impermeable wall, for fixed values of the other parameters, shows that it increases with increasing in $M_1$ or $M_2$. An increase in the thickness of the beds increases the stress for all values of the remaining parameters.

In the nonmagnetic case (tables 3 & 4) the stress on the lower boundary changes its sign from negative to positive or vice-versa for the thickness of porous bed $\tau \leq 0.40$. This change in sign depends on the relative magnitude of the Darcy parameters. For thickness $\tau > 0.4$, this stress does not change the sign and it is continuously positive for all variations of the governing parameters. The magnitude of the stress increases with increase in the Reynolds numbers with fixed pressure gradients. At the upper permeable bed, irrespective of the thickness, the stress is negative for $D_1 \leq 2 \times 10^4$ and positive for $D_1 > 2 \times 10^4$. In general the lower the permeability then the higher are the stresses on the boundaries for fixed values of all the other parameters.
### TABLE - 1
SHEAR STRESS ($\Gamma_1$) AT -$\left(1+\tau\right)$

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
<th>IX</th>
<th>X</th>
<th>XI</th>
<th>XII</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>1.22</td>
<td>0.98</td>
<td>0.88</td>
<td>0.00</td>
<td>0.18</td>
<td>-2.87</td>
<td>0.07</td>
<td>0.05</td>
<td>-1.98</td>
<td>3.48</td>
<td>-0.78</td>
<td>-0.93</td>
</tr>
<tr>
<td>0.40</td>
<td>6.75</td>
<td>4.13</td>
<td>-1.37</td>
<td>-2.83</td>
<td>-6.61</td>
<td>-25.82</td>
<td>0.86</td>
<td>-9.86</td>
<td>-23.14</td>
<td>140.85</td>
<td>-6.40</td>
<td>-7.46</td>
</tr>
<tr>
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<td>38.52</td>
<td>174.41</td>
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<td>123.28</td>
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<tr>
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<td>53.41</td>
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### TABLE - 2

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### TABLE 3

**SHEAR STRESS ($\Gamma_1$) AT $-\{1+\tau\}$**

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**TABLE - 4**

SHEAR STRESS ($\tau_2$) AT (1+\tau)

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Appendix

\[\psi_1^1 = [1 - 2\tau (y + 1 + \tau)] [1 - 4\tau (y + 1 + \tau)]\]

\[\psi_2^1 = 8\tau (y + 1 + \tau) [1 - 2\tau (y + 1 + \tau)]\]

\[\psi_3^1 = -2\tau (y + 1 + \tau) [1 - 4\tau (y + 1 + \tau)]\]

\[\psi_1^2 = [1 - 2(y + 1 + \tau/2)] [1 - 4\tau (y + 1 + \tau/2)]\]

\[\psi_2^2 = 8\tau (y + 1 + \tau/2) [1 - 4\tau (y + 1 + \tau/2)]\]

\[\psi_3^2 = -2\tau (y + 1 + \tau/2) [1 - 4\tau (y + 1 + \tau/2)]\]

\[\psi_1^3 = [1 - 2(y + 1)] [1 - 4(y + 1)]\]

\[\psi_2^3 = 8(y + 1) [1 - 2(y + 1)]\]

\[\psi_3^3 = -2(y + 1) [1 - 4(y + 1)]\]

\[\psi_1^4 = [1 - 2(y + 1/2)] [1 - 4(y + 1/2)]\]

\[\psi_2^4 = 8(y + 12) [1 - 2(y + 12)]\]

\[\psi_3^4 = -2(y + 1/2) [1 - 4(y + 1/2)]\]

\[\psi_1^5 = (1 - 2y) (1 - 4y)\]

\[\psi_2^5 = 8y(1 - 2)\]

\[\psi_3^5 = -2y(1 - 4y)\]

\[\psi_1^6 = [1 - 2(y - 1/2)] [1 - 4(y - 1/2)]\]

\[\psi_2^6 = 8(y - 1/2) [1 - 2(y - 1/2)]\]
\[ \psi_3^6 = -2(y-1/2) \left[ 1-4(y-1/2) \right] \]
\[ \psi_1^7 = \left[ 1-2/\tau (y-1) \right] \left[ 1-4/\tau (y-1) \right] \]
\[ \psi_2^7 = 8/\tau (y-1) \left[ 1-4/\tau (y-1) \right] \]
\[ \psi_3^7 = -2/\tau (y-1) \left[ 1-4/\tau (y-1) \right] \]
\[ \psi_1^8 = \left[ 1-2/\tau (y-1-\tau/2) \right] \left[ 1-4/\tau (y-1-\tau/2) \right] \]
\[ \psi_2^8 = 8/\tau \left[ y-1-\tau/2 \right] \left[ 1-2/\tau (y-1-\tau/2) \right] \]
\[ \psi_3^8 = -2/\tau \left[ y-1-\tau/2 \right] \left[ 1-4/\tau (y-1-\tau/2) \right] \]
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