CHAPTER II

REVIEW OF LITERATURE
2.0 In management and production economics, it is often assumed that the producer is cost minimizer or revenue maximizer or profit maximizer. However, these assumptions are not always realistic, especially in a competitive situation where different producers employ different techniques. For example, to produce a homogeneous product, the producer may have multiple goals instead of a single goal such as cost minimization. The various diversified goals may be, maintenance of stable profits and prices, improving market share and so on. Goal programming paves a way of striving toward several such multiple objectives simultaneously. Goal programming provides an objective function for each objective and consequently finds a solution that minimizes the weighted sum of deviations of these objective functions from their respective goals. One can come across three possible types of goals (i) A lower, one sided goal which sets a lower limit that we do not fall under. (ii) An upper, one-sided goal that sets an upper limit which we are not allowed to exceed. (iii) A two sided goal that specifies a lower limit that we do not fall under and an upper limit that we do not want to exceed.

A goal programming problem may be preemptive or non-preemptive. Preemptive goal programming problem is also called priority goal programming problem. Preemptive goal programming begins with the most important goal and continues until the achievement of a less
important goal would cause management to fail to achieve a more important one. In the final solution, all the goals may not be fulfilled to the fullest extent. All the goals are equally important in non-preemptive goal programming.

A Goal programming problem may be linear or non-linear. If it is linear, then we call it linear goal programming problem. Goal programming has applications in the theory of production.

Goal programming can be used to estimate frontier full and stochastic cost functions of a production unit in a competitive environment. The resulting estimation procedure is called Goal Programming Constrained Regression (GP-CR). Sueyoshi estimates stochastic frontier* cost function using data envelopment analysis.

2.1 To understand goal programming estimation of stochastic cost function, it is necessary to understand the postulates of Data Envelopment Analysis** (DEA), factor minimal cost function and its properties.

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DEA is based upon production possibility sets determined by the input and output vectors of production units, called the Decision Making Units (DMU), which compete with each other in a competitive environment.

For example,

\[ x_i : \text{Input vector of } i^{th} \text{ production unit.} \]
\[ x_i + R^x \]
\[ u_i : \text{Output vector of } i^{th} \text{ production unit} \]
\[ u_i + R^u \]
\[ p_i : \text{Input price vector} \]
\[ p_i + R^p - \{0\} \]

where 0 is null vector.

It is hypothesised that all producers are not equally efficient, since different producers employ different techniques even if they employ the same technique of production, they may differ in terms of managerial efficiency. Thus, one of the postulates of DEA is 'inefficiency'. The following are DEA postulates:

Let \( T \) be the production possibility set, which is defined as,

\[ T = \{(x,u) : x \text{ produces } u\} \tag{2.1.1} \]


(i) **Convexity:** \((x_i, u_i) + T, \ (i = 1,2, \ldots, k)\)
\[
\Rightarrow \left( \sum_{i=1}^{k} \lambda_i x_i, \sum_{i=1}^{k} \lambda_i u_i \right) + T
\]
where \(\lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1\)

(ii) **Inefficiency:**
\[
(x, \bar{u}) + T \Rightarrow \bar{x} \text{ produces } \bar{u} \Rightarrow x \text{ produces } u
\]
where \(x \geq \bar{x}, \ u \leq \bar{u}\)

(iii) **Minimum Extrapolation:**

\(T\) is the intersection of all input sets which contain \((x_i, u_i),\)
\(i = 1,2, \ldots, k.\)

\[
T = \bigcap_{i} T_i
\]

The production possibility set consistent with the DEA postulates may be expressed as,

\[
T = \{ (x, u) : \sum \lambda_i x_i \leq x, \ \sum \lambda_i u_i \geq u, \ \lambda_i \geq 0, \ \sum \lambda_i = 1 \} \quad \text{\text{\textcircled{2.1.5}}}\]

But, \((x, \bar{u}) + T \Rightarrow \bar{x} = \sum \lambda_i x_i, \ \bar{u} = \sum \lambda_i u_i, \ \lambda_i \geq 0, \ \sum \lambda_i = 1\)

Consequently, \(\sum \lambda_i x_i \leq x, \ \sum \lambda_i u_i \geq u\)

\[
\lambda_i \geq 0 \quad \sum \lambda_i = 1
\]
If $x$, and $u$, are scalar valued, then $T$ may be graphically expressed as,

The dotted region bound by the line segments AB, BC, CD and ordinates at $x_o$ and $x_w$ is the production possibility set. The production units which operate at A, B, C and D determine the production frontier. These are technically efficient DMU's. The production units that operate at E and F are inefficient.

Production technology may be examined in terms of its input sets $L(u)$, where

$$L(u) = \{x : x \text{ produces } u\}$$

In addition, if the production technology is piece wise linear, as seen above, the input sets are expressed as,*

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\[ L(u) = \left\{ x : \sum \lambda_i x_i \leq x \cdot \sum \lambda_i u_i \geq u, \lambda_i \geq 0, \sum \lambda_i = 1 \right\} \quad \text{--- (2.1.6)} \]

Let \( u + R, x + R^2 \), then a typical input set is as shown below:

The production units A and B which fall on the isoquant of \( L(u) \) are technically efficient. But, the DMU-C is technically inefficient. The DMU's A, B and C produce the same output quantity.
The producer who operate at A is technically efficient. \( PP^1 \) is the cost line,

\[
C^A = p_1 x_1^A + p_2 x_2^A
\]

Cost at 'A' is higher than the cost at B. Therefore, producer -A is cost inefficient.

2.2 For a production unit that employs the input vector \( x_\ast \) and produces the output vector \( u_\ast \), factor minimal cost can be obtained by solving the following problem:

\[
Q (u_\ast, p) = \text{Min } px
\]

such that \( x + L (u_\ast) \)

In terms of piecewise linear production technology, the factor minimal cost is obtained by solving the following linear programming problem:

\[
Q (u_\ast, p) = \text{Min } px
\]

such that \( \sum \lambda_i x_i \leq x \)

\( \sum \lambda_i u_i \geq u_\ast \)

\( \lambda_i \geq 0 \)

\( \sum \lambda_i = 1 \)
2.3 Production function:

A production function is a technological relationship between inputs and outputs. It has nothing to do with prices of inputs and outputs. For example, for scalar output $u$ and input vector $x$, a production function may be expressed as,

$$ u = \phi(x) \quad \text{--- (2.3.1)} $$

The production function $\phi(x)$ and input sets $L(u)$ determine each other completely*.

$$ \phi(x) = \text{Max} \{ u : x + L(u) \} \quad \text{--- (2.3.2)} $$

$$ L(u) = \{ x : \phi(x) \geq u \} \quad \text{--- (2.3.3)} $$

A production function is said to be homogeneous, if, $\phi(\lambda x) = \lambda^\theta \phi(x)$.

In addition suppose that, $\lambda \geq 1$.

$\theta < 1 \quad \Rightarrow \quad$ Returns to scale are decreasing

$\theta = 1 \quad \Rightarrow \quad$ Returns to scale are constant

$\theta > 1 \quad \Rightarrow \quad$ Returns to scale are increasing

For a non-homogeneous production function returns to scale are defined as,**

$$ \text{RTS} = \frac{x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2}}{f} \quad \text{--- (2.3.4)} $$


This definition holds good for homogeneous production functions too.

For piecewise linear technology the production possibility sets consistent with constant, non-increasing and variable returns to scale are respectively,*

\[ T(k) = \{(x,u) : \sum \lambda x, \leq x, \sum \lambda u, \geq u, \lambda, \geq 0 \} \]
\[ T(NI) = \{(x,u) : \sum \lambda x, \leq x, \sum \lambda u, \geq u, \lambda, \geq 0, \sum \lambda, \leq 1 \} \quad \text{(2.3.5)} \]
\[ T(v) = \{(x,u) : \sum \lambda x, \leq x, \sum \lambda u, \geq u, \lambda, \geq 0, \sum \lambda, = 1 \} \]

\[ T(v) \subseteq T(NI) \subseteq T(k) \]

![Graph](image)

Figure (2.3.1)

\[ T(v) = C \]
\[ T(NI) = A \cup C \]
\[ T(k) = A \cup B \cup C \]

The production possibility set consistent with variable returns to scale is the region \( C \). The PP set that admits non-increasing returns to scale is union of the regions \( A \) and \( C \). The ray that emanates from the

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* Fore, R.C., S. Grosskoff and Lovel, C.A.K., (1985), OP.Cit.,
origin is constant returns to scale frontier. The PP set that admits only constant returns to scale is union of the sets A, B and C.

In terms of input level sets, we have,

\[ L^* (u) \subseteq L^{**} (u) \subseteq L^t (u) \]

Where \( L^* (u) \), \( L^{**} (u) \) and \( L^t (u) \) are input level sets consistent with variable, non-increasing and constant returns to scale respectively.

**Factor minimal cost:**

Let \( Q (u, p) \) be the factor minimal cost function. It satisfies the following properties:

p.1 \( Q (u, \lambda p) = \lambda Q (u, p) \) \hspace{1cm} \cdots \hspace{1cm} (\ast)

p.2 \( p_1 \geq p_2 \Rightarrow Q (u, p_1) \geq Q (u, p_2) \) \hspace{1cm} \cdots \hspace{1cm} (\ast\ast)

\[ (\ast) \hspace{1cm} Q (u, p) = \min \{ px : \phi (x) = u \} \]

\[ Q (u, \lambda p) = \min \{ \lambda px : \phi (x) = u \} \]

\[ = \lambda \min \{ px : \phi (x) = u \} \]

\[ = \lambda Q (u, p) \]

\[ (\ast\ast) \hspace{1cm} p_1 \geq p_2 \Rightarrow p_1 x \geq p_2 x, \forall x \text{ such that } \phi (x) = u \]

\[ \Rightarrow \min \{ p_1 x : \phi (x) = u \} \geq \min \{ p_2 x : \phi (x) = u \} \]

\[ \Rightarrow Q (u, p_1) \geq Q (u, p_2) \]
\[ Q \{ u, \lambda p + (1-\lambda) p_2 \} \geq \lambda Q (u, p_1) + (1-\lambda) Q (u, p_2) \]

\[ 0 \leq \lambda \leq 1 \]

p.4  \( Q (\alpha, p) = 0 \)

To produce null output vector the factor minimal cost is zero.

p.5  \( Q (u, p) > 0, \text{ if } u > 0, \; p \geq 0 \)

The factor minimal cost to produce a positive output is positive if at least one input is not a free good.

The restrictions on the cost function imply restrictions on the first-order conditions of the optimization problem,

\[ \text{Min} \{ p x : \phi (x) = u \} \quad \text{--- (2.3.6)} \]

The most commonly used form of the first order conditions is the system of input demand equations.

If we introduce the axiom of inefficiency, then the factor minimal cost function may be expressed as the following optimization problem:

\[ Q (u, p) = \text{Min} \{ p x : x + L (u) \} \quad \text{--- (2.3.7)} \]

If output is scalar valued, equivalently we write,

\[ Q (u, p) = \text{Min} \{ p x : \phi (x) \geq u \} \]

where \( \phi (x) = \text{Max} \{ u : x + L (u) \} \)

For piecewise linear technology, factor minimal cost is obtained by solving the following linear programming problem:

\[ Q^k (u, p) = \text{Min} \; p x \]

subject to

\[ \sum \lambda x, \leq x \]

\[ \sum \lambda u, \geq u \quad \text{--- (2.3.8)} \]

\[ \lambda, \geq 0 \]
$Q^* (u,p)$ is factor minimal cost consistent with constant returns to scale.

Let $Q (u,p)$ be the factor minimal cost function and $C$ be the observed cost.

Define, $C = Q (u,p) + u + v$ \hspace{1cm} \text{--- (2.3.9)}

where $u$ is one sided error term. Failure to achieve cost efficiency which can be decomposed into the product of technical and allocative efficiencies leads to the presence of one sided error component $'u'$. $v$ is two sided Statistical error term. For example, one may assume $'u'$ follows half normal and $'v'$ follows normal distribution*.

$v = 0 \Rightarrow Q (u,p)$ is full frontier.


Observed costs fall on or above the full frontier.

Estimation of stochastic cost function involves two stages. In the first stage, the managerial errors \((u)\) are computed for the sake of which the following linear programming problem is solved for each DMU:

\[ Q(u,p) = \min \{ px : x + L(u) \} \]

\[ u = C - Q(u,p) \]

where \(C\) is observed cost of production

\[ u = C \left[ 1 - \frac{Q(u,p)}{C} \right] \]

\[ = C [1 - OE] \quad \text{--- (2.3.10)} \]

where \(OE\) is overall productive efficiency.

The concepts of technical, allocative and overall production efficiencies can be best understood by means of the following diagram:

![Diagram](image-url)

\[ Px = Q(u,p) \]

Fig. (2.3.3)
The producer who operates at \( P \) is technically inefficient. By reducing his inputs radially to the point \( Q \) he attains technical efficiency. Consequently, technical efficiency is defined as *

\[
TE = \frac{OQ}{OP}
\]

Overall productive efficiency is defined as the ratio of factor minimal cost to actual cost of production.

\[
u = C\left[1 - OE\right]
\]

The stochastic factor minimal cost function may be expressed as,

\[
C^* = \exp \left[ f(p,y) + v \right]
\]

\[
\ln C^* = f(p,y) + v
\]

\( C^* \) is obtained by the adjustment,

\[
C^* = C - u
\]

Thus, a knowledge of actual cost and cost efficiency leads to \( \{C^*\} \) series.

Before we estimate a stochastic cost frontier, a priori a functional form should be assigned to \( f(p,y) \), where \( p \) is the vector of input prices and \( y \) is the observed output. Let this functional form be approximated by a translog cost function which is a second order Taylor series approximation of an arbitrary functional form.

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\[ f(p, y) = \alpha_\nu + \sum \alpha_i \ln p_i + \frac{1}{2} \sum \sum \gamma_{ij} \ln p_i \ln p_j + \frac{1}{2} \sum \sum \delta_{ij} \ln y_i \ln y_j + \sum \sum \xi_{ij} \ln p_i \ln y_j \ldots \] (2.3.13)

This function should satisfy certain regularity conditions*.

R.1 It should be linear in input prices, since if all prices increase by \( \lambda \) and output is constant, the factor minimal cost should also increase by \( \lambda \).

This requires to impose the following parametric restrictions.

\[ \sum \alpha_i = 1, \sum \gamma_{ij} = 0, \sum \gamma_{ij} = 0, \sum \xi_{ij} = 0, \sum \xi_{ij} = 0 \]

R.2 Parameters satisfy the symmetry condition,

\[ \gamma_{ij} = \gamma_{ji}, \]
\[ \xi_{ij} = \xi_{ji} \]

R.3 Monotonicity of cost shares require non-positive own elasticities.

\[ \gamma_{ii} \leq 0 \]

But these are only necessary conditions.

Parameters of translog stochastic cost function are estimated by linear goal programming approach.


Diewart, W., (1976), 'Exact and Superlative Index Numbers, Journal of Econometrics, 4, pp 115-146.
For jth production unit, the logarithmic stochastic cost function may be expressed as,

\[ \ln C_j = f_j(p,u) + v. \]

Let \( v_i = d_i^r - d_i^l. \)

where \( d_i^r = \text{Max} \{a, v_i\} \)

\[ d_i^l = \text{Min} \{a, v_i\} \]

\( d_i^r, d_i^l \geq 0 \)

i\textsuperscript{th} cost share equation of j\textsuperscript{th} production unit*:

\[ g_v(p,y) + d_i^r - d_i^l = S_v. \]

\( d_i^r, d_i^l \geq 0 \)

\* \( \ln C_j = f_j(p,y) \)

i\textsuperscript{th} cost share equation is obtained as follows:

\[ \frac{\partial \ln C_j}{\partial \ln p_i} = \frac{\partial C_j}{\partial p_i} \frac{p_i}{C_j}. \]

By Shephard's lemma, we have \( \frac{\partial C_j}{\partial p_i} = X_u(p,y) \)

\[ \frac{\partial \ln C_j}{\partial \ln p_i} = \frac{p_i X_u}{C_j} = S_v \]

\[ g_v(p,y) = S_v \]

If stochastic cost function is manipulated,

\[ g_v(p,y) + v_i = S_v. \]

\[ g_v(p,y) + d_i^r - d_i^l = S_v. \]
The Goal Programming, Constrained Regression (GP/CR) model is:

Minimize \[ \sum (d_r^+ + d_r^-) + \sum \sum (d_r^+ + d_r^-) \]

Subject to

\[ f_r (p, y) + d_r^- - d_r^+ = \ln \xi_r \] \hspace{1cm} (2.3.14)

\[ g_r (p, u) + d_r^- - d_r^+ = S_r \]

\[ d_r^+, d_r^-, d_r^+, d_r^- \geq 0 \]

This is a goal programming problem with single priority. It belongs to robust estimation, since its estimators are less influenced by outliers than the least squares estimators. The resultant estimators are equivalent to maximum likelihood estimators under the assumption of Laplace error distribution. Augmentation of cost share equations in estimation the estimating procedure has the effect of adding many additional degrees of freedom without including any regression co-efficient.

2.4 Full frontier** cost function and its estimation:

Full frontier cost function is associated with the following specification:

\[ C = \exp \left[ f (p, u) + u \right] \] \hspace{1cm} (2.4.1)

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Estimation of full frontier requires to solve the following goal programming problem:

Minimize \[ \sum \delta^* + \sum (d'_j - d'_i) \]

subject to

\[ f_i(p, y) + \delta^*_i = \ln C_i \]
\[ g_n(p, u) + d'_v - d'_n = S_v \] \[(2.4.2)\]
\[ \delta^*_i, d'_v, d'_n \geq 0 \]

There are two differences between the stochastic and full cost function frontiers.

(a) Full frontier uses observed \( C_i \) and \( S_n \) as dependent variables, while stochastic frontier cost function uses \( C^*_i \) and \( S^*_n \).

(b) Full frontier maintains solely \( \delta^*_i \) in the objective function so as to yield the full frontier cost function. Stochastic frontier includes both \( \delta^*_i \) and \( \delta^*_j \) in the objective function so that it can estimate parameters of the stochastic frontier cost function, using observed cost values.

Linear goal programming approach is used to estimate stochastic production functions also*.

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Let $Y_j$ be observed or transformed output of $j^{th}$ production unit, $X_j$ be the vector of inputs of $j^{th}$ production unit. We postulate the following linear regression equation:

$$Y_j = X_j' \beta - u_j + v_j \quad (j = 1, 2, \ldots, n)$$

2.5 As an example consider the Cobb-Douglas production frontier* of $j^{th}$ production unit:

$$y_i = A \prod_{i=1}^{m} x_{ii}^{a_i} e^{\cdot \cdot \cdot v_i} \quad (\text{2.5.1})$$

where $x_{ij}$: $i^{th}$ input employed by the $j^{th}$ production unit.

$$\ln y_j = \ln A + \sum_{i=1}^{m} a_i \ln x_{ij} - u_j + v_j$$

$$Y_j = X_j' \beta - u_j + v_j \quad (\text{2.5.2})$$

where $Y_j$ is transformed output

$$\beta = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}$$

$X_j = \{ x_{ij}, x_{j2}, \ldots, x_{jm} \}$

$u_j (>0)$ is one sided disturbance term.

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$u_j$ is stochastic disturbance term.

The managerial error $u_j$ is interpreted as deviation from frontier production function.

![Graph](image)

**Fig. (2.5.1)**

$\delta$ is output technical efficiency, where $\delta \geq 1$. Let $y$ be observed output, then we have,

\[ \delta y = f(x) \]
\[ y = \delta^{-1} f(x) \]
\[ \ln y = \ln f(x) - \ln \delta \]

\[ Y = X\beta \cdot u \quad \cdots (2.5.3) \]

where $u = \ln \delta$

$f(x)$ is full frontier, in the sense that the outputs of all production units fall on or below the frontier production function.
The method of estimation is called Data Envelopment Analysis and Least Absolute Deviation (DEA/LAV) method.

Stochastic frontier production function*:

\[ Y_i = X_i \beta - u_i + v_i \]

Full frontier : \( X_i \beta \)

Average frontier : \( X_i \beta + v_i \)

DEA / LAV estimation requires first to estimate technical efficiency component, which can lead to the computation of \( \{ \bar{Y}_i \} \) series, where,

\[ Y_i = Y_i + u_i \]

For \( j^{th} \) production unit DEA estimate of \( u_j \) is obtained by solving the following linear programming problem:

Max \( \delta \) such that

\[ \sum \lambda_i x_i \leq x, \]

\[ \sum \lambda_i y_i \geq \delta y, \quad (2.5.5) \]

\[ \lambda_i \geq 0 \]

where \( x_i \) and \( y_i \) are input vector and output of \( j^{th} \) production unit respectively. \( \delta_j \) is output technical efficiency of \( j^{th} \) production unit.

Let \( \delta^* = \text{Max} \delta \)

To understand the above linear programming problem one should have knowledge of output sets. An output set may be designated as \( P(x) \). \[
\begin{align*}
P(x) &= \{ \, u : x \text{ produces } u \, \} \\
&= \{ \, u : x \in L(u) \, \}
\end{align*}
\]

Conversely, \( L(u) = \{ \, x : u \in P(x) \, \} \) \quad \text{--- (2.5.6)}

Thus, there is duality between input and output sets.

The output set \( P(x) \) in the above figure belongs to one input and two output production technology. The production unit that operates at A is output technical inefficient. By radially expanding its outputs the producer attains output technical efficiency if he operates at B. Thus, output technical efficiency may be defined as,

\[
\text{OTE} = \frac{OB}{OA} \geq 1
\]

OTE is viewed as the following optimization problem:

Maximize $\delta$

such that $\delta u - P(x)$

Further, if the production technology is piecewise linear, we can express $P(x)$ as,

$$P(x) = \{ u: \sum \lambda, u, \geq u, \sum \lambda x, \leq x, \lambda, \geq 0 \}$$

Output technical efficiency is estimated, solving the following linear programming problem*:

Max $\delta$

such that

$$\sum \lambda, x, \leq x.$$

$$\sum \lambda, u, \geq \delta u.$$

$$\lambda, \geq 0$$

where $x_o$ and $u_o$ are input and output vectors of the production unit whose efficiency is under evaluation.

* Fare, R.C., S. Grosskoff and Lovell, C.A.K., (1986), OP. Cit.,
For each production unit compute $\delta_i$ and $u_i$, where

$$u_i = \ln \delta_i$$

To estimate stochastic frontier production function compute the series,

$$\{Y_i\},$$

where $Y_i = Y_i + u_i$

The LAV estimation is carried out to obtain parametric vector estimates, the linear regression equation being,

$$Y_i = X_i \beta + v_i$$

Define $v_i = v_i^* - v_i$, $(j = 1,2,\ldots,n)$

The LAV, linear goal programming problem* is,

Minimize $\Pi = \sum_{i=1}^{n} (v_i^* + v_i)$

subject to $X_i \beta + v_i - v_i = Y_i$

$$x_i^*, x_i \geq 0, \quad j = 1,2,\ldots,n$$

Let $\hat{\beta}$ be the LAV estimator of $\beta$. Some of the properties of LAV estimator are as follows:

* $\Pi = \sum |v_i| = \sum (v_i^* + v_i)$
The goal programming problem is,

\[ \min \ a_1 \beta_1 + \ldots + a_m \beta_m + \eta^- \gamma_1 + \ldots + \eta^- \gamma_p + \eta^+ \gamma_q \]

\[
\begin{bmatrix}
x_{11} & x_{12} & \cdots & x_{1n} & 1 & -1 & 0 & \ldots & 0 & 0 \\
x_{21} & x_{22} & \cdots & x_{2n} & 0 & 0 & 1 & -1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
x_{n1} & x_{n2} & \cdots & x_{nn} & 0 & 0 & 0 & 0 & \ldots & -1 & 1
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_m \\
\eta^-_1 \\
\eta^-_2 \\
\vdots \\
\eta^-_p \\
\eta^+_1 \\
\eta^+_2 \\
\vdots \\
\eta^+_q
\end{bmatrix}
= 
\begin{bmatrix}
Y_1 \\
Y_2 \\
\vdots \\
Y_m \\
Y_p \\
\vdots \\
Y_q
\end{bmatrix}
\]

\[ \eta^-, \eta^+ \geq 0 \]

The dual of this problem is,

\[
\max \ Z = \sum w_j Y_j + \ldots + \sum w_j Y_q
\]

subject to \[ \sum w_j x_i = 0, \quad i = 1, 2, \ldots, m \] \quad --- (2.5.9)

\(-1 \leq w_j \leq 1, \quad j = 1, 2, \ldots, n \)

where \( w_j \) is the dual variable associated with the \( j \)th constraint.

It can be shown that,

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* Shiyuki Sueyoshi, (1991), OP. Cit.
$w_j = 1 \Rightarrow \text{$j$th observation lies above the LAV hyperplane}$

$-1 < w_j < 1 \Rightarrow \text{$j$th observation lies on the LAV hyperplane}$

$w_j = -1 \Rightarrow \text{$j$th observation lies below the LAV hyperplane}$

ii) LAV estimators are robust

iii) LAV can yield maximum likelyhood (ML) estimates under the assumption that the stochastic error term follows Laplace error distribution.

iv) DEA / LAV and DEA Estimates Of Output Technical Efficiency:

\[\begin{array}{c}
\text{A, B, C, D are data points. The piecewise linear production frontier is DEA efficiency frontier. A, B, D are DEA efficient. C is DEA inefficient. Output technical efficiency as implied by DEA frontier is,}
\end{array}\]
OTE (DEA) = \frac{y}{y}.

The smooth frontier is DEA/LAV frontier. Output technical efficiency of C measured relative to DEA/LAV frontier is,

OTE (DEA/LAV) = \frac{Y}{y}.

Thus, it is expected that there is some difference between the two efficiency measures.

The point B is DEA efficient but not DEA/LAV efficient.

The point D is both DEA and DEA/LAV efficient.

2.6 Goal programming has applications in discriminent analysis\(^*\) (DA). DA is a statistical technique used to predict group membership. In the DA approach, a group of observations whose memberships are already identified, are used for the measurement of a set of estimates by minimizing incorrect group classification. DA provides a set of estimates (weights), consequently yielding an evaluation score, that is compared with a threshold value for determining group membership.

DA can be formulated by the following model:

\[
\text{Min} \quad \sum_{i \in t_1} S_i^+ + \sum_{i \in t_2} S_i^-
\]

Subject to

\[
\sum_{i \in t} \lambda_i z_i + S_i^+ \geq d, \quad j \in G_1
\]

\[
\sum_{i \in t} \lambda_i z_i - S_i^- \leq d, \quad j \in G_2
\]

\[S_i^+, S_i^- \geq 0\]

\[d \text{ and } \alpha \text{ are unrestricted for sign.}\]

\[d \text{ is a threshold value.}\]

This is a goal programming problem. There are \(n\) decision making units (DMU). With each DMU there are \(k\) characteristic measurements. It is a priori known that \(n_1\) DMUs belong to first group \((G_1)\) and \(n_2\) DMUs belong to second group \((G_2)\). The threshold value may be unknown.

\[
\sum_{i \in t} \lambda_i z_i - S_i^+ \quad d \quad \sum_{i \in t} \lambda_i z_i + S_i^-
\]

\[n_1 + n_2 = n\]
This DA model produces an optimum set of weights ($\lambda^*$) which define a hyperplane for separating the two groups. Suppose a newly sampled DMU possesses the measurements $\{z_i\}$. To classify this DMU into $G_1$ or $G_2$, we combine $\lambda^*$ and $z_i^*$ and compute,

$$\sum \lambda^* z_i^*$$

we classify the DMU into $G_1$ or $G_2$ depending upon the value $\sum \lambda^* z_i^*$ falls below or above the estimated threshold value, $d^*$.

To avoid a trivial solution, $\lambda_i = 0, \forall i$ and $d=0$, to make a more clear separation between $G_1$ and $G_2$, we reformulate the previous goal programming problem as,

$$\text{Min} \quad \sum \sum S_i^r + \sum S_i^a$$

subject to

$$\sum \lambda_i z_i + S_i^r \geq d, \quad j \in G_1$$

$$\sum \lambda_i z_i - S_i^a \leq d - \eta, \quad j \in G_1$$

$$S_i^r \geq 0, S_i^a \geq 0$$

$d, \alpha$ unrestricted for sign
\[ \eta \text{ is a small number to impose a small gap between the two groups.} \]

\[
\sum \lambda, z_a - S_j - \eta \quad d \quad \sum \lambda, z_a + S_j
\]

Using additional slacks we reformulate the problem as,

\[
\text{Min } \sum_{j \in I^1} S_j^* + \sum_{j \in I^2} S_j^*
\]

subject to

\[
\sum \lambda, z_a + S_j^* - S_j = d \quad j \in G_1
\]

\[
\sum \lambda, z_a + S_j^* - S_j^* = d - \eta \quad j \in G_2
\]

\[ S_j^*, S_j^* \geq 0 \]

\[ d \text{ and } \lambda \text{ are unrestricted} \]

For a newly sampled DMU the classification into \( G_1 \) or \( G_2 \) is as follows:

(i) \( S_j^* > 0 \Rightarrow S_j^* = 0 \)

\[ \Rightarrow \sum \lambda, z_a - S_j = d \]

\[ \Rightarrow \sum \lambda, z_a > d \]

\[ \Rightarrow j \in G_1 \]
(ii) \( S_j^* > 0 \Rightarrow S_j^- = 0 \)
\[
\Rightarrow \sum \alpha_i z_{iu} + s_i^* = d - \eta \\
\Rightarrow \sum \alpha_i z_{iu} < d - \eta \Rightarrow j \in G_i.
\]

(iii) The non-negative slacks \( S_j^* \) and \( S_j^- \) may be interpreted as errors due to misclassifications.

\( j \notin G_i \), but suppose \( S_{ji}^* > 0 \Rightarrow S_{ji}^- = 0 \)
\[
\Rightarrow \sum \lambda_{ji} z_{ji} < d \Rightarrow j \in G_i.
\]
Thus, \( S_{ji}^- \) is an error due to misclassification.

Similarly, suppose \( j \notin G_2 \). But \( S_2 > 0 \)
\[
\Rightarrow S_{2i}^* = 0 \\
\Rightarrow \sum \alpha_i z_{2i} > d - \eta \\
\Rightarrow j \in G_i.
\]
\( S_{2i}^- \) is an error due to misclassification.

**DEA - additive model:**

\[
\text{Min } \sum_{i=1}^n S_i^* + \sum_{j=1}^p S_j^- \\
\text{subject to } \sum_{j=1}^n x_{ij} \lambda_j + S_i^* = x_i, \quad (i = 1,2,\ldots,k) \\
\quad \sum_{j=1}^n y_{ij} \lambda_j - S_j^- = y_j, \quad (r = 1,2,\ldots,s)
\]

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The various features of DEA problem are as follows:

(i) $n$ producers are in competition,

(ii) $k$ inputs are combined to produce $s$ outputs.

(iii) $x_{ij}$: $i^{th}$ input of $j^{th}$ production unit.

(iv) $y_{ij}$: $r^{th}$ output of $j^{th}$ production unit.

(v) $x_{io}, y_{io}$ are $i^{th}$ input and $r^{th}$ output of the production unit whose efficiency is under evaluation.

(vi) $\lambda_j (j = 1, 2, \ldots, n)$ are the non-negative intensity parameters.

By combining additive DEA and DA the following goal programming is formulated:

\[
\begin{align*}
\text{Min} & \quad \sum_{i \in G_1} S_{ij} + \sum_{i \in G_2} S_{ij}, \\
\text{such that,} & \quad \sum_{i} \alpha_i z_i + S_{ij} - S_{ij} = d, \quad j \in G_1, \\
& \quad \sum_{i} \beta_i z_i + S_{ij} - S_{ij} = d - n, \quad j \in G_2, \\
& \quad \sum \alpha_i = 1 \\
& \quad \sum \beta_i = 1
\end{align*}
\]
All slacks non-negative

\[ \alpha_i, \beta_i \geq 0 \]

\[ d : \text{unrestricted for sign.} \]

This problem has the following DEA features:

(i) Sum of slacks are minimized,

(ii) The intensity parameters are non-negative and sum to unity.

But, in the specification of the problem two separating functions,

\[ \sum \alpha_i z_{i*} \text{ and } \sum \beta_i z_i \text{ are used.} \]

The use of two separating functions have never been fully examined in the previous DA studies.

Let the optimal values of \( \alpha, \beta, \) and \( d \) be denoted by,

\( \alpha^*, \beta^*, \) and \( d^* \).

The following are the DEA - DA criterion for the classification of the observations.

(i) \[ \sum \beta^* z_m < d^* \leq \sum \alpha^* z_m \Rightarrow \text{m} \in G_i \cap \bar{G}_i, \] .... (2.6.6)

(ii) \[ \sum \alpha^* z_m < d^* < \sum \beta^* z_m \Rightarrow \text{m} \in G_i \cap \bar{G}_i, \]
for example, the first case arises if \( S_{1m} = 0 \) or \( S_{1m}^- = S_{2m} = 0 \) and \( S_{2m}^* > 0 \).

\[
\begin{align*}
S_{1m} > 0 \Rightarrow S_{1m}^- = 0 \Rightarrow \sum \alpha^i z_m > d^* \\
S_{1m}^- = S_{1m} = 0 \Rightarrow \sum \alpha^i z_m = d^* \\
\end{align*}
\]

combining we obtain, \( \sum \alpha^i z_m \geq d^* \)

\[
\begin{align*}
S_{1m} > 0 \Rightarrow S_{2m} = 0 \Rightarrow \sum \beta^i z_v < d^* - \eta < d^* \\
\Rightarrow \sum \beta^i z_v < d^*. \\
\end{align*}
\]

(b) \( \sum \alpha^i z_m < d^* \) \[ \Rightarrow m \in G_i \]

\[
\sum \beta^i z_v < d^* \] \[ \Rightarrow m \in G_i \] \[ \text{--- (2.6.7)} \]

This arises if \( S_{1m}^*, S_{1m}^- > 0 \)

\[
\begin{align*}
S_{1m}^* > 0 \Rightarrow S_{1m}^- = 0 \Rightarrow \sum \alpha^i z_v < d^* \\
S_{1m}^- > 0 \Rightarrow S_{1m} = 0 \Rightarrow \sum \beta^i z_v < d^* \\
\end{align*}
\]

(c) \( \sum \alpha^i z_m \geq d^* \) \[ \Rightarrow m \in G_i \]

\[
\sum \beta^i z_v \geq d^* \] \[ \Rightarrow m \in G_i \] \[ \text{--- (2.6.8)} \]

Thus, in the presence of (b) or (c) there is no overlap. However, when there is overlap, we solve the following problem,
Min \[ \sum_{i \in i} S_{i} + \sum_{j \in j} S_{j} \]

subject to

\[ \sum \alpha \cdot z_{u} + S_{u} - S_{u} = d, \quad i \in G_{1} \quad \text{(2.6.9)} \]
\[ \sum \alpha \cdot z_{v} + S_{v} - S_{v} = d - \eta, \quad j \in G_{2} \]
\[ \sum \alpha = 1 \]

All slacks are non-negative, \( d \) is unrestricted.

After obtaining \( \alpha^{*} \), the DEA-DA problem classify all the observations in \( G_{1} \cap G_{2} \) into either \( G_{1} \) or \( G_{2} \) by comparing \( \sum \alpha^{*} \cdot z_{u} \) with the threshold value \( d^{*} \).

\[ \sum \alpha^{*} \cdot z_{u} \geq d^{*}, \quad j \in G_{1} \cap G_{2}, \quad \text{then} \quad j \in G_{1} \quad \text{(2.6.10)} \]
\[ \sum \alpha^{*} \cdot z_{v} < d^{*}, \quad j \in G_{1} \cap G_{2}, \quad \text{then} \quad j \in G_{2} \]

2.7 Goal programming has applications in portfolio management.

In portfolio management the deviation of realized total return on all assets from expected total return on all assets subject to relevant linear constraints.

---

An individual invests his monitory resource on \( n \) assets, namely \( S_1, S_2, \ldots, S_n \).

Let \( R_j \) be rate of return on \( S_j \)

\[
R_j : r_{j1}, r_{j2}, \ldots, r_{jt}
\]

where \( R_j = r_{jt} \) is the rate of return on \( S_j \) for the time period \( t \).

\( x_j \) is investment of \( S_j \)

\[
\sum_{j=1}^{n} x_j = M_u
\] \hspace{1cm} \text{.... (2.7.1)}

Let \( u_j \) be maximum amount allowed for investment in \( S_j \).

\( 0 \leq x_j \leq u_j \)

Define mean rate of return on \( S_j \) as,

\[
E(R_j) = \frac{\sum_{t=1}^{T} r_{jt}}{T} = r_j
\] \hspace{1cm} \text{.... (2.7.2)}

(i) Return on \( j^{th} \) asset : \( r_j x_j \)

Return on total investment : \( \sum_{j=1}^{n} r_j x_j \)

Let \( \rho \) be the minimum rate of return.

Minimum return on total investment : \( \rho M_u \)

Rate of return constraint :

\[
\sum_{j=1}^{n} r_j x_j \geq \rho M_u
\] \hspace{1cm} \text{.... (2.7.3)}
(ii) Total investment on all the assets: \( \sum_{j=1}^{n} x_j \).

Investment constraint: \( \sum_{j=1}^{n} x_j = M_n \).

(iii) Investment constraint on \( j \)th asset:

\[
0 \leq x_j \leq u_j
\]

The following linear programming problem is postulated:

Minimize

\[
\frac{1}{T} \sum_{j=1}^{n} \left| \sum_{i=1}^{r} \alpha_{ij} x_i \right|
\]

subject to

\[
\sum_{j=1}^{n} r_j x_j \geq \rho M_n \quad \text{---- (2.7.4)}
\]

\[
\sum_{j=1}^{n} x_j = M_n
\]

\[
0 \leq x_j \leq u_j
\]

The objective function is the difference between realized total return on all assets, and expected total return on all assets.

Consider

\[
\frac{1}{T} \sum_{j=1}^{n} \sum_{i=1}^{r} x_i = 0
\]

\[
\frac{1}{T} \sum \sum (r_n - r_i) x_j = 0
\]
The above linear programming problem can be presented as a goal programming problem as shown below:

Let \[ \sum a_t x_i = v_i - w_i, \]

\[ v_i \geq 0, w_i \geq 0 \]

\[ v_i, w_i = 0, \forall t \]

\[ |\sum a_t x_i| = v_i + w_i. \]
\[
\text{Min } \frac{1}{T} \sum_{i=1}^{I} (v_i + w_i)
\]

subject to

\[
\sum_{j=1}^{n} a_{ij} x_j = v_i - w_i, \quad t = 1, 2, \ldots, T \]

\[
\sum_{j} x_j \geq \rho M_i
\]

\[
\sum x_j = M_i
\]

\[
0 \leq x_j \leq u_j, \quad j = 1, 2, \ldots, n
\]

\[
v_i, \ w_i \geq 0, \quad t = 1, 2, \ldots, T
\]

\[
v_i, \ w_i = 0, \quad t = 1, 2, \ldots, T
\]