CHAPTER – IV

HAGEN – POISEUILLE FLOW IN A CIRCULAR DUCT
BOUND BY A POROUS BED
Fig. A: Schematic diagram of the flow through circular duct bounded by a porous bed.
Fluid flow through porous pipes has been studied theoretically and experimentally by numerous researchers, such as Berman (4) and Terrill (31), because of many applications in fields such as diffusion technology, transpiration cooling, hemodialysis processes, desalination, fluid control in nuclear reactors, and numerous other fields.

In the understanding of the circulation of blood in lungs, the effect of porosity has to be included either in the medium or near the boundary of the ducts. In lungs, blood can be visualized as flowing between the opposing layers of capillary endothelium, held apart by endothelium-covered "post" of septal tissue. This capillary endothelium is covered in turn by a thin layer (interstitial space) lining the alveoli. The blood space in
the lung is idealized into a two-dimensional channel and the interstitial tissue space into a porous medium. As the endothelial layer between the two regions is permeable to water and certain other solutes, it can be considered as a permeable membrane of negligible thickness. The epithelial tissues between the air and vascular spaces are less permeable and can thus be treated as impermeable membrane. The irregular posts which keep apart the endothelial walls can be ignored for the time being.

Earlier theories by Weible (32) presented the pulmonary interalveolar vascular bed as a capillary bed in which endothelium lined tubular structures formed a network which was analysed analytically as a series of wedged cylinders. After extensive studies, Fung and Sobin (10) proposed a better model known as the "sheet model" in which the continuous space between the two endothelial layers provides the physical basis for blood to flow as a sheet. The morphometric basis for sheet flow was presented simultaneously by Sobin et al. (28). A theoretical study was reported by Lee (15) to find out the resistance caused by the inter-connecting posts. More recently, Tang and Feng (29) have studied the mass transfer problem in the alveolar sheet, considering the steady flow in a channel with permeable walls covered by porous media. Gopalan (11) has discussed pulsatile blood flow in the lung alveolar sheets by idealizing each of them as a channel covered by porous media. The flow in the porous region is governed by the generalized Darcy’s law. In the clear fluid region, the Navier Stokes equations region are considered neglecting the convective terms. The flow in the clear and porous regions are coupled through the
interfacial conditions. The analytical and numerical results are obtained for the velocity and pressure distribution are obtained in both clear fluid and porous region.

In this chapter, we discuss the flow in a circular duct bounded externally by a porous bed lining using a finite element analysis. The Brinkman equation is used in the porous bed while the Navier–Stokes equation governs the flow in the clear fluid region. The velocity and shear stress are evaluated and their behaviour is investigated for different values of the governing parameters.

2. Formulation

Consider the flow in a circular pipe of radius $h$, surrounded by a coaxial permeable bed of thickness $s$ bounded by an impermeable outer cylinder $r = s + h$. The flow is unidirectional under a prescribed axial pressure gradient. The entire flow region is divided into two zones. Zone-I ($0 \leq r \leq h$) corresponding to the clear fluid with viscosity $\mu$ and the porous flow zone—II ($h \leq r \leq h + s$) with permeability $k$, viscosity of the fluid $\mu_p$. $r = h$ serves as the interface between the two zones. $W$ and $W_p$ are the velocities of the fluid in the zones-I & II, respectively.

The following are the equations governing for the flow in each zone and the respective conditions including the interface conditions.
The boundary and interfacial conditions corresponding to zone-1 & II are
\[ W = 0, \quad r = h + s \] (2.3)
\[ W = W_p, \quad \mu \frac{dW}{dr} = \mu_p \frac{dW_p}{dr}, \quad r = h \] (2.4)
\[ \frac{dW}{dr} = 0, \quad \text{at } r = 0 \text{ in view of the axisymmetry} \] (2.5)

We assume that \( \mu = \mu_p \) since it has been found by Neale & Nader (23) that it provides good agreement with experimental observation.

We define the following nondimensional variables:
\[ W^* = \frac{W}{q/h^2}, \quad W_p^* = \frac{W_p}{q/h^2}, \quad p^* = \frac{p}{\rho q^2/h^3} \] (2.6)
\[ r^* = \frac{r}{h}, \quad z^* = \frac{z}{h}, \quad \tau^* = \frac{s}{h} \]

Where \( q \) is the flux of the fluid across the channel and \( \rho \) is the density of fluid.

Making use of these non-dimensional variables in equation (2.1) – (2.2) (dropping asterisks) we obtain the following
Zone - I

\[ \frac{d^2 W}{dr^2} + \frac{1}{r} \frac{dW}{dr} = R \frac{dp}{dz} \]  \hspace{1cm} (2.7)

Zone - II

\[ \frac{d^2 W_p}{dr^2} + \frac{1}{r} \frac{dW_p}{dr} - D W_p = R \frac{dp}{dz} \]  \hspace{1cm} (2.8)

Where \( R = \frac{gh}{\mu h} \) is the Reynolds number

\[ D = \frac{h^2}{k} \]  is the Darcy parameter.

\[ p = R \frac{dp}{dz} \]

The corresponding non-dimensional conditions (2.3 to 2.5) are given by

\[ W = 0 \quad ; \quad r = 1 + \tau \]  \hspace{1cm} (2.9)

\[ W = W_p \text{ and } \left. \frac{dW}{dr} = \frac{dW_p}{dr} \right|_{r = 1} \]  \hspace{1cm} (2.10)

\[ \left. \frac{dW}{dr} = 0 \right|_{r = 0} \]  \hspace{1cm} (2.11)

3. Finite Element Analysis

We divide the flow domain into line elements along the radical direction.

Let \( V_1(\tau) \) be the weight function in Zone-I. The corresponding variational formulation gives the following
Let $V_z(r)$ be the weight function in zone-11. The corresponding variational formulation gives the following

$$0 = \int_{r_{x_{11}}}^{r_{x_{12}}} \left[ \frac{dV_1}{dr} \frac{dW}{dr} + PV_1 \right] r dr - [rV_1(r)]_{r=r_{n_{1}}} Q_1 - [rV_1(r)]_{r=r_{n_{1}}} Q_0$$

where $Q_1 = \left( \frac{dW}{dy} \right)_{r=r_{n_{0}}} \quad \text{and} \quad Q_0 = \left( \frac{dW}{dy} \right)_{r=r_{n_{0}}}$

Let $V_2(r)$ be the weight function in zone-II. The corresponding variational formulation gives the following

$$0 = \int_{r_{x_{12}}}^{r_{x_{22}}} \left[ \frac{dV_2}{dr} \frac{dWp}{dy} + D V_2 Wp + PV_2 \right] r dr - V_2(r_{n_{1}}) Q_1 - V_2(r_{n_{1}}) Q_0$$

where $Q_1 = \left( \frac{dWp}{dr} \right)_{r=r_{n_{1}}} \quad \text{and} \quad Q_0 = \left( \frac{dWp}{dy} \right)_{r=r_{n_{1}}}$

**Linear Polynomial Approximation**

We make use of linear polynomial approximations and divide each zone into $n_i$ ($i=1,2,3,4$) elements.

We now obtain the global matrix for the velocity with reference to Zone-I.

Supposing $u_i^{(k)}$ are the local nodal values with reference to typical element $e_i(r_k, r_{k+1})$ under linear polynomial approximation, each element will have two end local nodal values. In case we divide each zone into $n_1$ elements, implementing the interelement continuity conditions, there will be $n_1+1$ global nodal values and calling then $U_{1}^{J}$ ($J=1,2,\ldots, n_{1+1}$), where we affix 1 to denote the zone-I, the velocity in zone-I may be represented as follows:

$$W = \sum_{J=1}^{n_{1+1}} U_{1}^{J} \psi_{J}^{1}$$
where $\Psi^k_j$ are the shape functions given in appendix - I

Now substituting for $W^k = \sum_{i=1}^{l} \int u_i^{(k)} \psi_j^{(k)}$ in (3.1) and integrating over $e_k$, we obtain the local stiffness matrix. Assembling these local matrices over $n_1$ such elements and using inter element continuity conditions, the global matrix for $W$ in terms of the $n_1+1$ respective global nodal values $U_{j}^{(I)} (J = 1,2, \ldots, n_1+1)$ is given by

$$
\begin{bmatrix}
  c_{11} & c_{12} & 0 & 0 & \ldots & 0 \\
  c_{21} & c_{22} & 0 & 0 & \ldots & 0 \\
  0 & c_{22} & c_{21} & c_{22} & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & 0 & c_{22} & c_{22} \\
  0 & 0 & \ldots & 0 & c_{21} & c_{22}
\end{bmatrix}
\begin{bmatrix}
  U_{1}^{(I)} \\
  U_{2}^{(I)} \\
  U_{3}^{(I)} \\
  \vdots \\
  U_{n_1+1}^{(I)}
\end{bmatrix}
= \begin{bmatrix}
  f_{1}^{(I)} \\
  f_{2}^{(I)} + f_{4}^{(I)} \\
  \vdots \\
  \vdots \\
  0
\end{bmatrix} + \begin{bmatrix}
  Q_{1}^{(I)} \\
  Q_{2}^{(I)} \\
  \vdots \\
  \vdots \\
  0
\end{bmatrix}
$$

where

$$
c_{ij} = \int_{e_k} \frac{d\psi^k_i}{dr} \frac{d\psi^k_j}{dr} dr, \quad f_{ij} = \int_{e_k} p^{(k)} \psi^k_i dr, \\
Q_{1}^{(I)} = \left( \frac{dW}{dr} \right)_{r=r_{I-1}}, \quad Q_{2}^{(I)} = \left( \frac{dW}{dr} \right)_{r=r_{I+1}}
$$

A similar global matrix can be obtain for zone-II.

These global matrices corresponding to each zone can be assembled to obtain the velocity global matrix for the entire flow region, making use of the interface continuity.
conditions with reference to the velocity as well as inter equilibrium conditions of the secondary variables.

For computational purpose, we choose four elements in each zone. The corresponding global matrices with reference to the velocity in Zone – I and II respectively are given by

\[
\begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
-1 & 4 & -3 & 0 & 0 \\
0 & -3 & 8 & -5 & 0 \\
0 & 0 & -5 & 12 & -7 \\
0 & 0 & 0 & 7 & 7
\end{bmatrix}
\begin{bmatrix}
U_1^1 \\
U_2^1 \\
U_3^1 \\
U_4^1 \\
U_5^1
\end{bmatrix}
= \begin{bmatrix}
\frac{p}{4} \\\
\frac{3p}{24} \\
\frac{12p}{48} \\
\frac{18p}{48} \\
\frac{11p}{48}
\end{bmatrix}
+ \begin{bmatrix}
Q_1^1 \\
0 \\
0 \\
0 \\
Q_2^4
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
a_1 & a_3 & 0 & 0 & 0 \\
a_2 & a_3 & a_4 & 0 & 0 \\
0 & a_4 & a_5 & a_6 & 0 \\
0 & 0 & a_6 & 2a_7 & a_8 \\
0 & 0 & 0 & a_9 & a_{10}
\end{bmatrix}
\begin{bmatrix}
U_1^2 \\
U_2^2 \\
U_3^2 \\
U_4^2 \\
U_5^2
\end{bmatrix}
= \begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4 \\
b_5
\end{bmatrix}
+ \begin{bmatrix}
Q_1^{11} \\
0 \\
0 \\
0 \\
Q_2^{14}
\end{bmatrix}
\]

(3.3)

where \(Q_1^{11} = \left( \frac{dW}{dr} \right)_{r=r_0} \) and \(Q_2^{14} = \left( \frac{dW}{dr} \right)_{r=r_1} \)

(3.4)

\[
Q_1^{21} = \left( \frac{dWp}{dr} \right)_{r=r_1} \quad Q_2^{24} = \left( \frac{dWp}{dr} \right)_{r=r_{10r}}
\]
The inter element continuity conditions gives

\[ U'_5 = U_{11} \]

The boundary conditions on the impermeable wall

\[ U_2 = 0. \]

The actual symmetric condition \( 2.11 \) reduces to

inter equilibrium conditions \( 2.10 \) gives

\[ Q'^{11} = 0 \quad Q_2^{14} + Q_1^{21} = 0. \]

Making use of these conditions and assembling the matrix equations of two zones \( 3.3 \)\&\( 3.4 \), we obtain a \( 9 \times 9 \) matrix for the unknown global nodal values of the velocity which we call \( U_i = (i=1,\ldots,8) \) and the unknown secondary variable \( Q_2^{24} \).
This 9 X 9 matrix equation can be partitioned in the form

\[
\begin{bmatrix}
D_{11} & D_{12} \\
D_{21} & D_{22}
\end{bmatrix}
\begin{bmatrix}
\Delta_U^1 \\
\Delta_U^2
\end{bmatrix}
= \begin{bmatrix}
F_U^1 \\
F_U^2
\end{bmatrix}
\]

(3.5)

where \( \Delta_U^1 \), \( \Delta_U^2 \), \( F_U^1 \), \( F_U^2 \) are column matrices given by

\[
\Delta_U^1 = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix}
\]

\[
\Delta_U^2 = \begin{bmatrix} U_5 \\ U_6 \\ U_7 \\ U_8 \\ 0 \end{bmatrix}
\]

\[
F_U^1 = \begin{bmatrix}
p \\
4 \\
3p \\
24 \\
12p \\
48 \\
18p \\
48
\end{bmatrix}
\]

\[
F_U^2 = \begin{bmatrix}
h_1 \\
h_2 \\
h_3 \\
h_4 \\
h_5 + Q^2
\end{bmatrix}
\]

\[
D^{11} = \begin{bmatrix}
1 & -1 & 0 & 0 \\
-1 & 4 & -3 & 0 \\
0 & -3 & 8 & -5 \\
0 & 0 & -5 & 12
\end{bmatrix}
\]

\[
D^{12} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-7 & 0 & 0 & 0
\end{bmatrix}
\]

\[
D^{21} = \begin{bmatrix}
0 & 0 & 0 & -7 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
D^{22} = \begin{bmatrix}
a_1 + 7 & 0 & 0 & 0 \\
0 & a_2 & a_3 & 0 \\
0 & a_4 & a_5 & 0 \\
0 & 0 & a_6 & a_7 & a_8 \\
0 & 0 & 0 & a_9 & a_{10}
\end{bmatrix}
\]
Solving the matrix equation (3.5) we obtain the solution for $U_i = (i = 1, 2, \ldots, 8)$

finite element solution in each zone is given by

$$W \approx \begin{cases} 
U_1 \psi_1^i(r) + U_2 \psi_2^i(r) & 0 \leq r \leq 1/4 \\
U_2 \psi_1^i(r) + U_3 \psi_2^i(r) & 1/4 \leq r \leq 1/2 \\
U_3 \psi_1^i(r) + U_4 \psi_2^i(r) & 1/2 \leq r \leq 3/4 \\
U_4 \psi_1^i(r) + U_5 \psi_2^i(r) & 3/4 \leq r \leq 1
\end{cases}$$

$$W_p \approx \begin{cases} 
U_5 \psi_1^i(r) + U_6 \psi_2^i(r); & 1 \leq r \leq 1 + \pi/4 \\
U_6 \psi_1^i(r) + U_7 \psi_2^i(r); & 1 + \pi/4 \leq r \leq 1 + \pi/2 \\
U_7 \psi_1^i(r) + U_8 \psi_2^i(r); & 1 + \pi/2 \leq r \leq 1 + 3\pi/4 \\
U_8 \psi_1^i(r); & 1 + 3\pi/4 \leq r \leq 1 + \pi
\end{cases}$$

**Quadratic polynomial approximation**

Assuming that $u_i^{(k)}$ are the local nodal values with reference to the typical element $e_i (r_{2k-1}, r_{2k+1})$ under a quadratic polynomial approximation, each element has three local nodal values. In this case we divide the zone-I into $m_1$ elements and implementing the inter element continuity conditions then there will be $2m_1+1$ global nodal values. Calling these $\ldots U_1^J (J = 1, 2, \ldots, 2m_1 + 1)$ where we affix 1 to denote the zone-I, we obtain

$$W = \sum_{J=1}^{2m_1+1} U_j^J \psi_j$$

Substitute for $W^{(n)} = \sum_{J=1}^{3} u_J^{(n)} \psi_J^{(n)}$ to equation (3.1) and integrating over $e_n$, we obtain the local stiffness matrices. Assembling these local matrices over $m_1$ elements and
using inter element continuity conditions. The global matrix for $W$ in terms of $2m_1+1$ respective global nodal values $U_j^1 (J=1,2, \ldots, 2m_1+1)$ is given by

\[
\begin{bmatrix}
  d_{11} & d_{12} & 0 & 0 & 0 & 0 & \cdots & 0 \\
  d_{21} & d_{22} & 0 & 0 & 0 & 0 & \cdots & 0 \\
  d_{31} & d_{32} + d_{11}^2 & d_{12} & d_{13} & 0 & 0 & \cdots & 0 \\
  0 & d_{32} & d_{33} & d_{13} & 0 & 0 & \cdots & 0 \\
  0 & 0 & d_{33} & d_{13} & d_{13} & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
  U_1^1 \\
  U_2^1 \\
  U_3^1 \\
  U_4^1 \\
  U_5^1 \\
  \vdots \\
  U_{n-1}^1 \\
  U_n^1 \\
\end{bmatrix} = \begin{bmatrix}
  f_1^1 \\
  f_2^1 + f_1^1 \\
  f_3^1 \\
  f_4^1 \\
  f_5^1 \\
  \vdots \\
  f_{n-1}^1 + f_1^1 \\
  f_n^1 \\
\end{bmatrix}
\begin{bmatrix}
  Q_1^{11} \\
  \vdots \\
  Q_n^{1n} \\
\end{bmatrix}
\]

where

\[
d_{jk}^1 = \int_{r_{jk+1}}^{r_{jk+2}} d\psi_j^1 \frac{d\psi_j^1}{dr} \, dr \\
f_{jk}^1 = \int_{r_{jk+1}}^{r_{jk+2}} P_j^1 \psi_j^1 \, dr \\
Q_{jk}^{11} = \left( \frac{dW}{dr} \right)_{r=r_{jk+1}} \\
Q_{jk}^{1n} = \left( \frac{dW}{dr} \right)_{r=r_{jk+1}}
\]

A similar global matrix can be obtained for zone-II.

These global matrices corresponding to each zone can be assembled to obtain the velocity global matrix for the entire flow region making use of the interface continuity conditions with reference to the velocity.

For computational purpose, we choose two quadratic elements in each zone, the corresponding global matrices with reference to the velocity in zone-I & II are given by
\[
\begin{bmatrix}
1 & -\frac{4}{3} & \frac{1}{3} & 0 & 0 \\
-\frac{4}{3} & 16/3 & -4 & 0 & 0 \\
-1/3 & -4 & 28/3 & -20/3 & 1 \\
0 & 0 & -20/3 & 16 & -28/3 \\
0 & 0 & 1 & -28/3 & 25/3
\end{bmatrix}
\begin{bmatrix}
U_1^1 \\
U_2^1 \\
U_3^1 \\
U_4^1 \\
U_5^1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
Q_1^1 \\
Q_1^2
\end{bmatrix}
\]  

(3.7)

and

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & 0 & 0 \\
a_{12} & a_{14} & a_{15} & 0 & 0 \\
a_{13} & a_{15} & a_{16} & a_{17} & a_{18} \\
0 & 0 & a_{17} & a_{19} & a_{20} \\
0 & 0 & a_{18} & a_{20} & a_{21}
\end{bmatrix}
\begin{bmatrix}
U_1^2 \\
U_2^2 \\
U_3^2 \\
U_4^2 \\
U_5^2
\end{bmatrix}
= \begin{bmatrix}
b_{11} \\
b_{12} \\
b_{13} \\
b_{14} \\
b_{15}
\end{bmatrix}
\begin{bmatrix}
Q_1^{31} \\
Q_1^{32}
\end{bmatrix}
\]  

(3.8)

where

\[
Q_1^{11} = \left(\frac{dW}{dr}\right)_{r=0}
\quad Q_1^{12} = \left(\frac{dW}{dr}\right)_{r=1}
\quad Q_1^{31} = \left(\frac{dW}{dr}\right)_{r=0}
\quad Q_1^{32} = \left(\frac{dW}{dr}\right)_{r=1}
\]

\[
a_{11} = \frac{2}{3r} \left(\frac{3r}{2} + 14\right) + \frac{D_{rr}}{6}
\quad a_{13} = -\frac{2}{3r} (2r + 16)
\]

\[
a_{13} = \frac{2r}{3r} + (\frac{r}{2} + 2)
\quad a_{14} = \frac{2}{3r} (8r + 32) + \frac{D_{rr}}{6} (r + 4)
\]

\[
a_{15} = -\frac{2}{3r} (6r + 16)
\quad a_{16} = \frac{2}{3r} (28 + 14r) + \frac{D_{rr}}{6} (\frac{r}{2} + r)
\]

\[
a_{17} = -\frac{2}{3r} (16 + 10r)
\quad a_{18} = \frac{2}{3r} (2 + \frac{r}{2})
\]
\[
\begin{align*}
\alpha_1 &= \frac{64 + 48\tau}{3\tau} + \frac{\tau(4 + 3\tau)D}{6} \\
\alpha_2 &= \frac{2}{3\tau} (14 + 25\tau) + \frac{D\tau}{6} (1 + \tau) \\
\beta_1 &= \frac{P\tau}{6} \\
\beta_2 &= \frac{P\tau}{6} (4 + \tau) \\
\beta_3 &= \frac{P\tau}{6} (1 + \tau) \\
\beta_4 &= \frac{P\tau}{6} (4 + 3\tau) \\
\beta_5 &= \frac{P\tau}{6} (1 + \tau)
\end{align*}
\]

Making use of inter element continuity, equilibrium conditions as well as the symmetry and boundary conditions as stated earlier in (3.4b) and assembling the matrices of the two zones, we obtain an 9 X 9 matrix equation for the unknown global nodes \( U_i \ (i = 1, 2, \ldots, 8) \) and the unknown secondary variables \( \Omega_{11} \).

The assembled matrix will be of order 9 X 9 in terms of unknown global nodal values of the axial velocity which we now represent as \( U_i \ (i = 1, 2, \ldots, 8) \) and the unknown secondary variables \( \Omega_{11} \).

This 9 X 9 matrix equation can be partitioned in the form

\[
\begin{bmatrix}
E_{11} & E_{12} \\
E_{21} & E_{22}
\end{bmatrix}
\begin{bmatrix}
\Delta_{1} \\
\Delta_{2}
\end{bmatrix} =
\begin{bmatrix}
F_{1} \\
F_{2}
\end{bmatrix}
\]

(3.9)

where \( \Delta_{1}, \Delta_{2}, F_{1}, F_{2} \) are column matrices given by

\[
\Delta_{1} = \begin{bmatrix}
U_1 \\
U_2 \\
U_3 \\
U_4
\end{bmatrix}
\]

\[
\Delta_{2} = \begin{bmatrix}
U_5 \\
U_6 \\
U_7 \\
U_8 \\
0
\end{bmatrix}
\]
Solving the matrix equation (3.9) we obtain \( U_i = (i = 1.8) \) and the finite element solution is given by

\[
W \approx \begin{cases} 
U_2 \psi_2^1(r) + U_3 \psi_3^1(r) & 0 \leq r \leq 1/2 \\
U_2 \psi_2^2(r) + U_4 \psi_2^2(r) + U_5 \psi_5^1(r) & 1/2 \leq r \leq 1 
\end{cases}
\]

\[
W_p \approx \begin{cases} 
U_5 \psi_1^1(r) + U_6 \psi_2^1(r) + U_7 \psi_5^1(r) & 1 \leq r \leq 1 + \frac{1}{2} \\
U_7 \psi_1^1(r) + U_8 \psi_2^2(r) & 1 + \frac{1}{2} \leq r \leq (1 + r)
\end{cases}
\]
4. **Discussions**

The flow region comprises of two zones, i.e., the clean fluid region and the porous bed. The velocity in these zones are evaluated computationally and their profiles are drawn for variation in the governing parameters \( \tau \), \( P \), and \( D \) (Fig. 12). In general, we observe that the magnitude of the velocity in the clean fluid region enhances with increase in the thickness of the porous bed. The fluid velocity experiences a directional transition from positive to negative in the vicinity of the central axis in the clean fluid region \( 0.25 \leq r \leq 0.5 \) for all values of the governing parameters \( P \) and \( D \) and relatively small thickness of the porous bed. However, for large thickness of the beds \( r \sim 0.8 \), this transition takes place near the interface of the clean fluid with the porous bed (Fig. 3). Such transition does not take place in the porous region (Fig. 4) – (6). The magnitude of the velocity in general enhances with the increase in \( P \). When the permeability of the medium is such that an increase in \( D \) through lower values less than \( 7 \times 10^1 \) or higher values greater than \( 10^4 \), the velocity reduces in the clean fluid region. But for values of \( D \) in the range \( 5 \times 10^3 \) to \( 7 \times 10^3 \), we find an appreciation in the velocity (Fig. 7) This phenomenon is observed for all thickness of the porous bed (Fig. 7) – (12)).
Fig. 5

$W_P$ with $P$ when $\tau = 0.40$, $D = 2 \times 10^4$

For Legend see Fig. 4

Fig. 6

$W_P$ with $P$ when $\tau = 0.80$, $D = 2 \times 10^4$

For Legend see Fig. 4
Fig. 7

$W$ with $D$ when $r = 0.20$ & $P = 40.0$

$D$  $10^3$  $5 \times 10^3$  $7 \times 10^3$  $10^4$  $2 \times 10^4$  $3 \times 10^4$  $4 \times 10^4$  $5 \times 10^4$

Fig. 8

$W$ with $D$ when $r = 0.40$ & $P = 40.0$

$D$  $10^3$  $5 \times 10^3$  $7 \times 10^3$  $10^4$  $2 \times 10^4$  $3 \times 10^4$  $4 \times 10^4$  $5 \times 10^4$. 
Fig. 9

W with D when $\tau = 0.80$ & $P = 40.0$

$D$  |  I  |  II  |  III  |  IV  |  V   |  VI  |  VII  |  VIII  \\
-----|-----|------|-------|-----|------|------|-------|--------
      | $10^3$ | $5 \times 10^3$ | $7 \times 10^3$ | $10^4$ | $2 \times 10^4$ | $3 \times 10^4$ | $4 \times 10^4$ | $5 \times 10^4$ \\

Fig. 10

$W$ with D when $\tau = 0.20$ & $P = 40.0$

$D$  |  I  |  II  |  III  |  IV  |  V   |  VI  |  VII  |  VIII  \\
-----|-----|------|-------|-----|------|------|-------|--------
      | $10^3$ | $5 \times 10^3$ | $7 \times 10^3$ | $10^4$ | $2 \times 10^4$ | $3 \times 10^4$ | $4 \times 10^4$ | $5 \times 10^4$ \\
The Stresses for variation in $D$ & $P$ are tabulated in table 1. We observe that the stress $\Gamma$ increase in magnitude with reference to increase in the thickness of the porous bed. Also $\Gamma$ enhances with increase in the Reynolds number lesser the permeability $\Gamma$ increases for Darcy parameter $D$ less than $10^4$, but for higher $D$ it reduces.
TABLE - I
SHEAR STRESS ($\Gamma$) AT ($1 + \tau$)

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
<th>IX</th>
<th>X</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>0.27</td>
<td>0.35</td>
<td>-0.66</td>
<td>-1.00</td>
<td>1.89</td>
<td>0.32</td>
<td>0.53</td>
<td>0.74</td>
<td>0.89</td>
<td>1.04</td>
</tr>
<tr>
<td>0.40</td>
<td>0.90</td>
<td>-1.30</td>
<td>-7.29</td>
<td>-2.44</td>
<td>-5.37</td>
<td>3.17</td>
<td>5.39</td>
<td>6.11</td>
<td>7.36</td>
<td>8.59</td>
</tr>
<tr>
<td>0.60</td>
<td>1.02</td>
<td>-2.46</td>
<td>-5.49</td>
<td>-3.21</td>
<td>-5.24</td>
<td>23.87</td>
<td>-78.52</td>
<td>-14.87</td>
<td>-7.24</td>
<td>-12.92</td>
</tr>
<tr>
<td>0.80</td>
<td>1.20</td>
<td>-3.49</td>
<td>-6.12</td>
<td>-4.16</td>
<td>-5.89</td>
<td>-9.84</td>
<td>-15.46</td>
<td>-14.34</td>
<td>-16.86</td>
<td>-19.36</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
<th>IX</th>
<th>X</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>$10^3$</td>
<td>$5\times10^3$</td>
<td>$7\times10^3$</td>
<td>$10^4$</td>
<td>$2\times10^4$</td>
<td>$10^4$</td>
<td>$10^4$</td>
<td>$3\times10^4$</td>
<td>$2\times10^4$</td>
<td>$2\times10^4$</td>
</tr>
<tr>
<td>P</td>
<td>40</td>
<td>40</td>
<td>40</td>
<td>40</td>
<td>40</td>
<td>30</td>
<td>50</td>
<td>50</td>
<td>60</td>
<td>70</td>
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</tbody>
</table>
### Appendix - I

<table>
<thead>
<tr>
<th>( \psi_{1}^{1} )</th>
<th>( = 1 - 4/\tau (y+1+\tau) )</th>
<th>( \psi_{2}^{1} = 4/\tau (y+1+\tau) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \psi_{1}^{2} )</td>
<td>( = 1 - 4/\tau (y+1+3\tau/4) )</td>
<td>( \psi_{2}^{2} = 4/\tau (y+1+3\tau/4) )</td>
</tr>
<tr>
<td>( \psi_{1}^{3} )</td>
<td>( = 1 - 4/\tau (y+1+\tau/2) )</td>
<td>( \psi_{2}^{3} = 4/\tau (y+1+\tau/2) )</td>
</tr>
<tr>
<td>( \psi_{1}^{4} )</td>
<td>( = 1 - 4/\tau (y+1+\tau/4) )</td>
<td>( \psi_{2}^{4} = 4/\tau (y+1+\tau/4) )</td>
</tr>
<tr>
<td>( \psi_{1}^{5} )</td>
<td>( = 1 - 4(y+1) )</td>
<td>( \psi_{2}^{5} = 4(y+1) )</td>
</tr>
<tr>
<td>( \psi_{1}^{6} )</td>
<td>( = 1 - 4(y+3/4) )</td>
<td>( \psi_{2}^{6} = 4(y+3/4) )</td>
</tr>
<tr>
<td>( \psi_{1}^{7} )</td>
<td>( = 1 - 4(y+1/2) )</td>
<td>( \psi_{2}^{7} = 4(y+1/2) )</td>
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<tr>
<td>( \psi_{1}^{8} )</td>
<td>( = 1 - 4(y+1/4) )</td>
<td>( \psi_{2}^{8} = 4(y+1/4) )</td>
</tr>
<tr>
<td>( \psi_{1}^{9} )</td>
<td>( = 1 - 4y )</td>
<td>( \psi_{2}^{9} = 4y )</td>
</tr>
<tr>
<td>( \psi_{1}^{10} )</td>
<td>( = 1 - 4(y-1/4) )</td>
<td>( \psi_{2}^{10} = 4(y-1/4) )</td>
</tr>
<tr>
<td>( \psi_{1}^{11} )</td>
<td>( = 1 - 4(y-1/2) )</td>
<td>( \psi_{2}^{11} = 4(y-1/2) )</td>
</tr>
<tr>
<td>( \psi_{1}^{12} )</td>
<td>( = 1 - 4(y-3/4) )</td>
<td>( \psi_{2}^{12} = 4(y-3/4) )</td>
</tr>
<tr>
<td>( \psi_{1}^{13} )</td>
<td>( = 1 - 4/\tau(y-1) )</td>
<td>( \psi_{2}^{13} = 4/\tau(y-1) )</td>
</tr>
<tr>
<td>( \psi_{1}^{14} )</td>
<td>( = 1 - 4/\tau[y-(1+\tau/4)] )</td>
<td>( \psi_{2}^{14} = 4/\tau[y-(1+\tau/4)] )</td>
</tr>
<tr>
<td>( \psi_{1}^{15} )</td>
<td>( = 1 - 4/\tau[y-(1+\tau/2)] )</td>
<td>( \psi_{2}^{15} = 4/\tau[y-(1+\tau/2)] )</td>
</tr>
<tr>
<td>( \psi_{1}^{16} )</td>
<td>( = 1 - 4/\tau[y-(1+3\tau/4)] )</td>
<td>( \psi_{2}^{16} = 4/\tau[y-(1+3\tau/4)] )</td>
</tr>
</tbody>
</table>
Appendix - 11

\[ \psi_1 \quad = \quad (1-2r)(1-4r) \]

\[ \psi_2 \quad = \quad 8r(1-2r) \]

\[ \psi_3 \quad = \quad -2r(1-4r) \]

\[ \psi_4 \quad = \quad [1-2(r-1/2)][1-4(r-1/2)] \]

\[ \psi_5 \quad = \quad 8(r-1/2)[1-2(r-1/2)] \]

\[ \psi_6 \quad = \quad -2(r-1/2)[1-4(r-1/2)] \]

\[ \psi_7 \quad = \quad [1-2/\tau(r-1)][1-4/\tau(r-1)] \]

\[ \psi_8 \quad = \quad 8/\tau(r-1)[1-4/\tau(r-1)] \]

\[ \psi_9 \quad = \quad -2/\tau(r-1)[1-4/\tau(r-1)] \]

\[ \psi_{10} \quad = \quad [1-2/\tau(r-1-\tau)][1-4/\tau(r-1-\tau)] \]

\[ \psi_{11} \quad = \quad 8/\tau(r-1-\tau/2)[1-2/\tau(r-1-\tau)] \]

\[ \psi_{12} \quad = \quad -2/\tau(r-1-\tau/2)[1-4/\tau(r-1-\tau/2)] \]
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