Chapter 1

INTRODUCTION

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The basic ideas of Graph theory are introduced during 18th century by the great mathematician Leonard Euler. Since then relatively in a short period the major development of Graph theory has occurred and inspired many mathematicians and it has become the source of interest to many researchers. To some extent this may be due to the ever growing importance of computer science and its connection with Graph theory.

Today Graph theory is one of the most flourishing branches of modern mathematics. The last 30 years have withnessed spectacular growth of Graph theory due to its wide applications to descrete optimization problems, combinatorial problems and classical algebraic problems.

Graph theory is a fascinating subject. One simple way of representing the structure of a system is to use graphs, which are simple diagrams consisting of points (vertices) and lines (edges). Graphs are useful in enhancing the understanding of the organization and behavioural characteristics of complex systems. It has a very wide range of applications to many fields like engineering, physical, social and biological sciences, linguistics etc.,

This thesis concentrates on the theory of domination in graphs.

The theory of domination has been the nucleus of research activity in graph theory in recent times. This is largely due to a variety of new parameters, that can be developed from the basic definition of domination. The NP-completeness of the basic domination problems and its close relationship to other NP-completeness problems have contributed to the enormous growth of research activity in domination theory. It is clearly established from the exclusive coverage of the "Topics on domination in graph" in the 86^{th} issue of the Journal of Discrete mathematics (1990), that the theory of domination is a very popular area for research activity in graph theory.

The rigorous study of dominating sets in graph theory began around 1960, even though the subject has historical roots dating back to 1862 when de Jaenisch (16) studied the problems of determining the minimum number of queens which are necessary to cover or dominate a n x n chessboard.

In 1958, Berge [7] defined the concept of the domination number of a graph, calling this as "coefficient of External Stability". In 1962, Ore [27] used the name 'dominating set' and 'domination number' for the same concept. In 1977 Cockayne and Hedetniemi [10] made an interesting and extensive survey of the results know at that time about dominating sets in graphs. They have used the notation γ (G) for the domination number of a graph, which has become very popular since then.

The survey paper of Cockayane and Hedetniemi has generated lot of interest in the study of domination in graphs. In a span of about twenty years after the survey, more than 1,200 research papers have been published on this topic, and the number of papers continued to be on the increase. Since then a number of graph theorists Konig [17], Ore [27], Bauer [6], Harary [6], Lasker [24], Berge [7], Cockayne [10], Hedetniemi [15], Alavi [2], Allan [3], Chartrand [2], Kulli [18], Sampthkumar [30], Walikar [35], Armugam [5], Acharya [1], Neeralgi [31], Nagaraja Rao [25], Vangipuram [32] many others have done very interesting and significant work in the domination numbers and the other related topics. Recent book on domination [14], has stimulated sufficient inspiration leading to the expansive growth of this field of study. It has also put some order into this huge collection of research papers, and organized the study of dominating sets in graphs into meaningful sub areas, placing the study of dominating sets in even broader mathematical and algorithmic contexts.

The split domination in graphs was introduced by Kulli & Janakiram [21]. They defined the split dominating set and the split domination number and obtained several interesting results regarding the split domination number of some standard graphs. They have also obtained relations of split domination number with the other parameters such as domination number, connected domination number, vertex covering number etc., Sampathkumar [29] obtained some interesting results on tensor product of graphs. Vasumathi & Vangipuram [32] and Vijayasaradhi & Vangipuram [33] obtained domination parameters of an arithmetic Graph and also they have obtained an elegant method for the construction of a arithmetic graph with the given domination parameter.

An interesting problem in Agricultural Science has generated considerable interest in us to apply the principle of domination in the field of Agriculture. The problem in the Agricultural Sciences is as follows :

In the field of agriculture control of pests plays a major role. Several pests interact among themselves resulting in the large scale production of pests causing wide spread damage to the agricultural products. The one reasonable solution to this problem will be to isolate the pests, so that the interaction among the several types of pests is prevented and it is easy for the agriculturists to eliminate the isolated pests. We identified the network of the several types of pests as a graph with a vertex in the graph being a specific type of the pest and the other pests with which this variety will interact denoting the adjacency of the vertex in the graph.

The problem reduces to the problem of removing a set of vertices in the graph such that the induced subgraph <V-D> contains only a set of independent vertices. To solve this problem we have introduced a new parameter called the annihilator dominating set.

Motivated by the study of domination, split domination and this problem of agricultural science, this new parameter 'Annihilator dominating set' is an extension of the concept of split domination set to this new concept called the Annihilator dominating set. We have investigated some properties of the split domination, annihilator domination numbers of some standard graphs, the product graphs and arithmetic graphs. We now give some basic definitions used in the thesis.

Basic Definitions :

Graph

A graph G is an ordered triple (V, E, ψ) consisting of a non-empty set of vertices (or nodes), a set E of edges (or lines or links) and an incidence function ψ that associates each edge e of G an unordered pair (u,v) of vertices of G.

If the edges are ordered pairs of vertices, then the graph is said to be a directed graph.

Incidence

If e = uv, then the edge e is said to join the vertices u and v. We also say that the edge e is incident with the vertices u and v, and u, v are said to be incident with the edge e.

Adjacency

If there is an edge e incident with the vertices u and v, then u and v are called adjacent vertices.

Similarly if there is a vertex incident with edges e and f, then e and f are called adjacent edges.

Degree

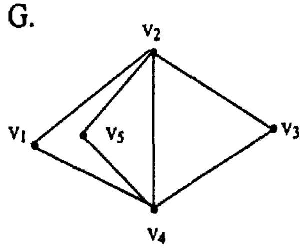
The number of edges incident with a vertex v is called the degree of v and is denoted by d (v).

Isolated vertex

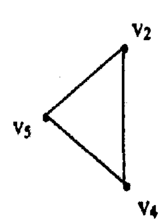
A vertex v in G such that d (v) = 0 is called an isolated vertex in G.

Subgraph

A subgraph of G is a graph having all of its vertices and edges in G.



Graph



Subgraph

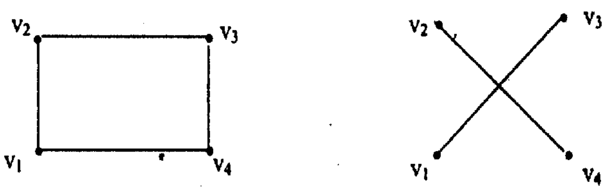
Fig. 1.1

Vertex induced subgraph

Let G (V, E) be a graph and V_1 is a subset of V. Then the subgraph of G whose vertex set is V_1 and edge set is the set of those edges in E whose both ends are in V_1 is called the vertex induced subgraph and in denoted by $<V_1>$.

Complementary graph

Let G (V, E) be a graph. The complementary graph G^C of G is a graph whose vertex set is V and two edges are adjacent in G if and only if they are non-adjacent in G^C.



Graph **Complementary Graph**

Fig. 1.2

Walk

A walk W in a graph G is a finite non-null sequence W = $v_o, e_1, v_1, e_2, ----, e_k, v_k$ whose terms are alternatively vertices and edges such that for $1 \leq i \leq k$, the end vertices of e_i are v_{i-1}, v_i and it is denoted as (v_0, v_k) – walk. v_0 is called the origin and v_k, terminus of the walk.

Trail

A walk is called a trail if all the edges of the walk are distinct.

Path

A trail in which the vertices are also distinct is called a path.

Directed Path

A directed path in a directed graph G is a non-null alternating sequence of distinct vertices and edges such that each edge is oriented from the vertex preceding it to the vertex following it.

Cycle

A path whose origin and terminus are the same is called a cycle.

Wheel

A graph is said to be a wheel when it is defined to be a graph $K_1 + C_{n-1}$, for $n \geq 4$. It is denoted by W_n.

$W_6 = K_1 + C_5$:

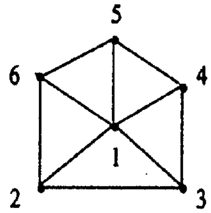


Fig. 1.3

Star

A graph is said to be a star if it is [illegible] a complete bipartite graph $k_{1,n}$. It is denoted by S_n.

S_6 : $k_{1,6}$:

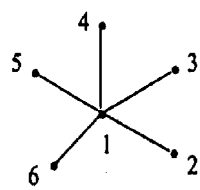


Fig. 1.4

Connected graph

A graph is said to be connected if there is a path between every pair of vertices. Otherwise it is said to be disconnected graph.

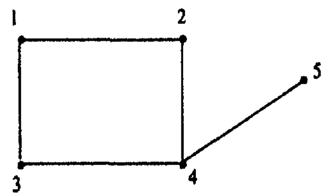
Directed Network

A directed Network D (N,L) is a directed graph whose vertex set is N and the vertices are called as nodes and the edge set is L and its elements are called as links.

Independent set

A subset S of V is called an independent set of G if no two vertices in S are adjacent.

A maximum independent set of G is an independent set whose cardinality is largest among all independent sets of G.



Independent set S = {1,4}

Fig. 1.5

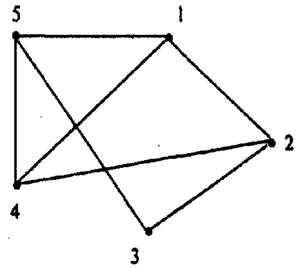
Neighbourhood of a vertex

Neighbourhood of a vertex $v \varepsilon V$ is a set consisting all vertices adjacent to v (including v). It is denoted by nbd (v)

nbd (v) = {the set of all vertices adjacent to v} $\cup$ {v}.

Dominating set

A subset D of V is said to be a dominating set of G if every vertex in V \ D is adjacent to a vertex in D.



Dominating set D = {1,3}

Fig. 1.6

Dominating number

The dominating number $\gamma(G)$ of G is the minimum cardinality of a dominating set of the Smallest size.

Split dominating set :

A dominating set D of graph G is called a split dominating set, if the induced subgraph <V-D> is disconnected.

The split dominating number $\gamma_s(G)$ of G is the minimum cardinality of a split dominating set of the Smallest size.

In this thesis, we have defined a new concept of domination viz., Annihilator dominating set and Annihilator domination number.

Annihilator dominating set

A dominating set D of graph G is said to be annihilator dominating set, if its induced subgraph <V-D> is a graph with isolated vertices or a graph with independent vertices.

Annihilator domination number

The annihilator domination number γ_a (G) of G is the minimum cardinality of an annihilator dominating set of the smallest size.

Kronecker Product of two graphs

If G_1, G_2 are two simple graphs with their vertex sets V_1 : {u_1, u_2,.....} and V_2 : {v_1, v_2, ...} respectively, then the Kronecker product of these two graphs is defined to be a graph with its vertex set as V_1 x V_2, where V_1 x V_2 is the cartesian product of the sets V_1 and V_2 and two vertices (u_i, v_j), (u_k, v_l) are adjacent if and only if u_i u_k and $v_j v_l$ are edges in G_1 and G_2 respectively.

This product graph is denoted by G_1 (k) G_2.

Cartesian Product of two graphs

If G_1, G_2 are two simple graphs with their vertex sets V_1 : $\{u_1, u_2, ...\}$ and V_2 : $\{v_1, v_2, ...\}$ respectively, then the Cartesian product of these two graphs is defined to be a graph with its vertex set as $V_1 \times V_2$: $\{w_1, w_2, ...\}$ and two vertices $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$ are adjacent if and only if either (i) $u_1 = u_2$ and $v_1v_2 \varepsilon E(G_2)$ or (ii) $u_1u_2 \varepsilon E(G_1)$ and $v_1=v_2$.

This product graph is denoted by G_1 (C) G_2.

Lexicograph product of two graphs

If G_1, G_2 one two simple graphs with their vertex sets V_1 : $\{u_1, u_2, ...\}$ and V_2 : $\{v_1, v_2, ...\}$ respectively, then the Lexicograph product of these two graphs is defined to be a graph with its vertex set as $V_1 \times V_2$: $\{w_1, w_2, ...\}$ and two vertices $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$ are adjacent if and only if either (i) $u_1 u_2 \varepsilon E(G_1)$ or (ii) $u_1 = u_2$ and $v_1v_2 \varepsilon E(G_2)$.

This product graph is denoted by G_1 (L) G_2.

Arithmetic graph

The Arithmetic graph V_m is defined as a graph with its vertex set as the set of all divisors of m (excluding 1), where m is a natural number and $m = p_1^{a_1} p_2^{a_2} \ldots\ldots p_r^{a_r}$, a canonical representation of m, where $p_i s$ are distinct primes and $a_i s \geq 1$ and two distinct vertices a,b which are not of the same parity are adjacent in this graph if $(a,b) = p_i$, for 1 i r.

The vertices a and b are said to be of the same parity if both a and b are the powers of the same prime, for instance $a=p^2$, $b = p^5$.

In the second chapter, we have obtained upper bounds for split domination number of some product graphs viz., Kronecker product graph, Cartesian product graph and Lexicograph product graph in terms of the split domination number of the given graphs.

The results are as follows :

If G_1, G_2 are any two graphs without isolated vertices, then

(i) $\gamma_s [G_1 (k) G_2] \leq \min \{ \gamma_s (G_1). |V_2|, |V_1|. \gamma_s (G_2)\}$

(ii) $\gamma_s [G_1 (C) G_2] \leq \gamma_s (G_1). |V_2|$

(iii) $\gamma_s [G_1 (L) G_2] \leq \gamma_s (G_1). |V_2|$

In the third chapter, we have obtained the split dominating set and split domination number of a Arithmetic graph V_m as given below.

If V_m is a graph, where $m = p_1^{a_1} p_2^{a_2} \ldots\ldots p_r^{a_r}$, for a_i's ≥ 1 and p_i's are distinct primes, then $\gamma_s [V_m] = r + 1$, where r is the core of m (the number of distinct prime factors of m).

Also we have evolved a method to construct a graph with a given split domination number t.

In the fourth chapter, we have defined a new concept of domination called the Annihilator dominating set and Annihilator domination number of a graph G, denoted by γ_a (G). We have obtained Annihilator domination number of some standard graphs such as tree, path, star, complete bipartite graph, etc.,

Also, we have obtained an interesting result on Annihilator domination number of a graph G in terms of split domination number and domination number. The result is as follows :

$$\gamma_a(G) = \gamma_s (G) + \sum_{i=1}^{t} \gamma (G_i)$$

Where G_1, G_2, ... G_t are components each component containing atleast two vertices in the induced subgraph <V-D>, where D is the split dominating set of G.

Further we have also obtained some upper bounds for the annihilator domination number of the product graphs.

The results are as follows :

If G_1, G_2 are any two graphs without isolated vertices, then

(i) $\gamma_a[G_1 (k) G_2] \leq \min\{ \gamma_a(G_1). |V_2|, |V_1|. \gamma_a(G_2)\}$

(ii) $\gamma_a[G_1 (L) G_2] \leq \gamma_a(G_1). |V_2| + |V_1|. \gamma_a(G_2) - \gamma_a(G_1). \gamma_a(G_2)$

(iii) $\gamma_a[G_1 (C) G_2] \leq \gamma_a(G_1). |V_2| + |V_1|. \gamma_a(G_2) - \gamma_a(G_1). \gamma_a(G_2)$

In the fifth chapter, we have obtained the upper bound for annihilator domination numbers of an arithmetic graph V_m. The result obtained is :

If V_m is a graph, where m is positive integer and $m = p_1^{a_1} p_2^{a_2}$, for a_i's≥ 1 and p_1, p_2 are distinct primes,

then (i) $\gamma_a[V_m] \leq 2a_1$, if $a_1 = a_2$

(ii) $\gamma_a[V_m] \leq 2a_1 + 1$, if $a_1 < a_2$

We have also evolved a method of construction of a graph with a given number as the cardinality of a annihilator dominating set of the graph.

The terminology and notations used in this thesis are the same as in Apostol [4], Bondy and Murty [8].

We have made a modest attempt in this thesis to reaffirm that the domination theory in graph theory generates new parameters giving a great fillip to the growth of research activity. We hope that we have sufficiently stimulated the researchers in this area by demonstrating that the tools of number theory can be usefully exploited in the matter of construction of graphs with required conditions in a very simple and elegant manner.

We have also demonstrated that some domination parameters like the annihilator domination number, go a long way in getting a solution for some devastating problems of eradication of pests in Agriculture.