CHAPTER -4

WORK SAMPLING

Usually work sampling study would be undertaken only upon the request from line or staff official. In many organizations, industrial engineering or methods and standards department would be asked to conduct it. However, unless the study requires the rating of operational performance and the establishment of time standards, it would not be necessary to use trained analysts or industrial engineers to do it. In fact, supervisors themselves often make-work sampling studies[1]. The objective of the proposed study is to work out following the initial request. A full statement of purpose should be prepared, so that the study can be properly designed. The analyst should try to visualize that the final report contains the study. This will help in determining the degree of accuracy required and the length of period over which the study should be made.

The purpose of study will indicate how the activities and delays should be broken down. When overall information is needed, a few elements may be satisfactory. In other situations, a finer break down may be called for, and consequently each element will represent a smaller percentage of the whole. This calls for more observations and leads to increased cost. If a work sampling study is undertaken to aid in reducing non-working time and increasing output per man-hour, the elements should be such that they will reveal delays within the control of the operator like late starting and early quitting. The elements should also reveal delays within the control of management such as shortage of materials or machine break down for repair or adjustment. If a work sampling study is being made to establish time standards, the unit of measure is considered first as in the case of time study. The units, produced must permit to count easily and positively. Irrespective of the purpose of study, nature of break down and each element to be measured must be carefully defined so that there can be no mistake in identifying it. A carefully prepared written definition is desirable. Work sampling was first used by Tippet[2] in the British textile industry. It was introduced under the name of "ration delay" during 1940. Work sampling is a fact-finding tool. In many cases, needed information about men or machines can be obtained in less time and at lower cost under this method than other means.

Significance

Working sampling has three uses.

1.1 **Ratio delay** : This is used to measure the activities and delays of men or machines – for example, to determine the percentage of day that a man is working or not working[3]

1.2 **Performance sampling**: This is employed to measure the working time/non-working time of a person on a manual task and to establish a performance index or performance level for the person during his working time[4].

1.3 **Work measurement**: Under certain circumstances, to measure manual tasks, that is, to establish standard time for an operation, this is used. Work sampling is based upon the laws of probability. A sample taken at random from a large group tends to have the same pattern of distribution as the large group or universe. If the sample is large enough, characteristics of the sample will differ but little from the characteristics of the group. Sample is the term used for this small number and population or universe is the term used for the large group. Obtaining and analyzing only a part of the universe is known as sampling. For example, determination of percentage of working day that the operator or machine is working or idle is based on the theory that the percentage number of observations recording the man or machine as idle is a reliable measure of percentage time that the operation is in the delay state. The accuracy of result is a function of number of observations taken.

The work sampling procedure in its simplest form, consists of making observations at random intervals of one or more operators or machines and noting whether they are working or idle. If the operator is

working, he is given a tally mark under "working": If he is idle, he is

given a tally mark under "idle". The percentage of day that the worker is

idle is the ratio of number idle tally marks to the total number of idle

Table-4.1: Details of Working and Idle Tally Marks

State	Tally marks			Total
Working	ᵗᴴᴸᴶ	ᵗᴴᴸᴶ	ᵗᴴᴸᴶ	
	ᵗᴴᴸᴶ			
	ᵗᴴᴸᴶ	ᵗᴴᴸᴶ	ᵗᴴᴸᴶ	51
	ᵗᴴᴸᴶ			
	ᵗᴴᴸᴶ	ᵗᴴᴸᴶ	1	
Idle	ᵗᴴᴸᴶ	1111		9

and working tally marks. It can be observed from Table 4.1 that out of

sixty, nine are idle observations. The percentage of idle time is 15

and the working time 85. If we cover 8-hours in a day, the total time is

480 minutes, idle time 72 minutes and the rest 408 minutes working

time.

In random sampling, there is no bias in the process. Each

part comprising the universe must have as much change or being

drawn as any other. It is important that the concept of randomness be

understood and carefully followed in work sampling studies. There is

some evidence that the intervals may be regular if the activity or

process being observed in random[5].

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1.4 **Confidence level:** At the outset, it is necessary to decide what level of confidence is desired in the final work sampling results. The most common confidence interval is 95%. The area under the normal distribution curve at 2 σ or two standard deviations is 95.45%, which, if rounded off, gives 95%. This means that the probability is that in 95% of time, random observations will present the facts and 5% of time they will not. 1σ would give a confidence interval of 68%. This means that the data obtained by random sampling has 68% chance of respresenting facts and 32% of time.

The formula for determining the sample size for confidence level of 68% or 1 σ is

$$SP = \sqrt{\frac{p(1-p)}{N}}$$

Where

S = Desired relative accuracy

P = Percentage expressed as a decimal

N = No.of random observations

1.5 **Accuracy of work sampling measurement**

When we determine the degree of accuracy desired, we are in effect determining the number of observations required. Number of observations affect time and cost of making the study. The purpose of work-sampling study suggests the degree of accuracy of results desired but there may be considerble latitude in specifying accuracy. In designinig the work-sampling study, the analyst will size up the entire situation. He wants satisfactory results from the view point of

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accuracy and at the same time will not require an unreasonably large number of observations. In a work-sampoling study, analyst can determine in advance the number of observatios needed for a given degree of accuracy. One of the things that the analyst considers is (consciously or unconsciously) the inherent variability of men, machines or processes being measured. A department that is operatating week in and week out with a steady volume of ouput, with raw material of uniform quality, low labour turnover and good supervision, presents an idle subject for work sampling or for measurement of that matter through time study.

Usually an accuracy of ± 5% is considered satisfactory. This is sometimes referred to as standard error. Assuming that the bionomial distribution is used as the basis for determining the error, then the formula for determining the number of observations required is

$$SP = 2\sqrt{\frac{p(1-p)}{N}}$$

Where   S = desired relative accuracy

P = % of occurrence of an activity

N= total number of random observations.

Even if the desired accuracy is known, there are still two unknowns in the equation ; P, the percentage occurrence and N, the total number of observations. In order to find N.P is generally assumed or estimated by a preliminary study.

After the preliminary study is completed,a calculation is made to determine whether the resuluts are within the desired

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accuracy. This can be done by calculating 'S' in the formula instead of 'N' as was calculated previously.

$$S = \left[2\sqrt{\frac{p(1-p)}{N}} \right] + p$$

If the value of 'S' is below ± 5% required accuracy, the number of observations is suffiucient. Here we have 95% confidence that men or machines were idle p% of time. The accuracy or standard error (S) means that the results are correct within S% of P%, or true value was between (S% - ϕ%) and (S% +ϕ%), where ϕ % = (S%) x (P%). The 95% confidence level means that the probability is that in 95 cases out of 100, the above results represent facts.

Profit contribution

Contribution is one of the vital important elements in the system of marginal costing. The rationale of contribution lies in the fact that where a business manufactures more than one product, the profit realised on individual products can not possibly be calculated due to the problem of apportionment of fixed cost to different products, which is done away under marginal costing. Therefore, some method is required for the treatment of fixed costs and marginal costing's answer to this challenge is contribution. Contribution is the difference between sales and variable cost or marginal cost of slaes. It may also be defined as the excess of selling price over variable cost per unit. Contribution is also known as 'contribution margin' or 'gross margin'. It is visualised as some sort of 'fund' or 'Pool' out of which, all fixed costs irrespective of their nature, are to be met and to which each

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product has to contribute its share. Contribution being excess of sales over variable cost is the amount that is contributed towards fixed expenses and profit. The difference between contribution and fixed costs is either profit or loss as the case may be.

The concept of contribution is a valuable aid to management in managerial decision making. It is useful in the fixation of selling price, determination of break even point, selection of a suitable product mix for profit maximisation, selection of method of production which gives highest contribution per limiting factor is adopted from among alternative methods of production, taking a decision as regards to purchase or manufacture a product or component, taking a decision as regards adding a new product in the market and ascertainment of profitability of products, departments etc. From the concept, it is evident that the essence of marginal costing technique lies in considering fixed costs on the whole as separate, quite distinct from vairable costs, which only are relevant to current operations and form part of product cost. Variable costs only are matched with revenues under different conditions of production and sales to compute what is known as 'contribution' towards recovery of fixed costs and yielding profits.

The marginal cost includes prime cost (direct material + direct labour + direct expenses) and variable overheads. Difference between selling price and total cost is profit or loss and the differnce between selling price and marginal cost is contribution. Hence, we can compute contribution and profit if sales figures are known. The

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relationship between contribution and sales is of vital importnace for studying the profitability of operations of a buiness. It revelas the effect on profit of changes in the volume and type of output. The relationship of contribution to sales will remain constant under different levels of sales only if, variable cost per unit remains constant; total fixed cost remains the same; and selling price per unit does not change.

The amount of any giyen volume of output by which aggregate costs are changed if the volume of output is increased by one unit.In practice, this is measured by total variable costs attributable to one unit. In this context, a unit may be a single article, a batch of articles, an order, a stage of production capactiy, a process or department. It relates to changes in output in particular circumstances under consideration. Variable costs are those costs which fluctuate, in total, in direct proportion to the volume of output. Such costs increase in aggregate as the output increases and decreases in the same proportion when output falls. The costs of direct material, direct labour, supplies and direct-expenses like sales commission are perfect examples of varaible cost

3. **Linear programming technique**

Many business and economic situations are concerned with a problem of planning activity. In each case, there are limited resources at disposal and the problem is to make such a use of these resources so as to yield the maximum production or minimise the cost of production or give the maximum profit etc. Such things are referred to as problems of constrained problems. Linear Programming (LP) is a

mathematical programming technique to optimise performance, for example: Profit or cost under a set of resource constraints such as: machine hours, man hours, money, materials etc., as specified by an organisation. LP can be successfully applied in the area of military, industry, agriculture, transportation, economics, health systems and even behavioural and social sciences.

It was in 1947 that George Dantzig and his associates found out a technique for solving military planning problems while they were working on a project for U.S. airforce. This technique consisted of representing various activities of an organisation as a L.P model and arriving at the optimal programme by minimising a linear objective function. Afterwards, Dantzig suggested this approach for solving business and industrial problems. He also developed the most powerful mathematical tool known as simplex method to solve LP problems. Much progress has been made in the theoretical development and in the practical applications of L.P. The works of Charnes and Cooper were responsible for the industrial applications of L.P. L.P is a quantitative technique, which is helpful to management in decision-making to determine the optimum use of various resources such as capital, raw materials, manpower, floor space, machine hours and other plant facilities to attain a particular objective, when there are alternate business activities aimed at. In recent years, L.P.was successfully employed in different areas of research including industry.

The model of any L.P problem will contain, objective function, set of constraints and non-negativity restrictions. Each of the components may consist of one or more of the following: decision variables; objective function coefficients; technological coefficients; and availability of resources. The unknown vairables called decision variables used in the problem are non-negative i.e. positive or zero or positve definite. The rule of choosing the 'best' values of decision variables can be described by a linear function of decision variables, that is, mathematical function involving only the first powers of variables with no cross products. This function is referred to as the objective function. The rules used for operating the process i.e.scarctiy of resources are expressed as a set of linear equations or linear inequalities. The set is called the constraint set.

L.P can be used for optimisation problems if the following conditions are satisfied: (i) there must be a well defined objective fucntion (Profit, cost or quantities Produced) which is to be either maximised or minimised and which can be expressed as a linear function of decision-variables; (ii) there must be constraints on the amount or extent of attainment of the objective and these constraints must be capable of being expressed as linear equalities or inequalities in terms of variables; (iii) there must be alternative courses of action; (iv) decision variables should be interrelated and non-negative. The non-negativity condition shows that L.P deals with real life situations for which negative quantities are generally illogical; and

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(v) resources viz., men, machines, money and materials etc., must be limited to supply

The general L.P problem is to find a vector $(X_1, X_2, X_j, ..., X_n)$ which maximises or minimises the linear form (i.e. objective function).

$$Z : C_1 X_1 + C_2 X_2 + ... + C_j X_j + ... C_n X_n$$

subject to constraints.

$$a_{11} X_1 + a_{12} X_2 + ... + a_{1j} X_j + ... + a_{1n} X_n \ (^*) \ b_1$$

$$a_{21} X_1 + a_{22} X_2 + ... + a_{2j}. X_j + ... + a_{2n} X_n \ (^*) \ b_2$$

.   .   .   .

.   .   .   .

$$a_{i1} X_1 + a_{i2} X_2 + ... + a_{ij}. X_j + ... + a_{in} X_n \ (^*) \ b_i$$

....

$$a_{m1} X_1 + a_{m2} X_2 + ... + a_{mj}. X_j + ... + a_{mn} X_n \ (^*) \ b_m$$

and $X_j \geq O$ for $J = 1,2,3.........n$

Where C_j is a known 'profit' or 'cost' co-efficient of X_j ; X_j is an unknown variable (decision variable, a_{ij} is a known constant, b_j is a known constant and (*) means the symbol $\geq$, $=$ or $\leq$ for each constraint.

The L.P suffers from the following limitations; (i) for large problems with many limitations and constraints, the computational difficulties are enormous, even when assistance of large digital computers is available. It may be sometimes possible to get over this difficulty by splitting the main problem into smaller ones, deriving solutions for them and then combining the results; (ii) according to L.P problem, there can be any value whereas sometimes it happens that

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some of the variables can have only integral value; (iii) the model does not take into account the effect of time. The operations research team must define the objective function and constraints, which can change over night due to internal as well as external factors; and (iv) many times, it is not possible to express both the objective function and of constraints in linear form. For example, in production planning, we often have non-linear constraints in production capacities like set-up and take-down times, which are often independent of quantities produced. The misapplication of L.P under non-linear conditions usually results in an incorrect solution.

3 1   Methods of solving L.P problem

Programming problems are concerned with the efficient use or allocation of limited resources to meet desired objectives. These problems are characterised by a large number of solutions that satisfy the basic conditions of each problem. The selection of a particular solution as the best solution to a problem depends on some   aim   or over-all objective, i.e., implied in the statement of the problem. A solution that satisfies both the conditions of the problem and the given objective is termed as optimum solution. The various methods available to solve this type of problem are graphical; systematic trial and error; vector; and simplex.

L.P problems which involve only two variables can be solved graphically. The procedure is as follows: (i) formulate the mathematical L.P problem. There will be 2 variables   x and y; (ii) since both x and y are non negative, graphic solution will be restricted

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to the first quadrant; (iii) choose an appropriate scale for x and y axis; (iv) each inequality in constraint equation can be written as equality; (v) Give any arbitrary value to one variable and get the value of other variable by solving the equation. Similarly give another arbitrary value to the variable and find out the corresponding value of other variable; (vi) now plot these two sets of values. Connect these points by a straight line. That divides the first quadrant into two parts. Since the constraint is an inequality, one of the two sides satisfies inequality; (vii) repeat these steps for every constraint stated in the L.P problem; (viii) There forms a common area called feasible area; (ix) for greater than or equal to constraints, feasible region will be the area which lives above the constraints; and (x) for less than or equal, the area is below these lines.

Any L.P problem in which either the number of competing candidates is limited to two or, if three or more than three competing candidates, the number of resources is limited to two, can always be solved by trial and error. Some systematic trial and error method is needed to solve the problems of management involving a large number of candidates and resources. In this method, an objective function is used to test the optimality of a given solution. The objective function is modified to yield information as to: (i) whether or not the given programme can be improved; and (ii) how to design a new programme? We can design an initial programme such that the given constraints are not violated. We can test the initial programme for optimality by examining the associated objective function. If the test

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indicates that a better programme can be designed, then the initial programme is revised. We can test the revised programme for optimality and if further improvement in the objective function is possible, then we can design another programme. We can repeat this process until we obtain an optimal solution. A set of linear equations can be represented as vectors and vice – versa. Any L.P problem can be stated in terms of linear equations and hence it can be represented in vector notation. This vector equation can be solved by employing certain vector opeations.

3.2 Simplex method

The simplex method is the most general and powerful method of all the methods of solving L.P problems. By simplex procedure, feasible solution can be determined by a finite number of steps. These setps are Iterations which consist in finding a new feasible solution whose corresponding value of the objective function is less than the value of the objective function for the preceding solution. This process is continued until a minimum solution has been reached. Mathematically, a L.P problem can be formulated as follows :

Given real numbers $C_1, C_2, \ldots\ldots\ldots C_n$

$b_1, b_2, \ldots\ldots\ldots b_n$ and

a_{ij} (i= 1,2,......m; j =1,2,.....n)

We wish to maximize the function 'Z' of n real variables defined by

$z (x_1, x_2 \ldots\ldots x_n) = c_1 x_1 + c_2 x_2 + \ldots\ldots\ldots + c_n x_n$

Subject to the constraints

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$$a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots\ldots + a_{1n} x_n \leq b_1$$

$$a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots\ldots + a_{2n} x_n \leq b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots\ldots\ldots + a_{mn} x_n \leq b_m$$

$$x_j \geq 0 \; ; \, J = 1, 2, \ldots\ldots n$$

The first step in simplex method for solving the general problem is to convert the system of 'm' linear inequalities (i) into a system of linear equations. This can be done by introducing 'slack variables' $x_{n+1}, x_{n+2}, \ldots\ldots\ldots x_{n+m}$ one for each of the inequalities so that the set of inequalities become

$$a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots\ldots + a_{1n} x_n + x_{n+1} = \quad b_1$$

$$a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots\ldots + a_{1n} x_n + x_{n+2} = \quad b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots\ldots\ldots + a_{mn} x_{n+m} \quad = \quad b_m$$

Now we will restate the L.P problem as follows :

To find among all sets of points ($x_1, x_2, \ldots\ldots x_{n+m}$)   satisfying   the liinear equations with each x_j non-negative, that point ($\bar{x}_1, \bar{x} , \ldots.. \bar{x}_{n+m}$) which maximises the linear funciton.

$$Z = C_1 x_1 + C_2 x_2 + \ldots\ldots\ldots + C_n x_n$$

The slack variables are merely a device used in arriving at the optimal solution, and may or may not have any physical meaning in individual problems. There are three phases in the solution of a L.P

problem. The first phase is to formulate the problem and the objective function. The second phase invovles the design of an initial programme which includes only slack variables. The third phase consists of applying a test of optimality to determine whether a given programme can be improved. The third phase consists of two parts such as testing the optimality of current programme and revising the current programme, if necessary, according to definite rules of transformation for the key row and non-key rows of simplex Table containing the current programme. A schematic diagram of Iterative procedure is given in figure 4.1.

All the constraints and objective functions are depicted in a matrix form. Different notations are developed in formulating the initial Simplex Table and subsequent table. It is a matter of convenience only. The procedure is shown schematically in figure 4.2.

3.2.1 Procedure

The procedure of simplex method in the case of maximization is as follows:

Step-I: Formulation of the problem:

(a) translating the technical specifications of the problem into inequalities, and making a precise statement of objective function; (b) converting inequalities into equalities by addition of non-negative slack variables; and (c) modifying the objective function to include slack variables.

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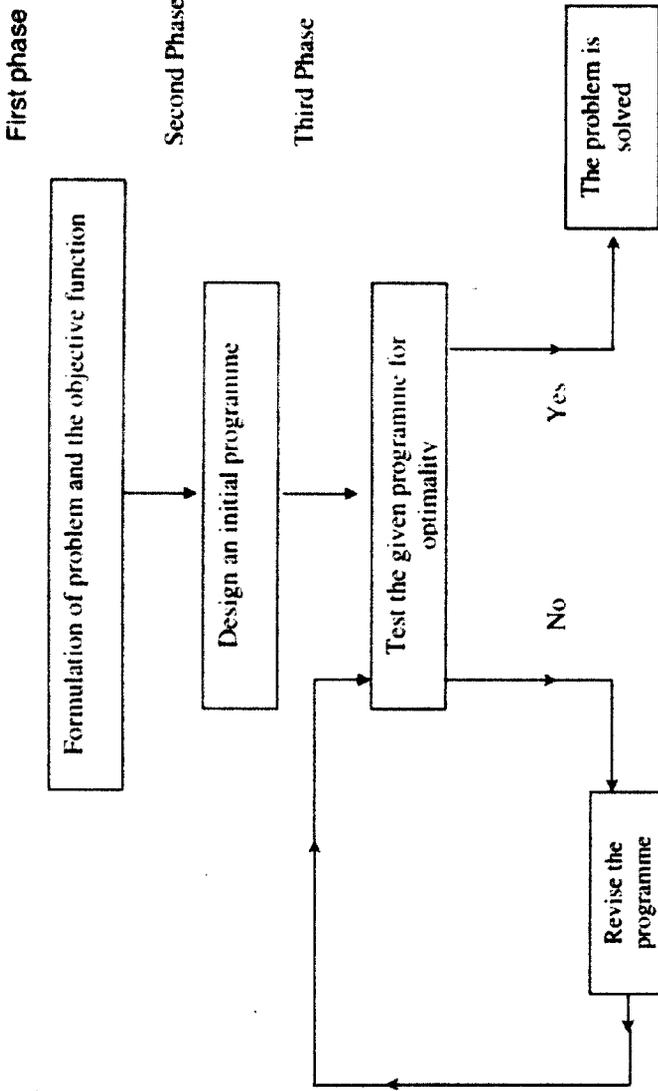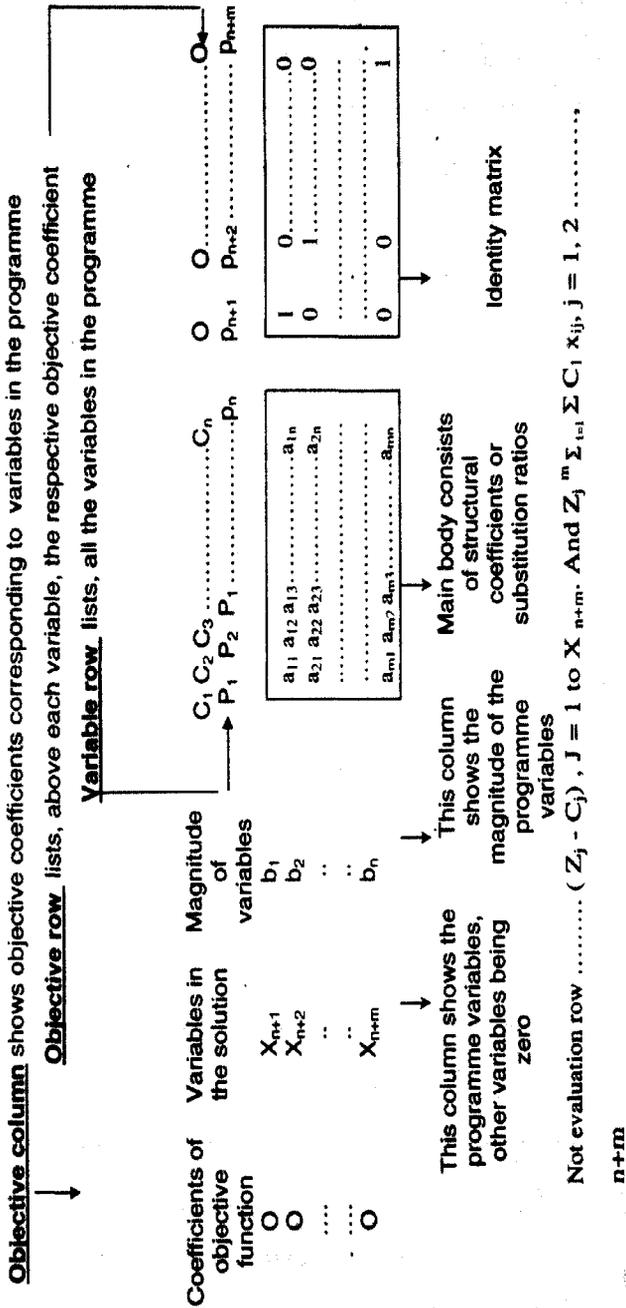**Figure 4.1: Schematic Representation of Simplex Algorithm**

First phase

Second Phase

Third Phase

Formulation of problem and the objective function

Design an initial programme

Test the given programme for optimality

The problem is solved

Revise the programme

Yes

No

Figure 4.2 : Nomenclature of Simplex Table

Objective column shows objective coefficients corresponding to variables in the programme

Objective row lists, above each variable, the respective objective coefficient

Variable row lists, all the variables in the programme

Coefficients of objective function	Variables in the solution	Magnitude of variables	$C_1 C_2 C_3 \ldots\ldots\ldots C_n$ $P_1 P_2 P_1 \ldots\ldots\ldots P_n$	$0 \quad 0 \ldots\ldots\ldots 0$ $P_{n+1} P_{n-2} \ldots\ldots P_{n+m}$
0	X_{n+1}	b_1	$a_{11}\, a_{12}\, a_{13} \ldots\ldots\ldots\ldots a_{1n}$	$1 \quad 0 \ldots\ldots\ldots 0$
0	X_{n+2}	b_2	$a_{21}\, a_{22}\, a_{23} \ldots\ldots\ldots\ldots a_{2n}$	$0 \quad 1 \ldots\ldots\ldots 0$
$\ldots$	$\vdots$	$\vdots$		
0	X_{n+m}	b_n	$a_{m1}\, a_{m7}\, a_{m1} \ldots\ldots\ldots a_{mn}$	$0 \quad 0 \ldots\ldots\ldots 1$

Main body consists of structural coefficients or substitution ratios

Identity matrix

This column shows the magnitude of the programme variables

This column shows the programme variables, other variables being zero

Not evaluation row $\ldots\ldots\ldots$ ($Z_j - C_j$) , $J = 1$ to X_{n+m}. And $Z_j {}^m \Sigma_{t=1} \Sigma C_1 x_{ij}, j = 1, 2 \ldots\ldots$, $n+m$

The numbers in the net-evaluation row, under each column of main body and the identity represents the opportunity cost of not having one unit of the respective column variables in the solution. In other words, numbers represent the potential improvement in the objective function, which will result by introducing, into the programme, one unit of the respective column variable.

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Step II: Designing an initial programme:

The first programme is designed to include only slack variables in the solution. This programme is to be placed in the Simplex Table. In the objective row, above each column variable, corresponding coefficient of that variable from step I (c) is to be placed.

Step III: Testing and revising the programme: (a) calculating the net evaluation row: To get a number in the net evaluation row under a column, entries in that column should be multiplied by the corresponding numbers in the objective column and add the products. Then the number listed in the objective row at the top of the column is to be subtracted from the above sum. The result is entered in the net-evaluation row under the column; (b)**Test:** The entries in the net-evaluation row for the given Simplex Table are examined. If all the entries are zero or positive, the optimal solution has been obtained. Otherwise, presence of any negative entry in the net evaluation row indicates that a better programme can be obtained.

Revising the programme:(i) To find the key column: The column under which falls the largest negative net-evaluation row entry is the key column. (ii) To find the key row and the key number: By dividing the entries in the 'quantity' column by the corresponding non-negative entries of the key column, replacement ratios are formed and these ratios are to be compared. The row in which the smallest replacement ratio falls is the key row. The number which lies at the intersection of the key row and key column is the key number. (iii) To transform the key row: All the numbers in the key row starting with and to the right of

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the quantity column are divided by the key number. The resulting numbers form the corresponding row of the next Table. (iv) To transform the non-key rows: Buy subtracting from the old row number of a given non-key row (in each column), the product of the corresponding key row number and the corresponding fixed ratio is formed by dividing the old row number in the key column by the key number. The result will give the corresponding new number.

The above rule can be placed in the following equation:

New row number = Old number – corresponding number in key
row x fixed ratio

Where

Fixed ratio = Old row number in key column
Key number

This transformation is to be made for all the non-key rows.

(v)The results of (iii) and (iv) entered in a Table represent the revised programme.

Step IV : To obtain the optimal programme, steps (iii) and (iv) must be repeated until an optimal programme has been derived.

3.2.2 **Dual of L.P Problem** : With every L.P maximization problem, it has proved useful to associate a closely related minimization problem and vice versa. Such pairs of problems are called dual L.P problems. The duality theorm states that "for every maximization (or minimization) problem in L.P, there is a unique similar problem of minimization (or maximization) involving same data which describes the orginal problem". The original problem is referred to as the "primal". The optimum solution of either problem indicates information concerning

the optimum solution of the other. In fact, if the initial Simplex Table for the primal problem contains on m x m, matrix, then solution of either problem by the simplex procedure yields an explicit solution to the other. Although every general L.P problem has a dual problem, the interpretation and inter-relationship of solutions of the primal and the dual are not very straight forward unless the problem is in the standard form. Further, if we convert a general L.P problem into its dual, the variables may become negative or unrestricted in sign. Therefore, it is useful to convert any L.P problem into a standard form before studying the duality and other related issues. In a standard L.P problem, we have: objective function which is a maximizing function; all the constraints are of the $\leq$ type;and all the variables are non-negative.

Consider the following standard L.P problem. This will be called the primal problem.

Maximize $Z = C_1 X_1 + C_2 X_2 + \ldots\ldots + C_n X_n$

Subject to =

$a_{11} X_1 + a_{12} X_2 \ldots\ldots\ldots\ldots + a_{1n} X_n \leq b_1$

$a_{21} X_1 + a_{22} X_2 \ldots\ldots\ldots + a_{1n} X_n \leq b_2$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$a_{m1} X_1 + a_{m2} X_2 + \ldots\ldots\ldots\ldots\ldots + a_{mn} X_n \leq b_m$

and $X_1 \geq 0, X_2 \geq 0 \ldots\ldots, X_n \geq 0$

Where : C = row matrix containing coefficients in the objective function.

X = Column matrix containing the decision variables

a = matrix containing co-efficients of constraints

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b= column matrix containig the RHS values of constraints.

The corresponding dual problem is

Minimize $F = b_1 Y_1 + b_2 Y_2 + \ldots\ldots + b_m Y_m$

Subject to $\quad a_{11} Y_1 + a_{21} Y_2 \ldots\ldots\ldots + a_m Y_m \geq C_1$

$\qquad\qquad a_{12} Y_1 + a_{22} Y_2 \ldots\ldots\ldots + a_{m2} Y_m \geq C_2$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\qquad a_{1n} Y_1 + a_{2n} Y_2 \ldots\ldots\ldots + a_{mn} Y_m \geq C_n$

and $\;Y_1 \geq 0, Y_2 \geq 0 \ldots\ldots, Y_m \geq 0$

where : b = Transpose of b matrix of the primal problem

$\qquad$ a = Transpose of coefficients matrix of constraints of primal problem

$\qquad$ c = Column matrix of objective function coefficients of primal problem.

$\;$ Y = matrix of dual variables.

Observing the general L.P problems, we find the following characteristics of primal and dual problems which give their remarkable symmetry: If the primal problem is a maximization, the dual problem is a minimization and vice – versa; the profit constraints C_j in the primal problem replace the capacity constraints and vice versa; If the primal problem involves $\leq$ signs, the dual problem involves $\geq$ signs and vice versa; a new set of variables appears in the dual problem; neglecting the number of non-negativity conditions if there are n variables and m inequalities in the primal problem, then in the dual problem, there will be 'm' variables and 'n' inequalities; and finally, the dual of the dual problem is the original

programming problem itself. It is instructive to note that the simplex method automatically identifies the dual basic solution. The optimal value of objective function remains the same as in the primal problem. Given an optimal solution of the primal problem, the ith dual variable acquires the coeffficient of the ith slack variable in the optimal objective function equation as its optimal value. In view of all this, it is possible to identify the dual solution from the primal solution.

Sensitivity analysis

Once some L.P problem of practical interest has been solved, it is often desired to study the effect of changes (discrete as well as continuous) in different parameters of the problem on the current optimum solution. With practical problems, it often happens that we are not only interested in the solution to the given problem but also desire to know how the solution will change if some of the parameters such as prices or elements of requirement vector are changed. The situation is rather unpleasant but it occurs nonetheless quite frequently. After solving the problem, we may discover that one or more of prices were incorrect. One and more of b_i were wrong and perhaps a decimal point was misplaced in some of a_{ij}. it may even turn out that some variables of interest or some constraint was omitted from the problem. The aforesaid two situations may arise, in general, when slight changes are made in the parameters or structure of a given L.P problem after optimum solution has been attained. An analysis of such post optimal problem can thus be termed as post-optimality analysis or sensitivity analysis or parametric programming. Thus, the

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investigations that deal with changes in the optimal solution due to changes in the co-efficients of A matrix (a_{ij}), the objective function cost coefficients (C_j) and the constraints of the right hand side of the equations (b_i) which can occur in a known fashion, i.e., where some of the co-efficients are assumed to be linear functions of a parameter is called "parametric linear programming".

Sensitivity analysis is carried out after the current optimum solution of the L.P. model is obtained. The goal is to determine whether changes in the model's coefficeints will have the current solution unchanged, and if not, how a new optimum can be obtained efficiently. The objective of sensitivity analysis is to reduce additional computational effort considerably which arises in solving the problem as a new method. The changes in the L.P problem which are usually studied by sensitivity analysis include.

1.   Coefficients (c_j) of the objective function.

   (a) Coefficients of non-basic variables ($C_j \notin C_B$)

   (b) Coefficients of basic variables ($C_j \in C_B$)

2.   Changes in the right-hand side constraints (b_i)

3.   Change in a_{ij} (the components of matrix A).

   (a) Coefficients of non-basic variables ($a_{ij} \notin c_B$)

   (b) Coefficeints of basic variables ($a_{ij} \in c_B$)

4.   Addition of new variables to the problem

5.   Addition of new constraint (S)

In general, these five changes may result in one of the following three cases :

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Case (i) The optimal solution remains unchanged; that is, the basic variables and their values remain essentially unchanged .

Case (ii) The basic variables remain the same but their values are changed.

Case (iii) The basic solution changes completely.

The detection of these cases by sensitivity analysis is mainly based on primal-dual properties. A study of these properties shows that: changes in the tightness of constraints in the primal (dual) can only affect the feasibility of primal (dual) or equivalently optimality of dual (primal); changes in the coefficients of objective function in the primal (dual) can only affect the optimality of primal (dual) or equivalently the feasibility of dual (primal); changes in the technological coefficients of non-basic variables of primal (dual) can only affect   the optimality of primal (dual), or equivalently the feasibility of dual (primal); These remarks provide key ideas in post optimality analysis. The techniques for treating the different changes in the L.P problems will now be introduced.

4.1   Changes in the coefficients (C_j) of objective function

(a)   Variation in $C_j \notin C_B$:

The optimality conditions $(Z_j - C_j) \geq O \ \forall \ j$ assume that C_j changed to $(C_j + \Delta C_j)$; since X_B is an optimal

$(Z_j - C_j) \geq O \ \forall \ j$

then ΔC_j must satisfy the optimality condition

$Zj - (C_j + \Delta C_j) \geq 0 \Rightarrow (Z_j - C_j) \geq \Delta C_j$

$$\Delta C_j \leq (Z_j - C_j)$$

Then the optimal value of objective function as well as optimal solution is not affected.

(b). Variation in $C_j \in C_B$

$$Z_j = C_B Y_j = \sum_{i=1}^{m} C_{Bi} Y_{ij}$$

If we change C_{Br} to $(C_{Br} + \Delta C_{Br})$, then Z_j is affected and is given

$$\hat{Z}_j = \sum_{\substack{i=1 \\ i \neq r}}^{m} C_{Bi} Y_{ij} + (C_{Br} + \Delta C_{Br}) Y_{rj}$$

$$(\hat{Z}_j - C_j) = (Z_j - C_j) + (Y_{rj} \Delta C_{Br}) \quad \text{.......} \quad (1)$$

If the solution x_B remains optimal after variation, then ΔC_{Br} must satisfy the optimality condition

$$(\hat{Z}_j - Cj) \geq 0$$

(1) implies $(Z_j - C_J) + \Delta C_{Br} Y_{rj} \geq 0$

The limitations of ΔC_{Br} are

$$\underset{y_{rj}^j \geq 0}{\text{Max}} \left[\frac{-(Z_j - C_j)}{Y_{rj}} \right] \leq \Delta C_{Br} \leq \underset{y_{rj}^j < 0}{\text{Min}} \left[\left[\frac{-(Z_j - C_j)}{Y_{rj}} \right] \right]$$

Further, the value of objective function may change when C_{Br} is changed to $C_{Br} + \Delta X C_{Br}$. The new value of objective function. $\hat{z} = Z + \Delta C_{Br} X_{Br}$.

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4.2 Changes in the component (b_i) of vector b.

If any component of b is changed, the optimality condition($z_j - c_j$) ≥ 0 is not affected. But the change in b will affect the optimal solution x_B = $B^{-1}b \geq 0$. Further, the new solution $\hat{X}B$ can also be affected. Let the change is Δb_1 in b_1.

$b = [b_1, b_2,......b_1......b_m$; $\hat{x}_* = B^{-1}\hat{b}$;

Let $B^{-1} = [\beta_1, \beta_2,......\beta_m]$

$\hat{b} = b + [0,0,........\Delta b,0....0]$

$\hat{X}_B = X_B + \beta_{i1}\Delta b_1 \geq 0; i = 1,2,.......m.$

$X_{Bi} \geq 0$

Therefore, the limits of Δb_1 are

$$\overset{Max}{\beta_{i1}^i > 0}\left[\frac{-X_{Bi}}{\beta_{i1}}\right] \leq \Delta b_1 \leq \overset{Min}{\beta_{i1}^i < 0}\left[\frac{-X_{Bi}}{\beta_{i1}}\right]$$

4.3 Variation in the component a_{ij} of matrix A

Assume that the component a_{ij} in the ith row and j^{th} column of matrix A is changed to ($a_{ij} + \Delta a_{ij}$). Further such variations can be made in two possible ways: Only that a_{ij} is changed to $a_{ij} + \Delta a_{ij}$ which does not belong to basic matrix. B; only that a_{ij} is changed to $a_{ij} + \Delta a_{ij}$ which belongs to the basic matrix B.

a)   Variation in a_{ij} when $a_{ij} \notin B$

If $a_{ij} \notin B$, then the change in a_{ij} will not affect the optimal solution.

$X_B = B^{-1}b.$

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Let $B^{-1} = [\beta_1, \beta_2 \ldots\ldots\ldots \beta_m]$

$a_j = [a_{1j}, a_{2j}\ldots\ldots\ldots a_{mj}]$

$\hat{a}_j = a_j + [0,0\ldots\ldots\Delta a_{ij}\ldots\ldots.0]$

$\hat{Z}_j = Z_j + [C_B \beta_i \Delta a_{ij}]$

The optimality condition remains satisfied $[\hat{Z}_j - c_j] \geq 0$

i.e. $(Z_j - c_j) + \Delta a_{ij} (C_{Bi} \beta)_i \geq 0$

The bounds of Δa_{ij} are given by

$$\overset{\text{Max}}{C^i_B} \beta_i > 0 \left[\frac{-(Z_j - C_j)}{C_B \beta_i}\right] \leq \Delta a_{ij} \leq \overset{\text{Min}}{C^i_B} \beta_i < 0 \left[\frac{-(Z_j - C_j)}{C_B \beta_i}\right]$$

Δa_{ij} is unrestricted if $C_B \beta_i = 0$

b)   Variation a_{ij} when $a_{ij} \in B$

The bounds of Δa_{ij} are given by when $a_{ij} \in B$.

$$\overset{\text{Max}}{P^q} > 0 \left[\frac{-(Z_q - C_q)}{p}\right] \leq \Delta a_{ij} \leq \overset{\text{Min}}{P^q} < 0 \left[\frac{-(Z_q - C_q)}{p}\right]$$

Where $p = \beta_{pi} (Z_q - C_q) - Y_{pq} C_B \beta_i$

4.4   Addition of new variables to the problem

Let a new variable be introduced in the L.P problem whose optimal solution is known. We shall answer the question - is there any way to obtain the new optimal solution without restarting the simplex method from the beginning? Adding a new variable is really just a special type of simultaneous changes in the objective function coefficients (C_j) and the coefficients (a_{ij}) of a corresponding non-basic variable. Consequently, addition of a new variable can only affect the optimality of the problem. This means that the variable will enter into

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the solution only where it improves the value of objective function. Otherwise, the new variable becomes just like a non-basic variable.

4.5   Addition of a new constraint

If we introduce a new constraint to the problem after obtaining the optimal solution, this new constraint can affect the feasibility of the present optimal solution, only if it is active. At first to judge whether the new constraint is satisfied by the present optimal solution or not. If the constraint is satisfied the newly added constraint is redundant and the optimal solution remains unchanged. If the constraint is not satisfied, the new constraint must be introduced to the system of constraints. We add a slack variable and also add the resulting equation to the final set of equations. Let the new slack variable be the basic variable for the new equation and we algebraically eliminate any other basic variable that appears in the equation. Although the conditions of optimality are satisfied, the solution is infeasible since the new slack variable is negative. In fact, to start with an infeasible optimal solution and to programme towards feasible optimal solution is the situation where the dual simplex method should be employed.

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