Chapter 5

Peristaltic transport of a conducting Bingham fluid in an inclined channel.

5.1 Introduction

Physiological fluids in animal and human bodies are in general, pumped by the continuous contraction and expansion of the ducts. These contractions and expansions are expected to be caused by peristaltic waves that propagate along the walls of the ducts. In general during peristalsis the fluid is pumped from lower pressure to higher pressure. Peristaltic transport occurs widely in the stomach, ureter, bile duct, small vessels etc. The principle of peristalsis is used by roller pumps for pumping fluids without being contaminated due to the contact with the pumping machinery. The initial work on peristalsis is done by using lab frame analysis. The important characteristics of peristaltic pumping namely trapping and reflux are studied in detail by Shapiro et al. (1969) for the peristaltic flow of a viscous fluid through a tube and a channel.

In physiological peristalsis the pumping fluid cannot always be treated as a Newtonian fluid. Kapur (1985) suggested several non–Newtonian models for physiological flows. He made theoretical investigations regarding blood as a Casson and Herschel–Bulkley fluids. Scott–Blair et al. (1974) reported that blood obeys Casson model for moderate shear rate flows. Lew et al. (1971)

reported that chyme is a non- Newtonian material having plastic like properties. In view of this Sreenadh et al. (2005) studied the effect of yield stress on peristaltic pumping of non-Newtonian fluids in a channel. The non- Newtonian fluids are Bingham and Herschel- Bulkley fluids. Vajravelu et al. (2005 a, 2005 b) made a detailed study on the effect of yield stress on peristaltic pumping of a Herschel – Bulkley fluid in an inclined tube and a channel. All these investigations are confined to hydromagnetic study of a physiological fluid obeying some yield stress model. It is reported that some physiological fluids like blood are conducting fluids. Motivated by this peristaltic transport of a conducting Bingham fluid in an inclined channel is investigated in this chapter under long wavelength and low Reynolds number assumptions. The expressions for the velocity field in the plug flow and non- plug flow regions, the pressure rise in the channel and the volume flow rate are obtained. The effects of the magnetic field, yield stress and amplitude ratio on the pumping characteristics are discussed.

5.2 Mathematical formulation and solution

Consider the peristaltic pumping of a conducting Bingham flow in a channel of half-width a. The channel is inclined at an angle θ with the horizontal. A longitudinal train of progressive sinusoidal waves takes place on the upper and lower walls of the channel. For simplicity, we restrict our discussion to the half-width of the channel as shown in the Fig 5.1. The region between

$y = 0$ and $y = y_0$ is called plug flow region. In the plug flow region, $|\tau_{yx}| \le \tau_0$. In the region between $y = y_0$ and $y = H$, $|\tau_{yx}| > \tau_0$.

The wall deformation is given by

$$H(X,t) = a + b\sin\frac{2\pi}{\lambda}(X - ct) \tag{5.1}$$

where b is the amplitude, λ the wavelength and c is the wave speed.

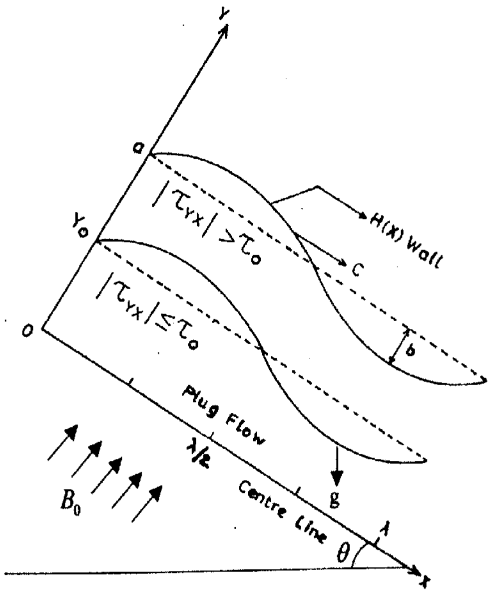

Fig 5.1. Physical Model

Under the assumptions that the channel length is an integral multiple of the wavelength λ and the pressure difference across the ends of the channel is a constant, the flow becomes steady in the wave frame (x, y) moving with velocity c away from the fixed (laboratory) frame (X,Y). The transformation between these two frames is given by

$$x = X - ct,\ y = Y,\ u(x, y) = U(X - ct, Y) \text{ and } v(x, y) = V(X - ct, Y) \tag{5.2}$$

Where U and V are velocity components in the laboratory frame and u and v are velocity components in the wave frame. In many physiological situations it is proved experimentally that the Reynolds number of the flow is very small. So, we assume that the wavelength is infinite. So the flow is of Poiseuille type at each local cross- section. Using the non-dimensional quantities

$$\bar{x} = \frac{x}{\lambda},\ \bar{y} = \frac{y}{a},\ \bar{z} = \frac{z}{a},\ \bar{u} = \frac{u}{c},\ \bar{w} = \frac{w}{c},\ \delta = \frac{a}{\lambda},\ \overline{p^1} = \frac{p^1 a^2}{\mu c \lambda},$$

$$\bar{t} = \frac{ct}{\lambda},\quad h = \frac{H}{a},\ \phi = \frac{b}{a},\ \overline{\tau_0} = \frac{a\tau_0}{\mu c},\ \overline{y_0} = \frac{y_0}{a}\ \bar{q} = \frac{q}{ac}.$$

in the equations governing the motion (dropping the bars) are

$$\left.\begin{aligned} &\frac{\partial}{\partial y}\left(\tau_0 - \frac{\partial u}{\partial y}\right) = -\frac{\partial p^1}{\partial x} - M^2 u - \eta_1 \sin\theta, \\ &0 = \frac{\partial p}{\partial y} + \eta_2 \cos\theta. \end{aligned}\right\} \tag{5.3}$$

where $\eta_1 = \frac{\rho g a^2}{\mu c}, \eta_2 = \frac{\rho g a^3}{\mu c \lambda}$, $M = B_0 a\sqrt{\frac{\sigma_e}{\mu}}$ and g is the acceleration due to gravity.

The non-dimensional boundary conditions are

$$\frac{\partial u}{\partial y} = \tau_0 \quad \text{at} \quad y = 0, \tag{5.4}$$

$$u = -1 \quad \text{at} \quad y = h, \tag{5.5}$$

where τ_0 is the yield stress.

Let $p^1(x, y) = p(x) - (\eta_2 \cos\theta) y$. Then (5.3) becomes

$$\frac{\partial}{\partial y}\left(\tau_0 - \frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} - M^2 u - \eta_1 \sin\theta, \frac{\partial p}{\partial y} = 0. \tag{5.6}$$

Solving equation (5.6) using the boundary conditions (5.4) and (5.5) we obtain the velocity as

$$u = \left[-1 + \left(\frac{(\partial p/\partial x - \eta_1 \sin\theta)}{M^2}\right) - \frac{\tau_0}{M}\sinh Mh\right]\frac{\cosh My}{\cosh Mh} + \frac{\tau_0}{M} Sinh My - \left(\frac{(\partial p/\partial x - \eta_1 \sin\theta)}{M^2}\right) \tag{5.7}$$

We find the upper limit of plug flow region using the boundary condition $\partial u/\partial y = 0$ at $y = y_0$. So, we have

$$\tau_0 = M\left[1 - (\partial p/\partial x - \eta_1 \sin\theta)/M^2\right]\sinh My_0 / \cosh M(h - y_0) \tag{5.8}$$

Taking $y = y_0$ in equation (5.7), we get the velocity in plug flow region as

$$u_p = \frac{-1+\left[\left(\partial p/\partial x-\eta_1 \sin\theta\right)/M^2\right]\left[1-\cosh M\left(h-y_0\right)\right]}{\cosh M\left(h-y_0\right)} \tag{5.9}$$

The volume flux q through each cross section in the wave frame is given by

$$\begin{aligned} q &= \int_0^{y_0} u_p dy + \int_{y_0}^{h} u\, dy \\ &= \frac{\left(\partial p/\partial x-\eta_1 \sin\theta\right)}{M^2} k_1 - k_2 \end{aligned} \tag{5.10}$$

Where $k_1 = \dfrac{M\, y_0 \cosh Mh - \sinh My_0 + \left(\sinh Mh - Mh\cosh Mh\right)\cosh M\left(h-y_0\right)}{M \cosh Mh \cosh M\left(h-y_0\right)}$

and $k_2 = \left(\dfrac{y_0}{\cosh M\left(h-y_0\right)} + \dfrac{\sinh Mh \cosh M\left(h-y_0\right) - \sinh My_0}{M \cosh Mh \cosh M\left(h-y_0\right)}\right)$

The instantaneous volume flow rate $Q(x,t)$ in the laboratory frame between the centre line and the wall is

$$Q(X,t) = \int_0^H U(X,Y,t)dY = \int_0^h (u+1)dy = q+h \tag{5.11}$$

The average volume flow rate $\overline{Q}$ over one wave period. $\left(T=\lambda/c\right)$ of the peristaltic wave is defined as

$$\overline{Q} = \frac{1}{T}\int_0^T Q(X,t)\, dt = q+1 \tag{5.12}$$

From equation (5.10), we have

$$\frac{\partial p}{\partial x}=\frac{(q+k_2)M^2}{k_1}+\eta_1\sin\theta \tag{5.13}$$

The pressure over one wavelength of the peristaltic is given by

$$\begin{aligned}\Delta p&=\int_0^1\frac{\partial p}{\partial x}dx=\int_0^1\left[\frac{M^2(q+k_2)}{k_1}+\eta_1\sin\theta\right]dx\\&=\int_0^1\left[\frac{M^2(\overline{Q}-1+k_2)}{k_1}+\eta_1\sin\theta\right]dx\end{aligned} \tag{5.14}$$

5.3 Discussion of the results

From equation (5.7), we have calculated the variation of axial velocity u with y at $x=0.25$ for different values of the angle of inclination θ with $\phi=0.6, M=0.5, \eta_1=0.5$ and $y_0=0.2$ for $(i)\frac{dp}{dx}=-4, (ii)\frac{dp}{dx}=0$ and $(iii)\frac{dp}{dx}=1$ as shown in Fig 5.2. It is observed that as θ increases the maximum velocity u increases for $\frac{dp}{dx}=-4$ and $\frac{dp}{dx}=0$, whereas for $\frac{dp}{dx}=1$ the minimum velocity decreases as θ increases, which may ascribable as the effect of the inclination of the channel with the horizontal.

The variation of axial velocity u with y at $x=0.25$ for different values of the Hartmann number M with $\theta=\frac{\pi}{6}, \phi=0.6, \eta_1=0.5, y_0=0.2$ and for

$(i)\frac{dp}{dx}=-3,(ii)\frac{dp}{dx}=0$ and $(iii)\frac{dp}{dx}=1$ as shown in Fig 5.3. We notice that as M increases the maximum velocity increases for all values of $\frac{dp}{dx}$. Further for $\frac{dp}{dx}=1$ and $M=0.5$ the flow reverses to that of the direction of propagation of the peristaltic wave.

Using equation (5.14), we have evaluated the variation of pressure rise Δp with time averaged flux $\overline{Q}$ with $\phi=0.6, \theta=\frac{\pi}{4}, \eta_1=0.5$ *and* $y_0=0.2$ and is shown in Fig 5.4. It is observed that as M increases, the pumping and free pumping both increases. In the co-pumping region $\overline{Q}$ decreases as M increases for an appropriately chosen $\Delta p<0$.

Fig 5.5 shows the variation of pressure rise Δp with time averaged flux $\overline{Q}$ for different values of angle of inclination θ with $\phi=0.6, M=0.5, \eta_1=1$ and $y_0=0.01$. As θ increases, we observe that the pumping, free pumping, co-pumping increase. Here we have plotted only for $\theta=0$ and $\theta=\frac{\pi}{3}$ because for a Bingham fluid $\overline{Q}$ and Δp do not vary much as θ increases in the interval $\left(0,\frac{\pi}{2}\right)$.

The variation of pressure rise Δp with time mean flow rate $\overline{Q}$ for different values of the amplitude ratio ϕ with $M=0.5, \eta_1=0.5, \theta=\frac{\pi}{4}$ and $y_0=0.2$ and is shown in Fig 5.6. As the amplitude of the peristaltic wave increases, the pumping and free pumping both increases. In the co-pumping region $\bar{Q}$ decreases as ϕ increases for an appropriately chosen $\Delta p < 0$.

The variation of Δp with $\bar{Q}$ for different values of the half width of the plug flow region y_0 with $M = 0.5, \eta_1 = 0.5, \theta = \frac{\pi}{4}$ and $\phi = 0.6$ and is shown in Fig 5.7. As y_0 increases the pumping increases. Any two curves intersect nearer to the line $\Delta p = 0$ and afterwards the behaviour of the flow is found to be reversed.

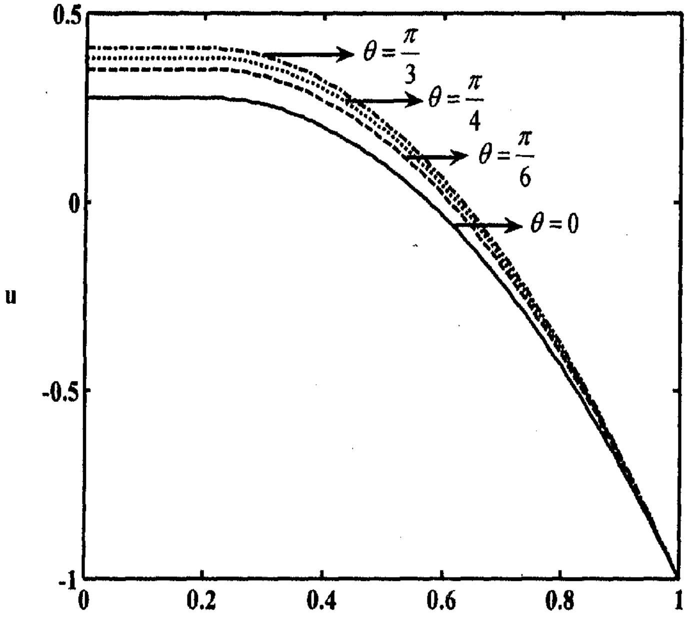

Fig 5.2(i). The variation of axial velocity u with y at $x = 0.25$ for different values of inclination angle θ with $\phi = 0.6, M = 0.5,\ \eta_1 = 0.5,$ $y_0 = 0.2$ and for $\frac{\partial p}{\partial x} = -4$.

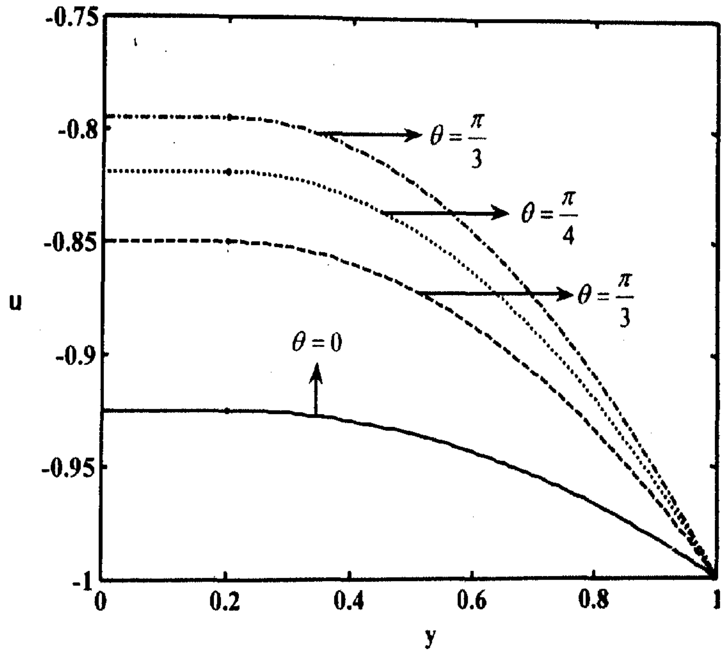

Fig 5.2(ii). The variation of axial velocity u with y at $x = 0.25$ for differen values of inclination angle θ with $\phi = 0.6, M = 0.5,\ \eta_1 = 0.5$, $y_0 = 0.2$ and for $\frac{\partial p}{\partial x} = 0$.

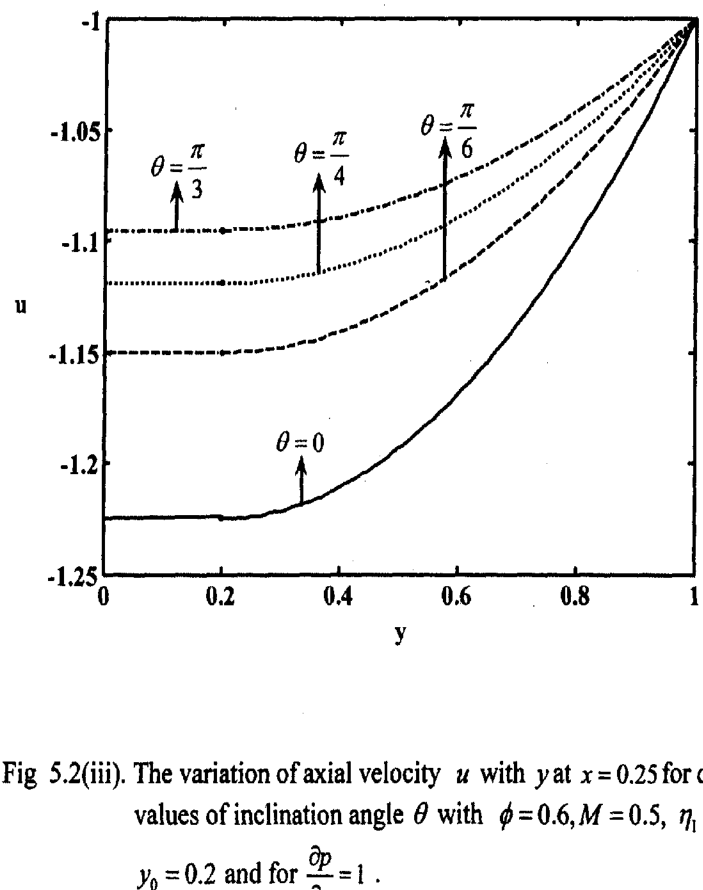

Fig 5.2(iii). The variation of axial velocity u with y at $x = 0.25$ for different values of inclination angle θ with $\phi = 0.6, M = 0.5, \eta_1 = 0.5$, $y_0 = 0.2$ and for $\frac{\partial p}{\partial x} = 1$.

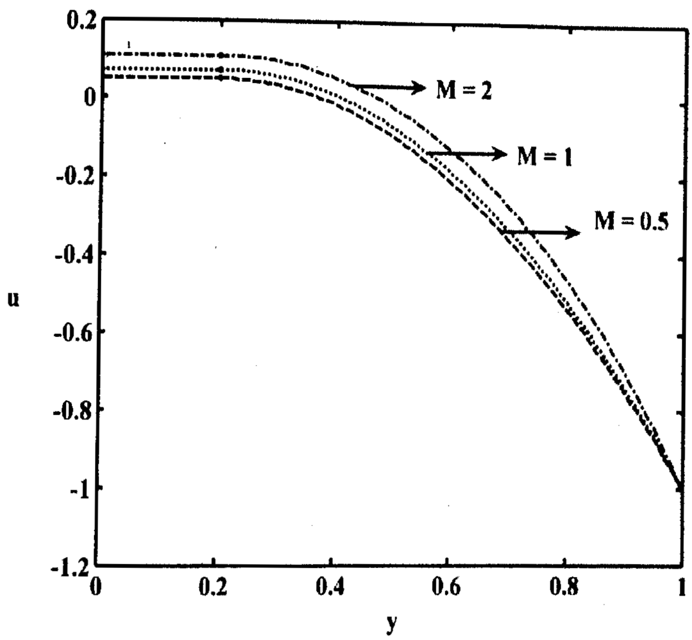

Fig 5.3(i). The variation of axial velocity u with y at $x = 0.25$ for different values of Hartmann number M with $\phi = 0.6, \theta = \pi/6$, $\eta_1 = 0.5$, $y_0 = 0.2$ and for $\frac{\partial p}{\partial x} = -3$.

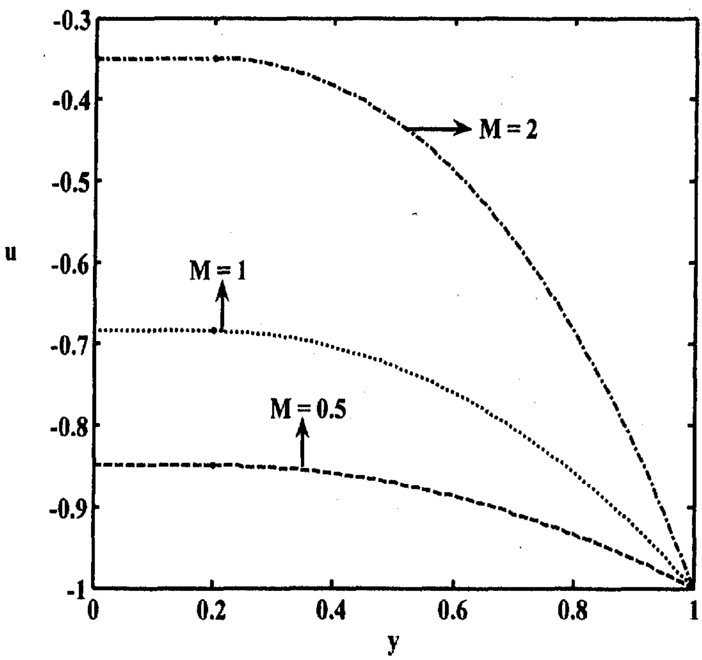

Fig 5.3(ii). The variation of axial velocity u wit y at $x = 0.25$ for different values of inclination angle θ with $\phi = 0.6, \theta = \pi/6$, $\eta_1 = 0.5$, $y_0 = 0.2$ and for $\frac{\partial p}{\partial x} = 0$.

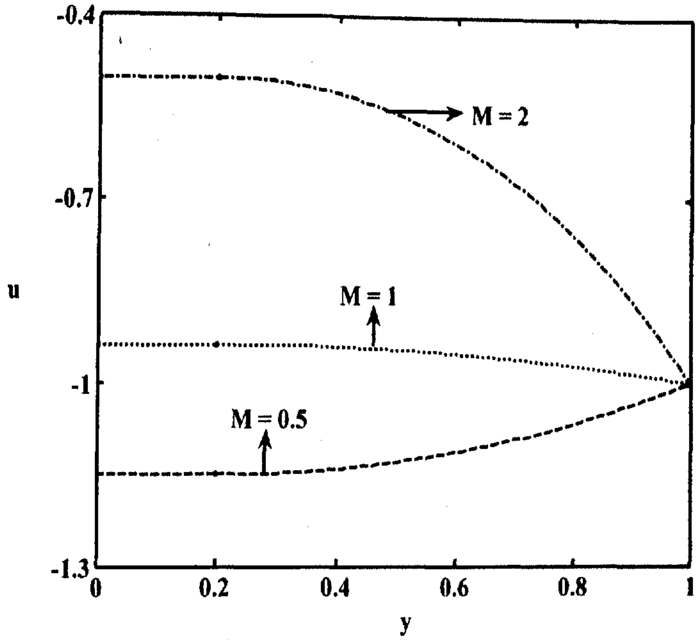

Fig 5.3(iii). The variation of axial velocity u with y at $x = 0.25$ for different values of inclination angle θ with $\phi = 0.6, \theta = \pi/6$, $\eta_1 = 0.5$, $y_0 = 0.2$ and for $\frac{\partial p}{\partial x} = 1$.

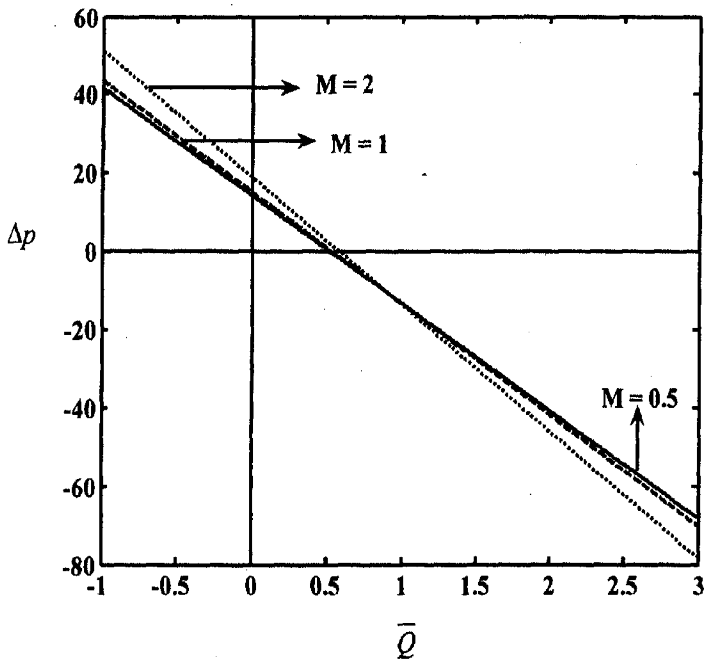

Fig 5.4. The variation of pressure rise Δp with average volume flow rate $\overline{Q}$ for different values of M with $\phi = 0.6, y_0 = 2, \theta = \pi/4$, and $\eta_1 = 0.5$.

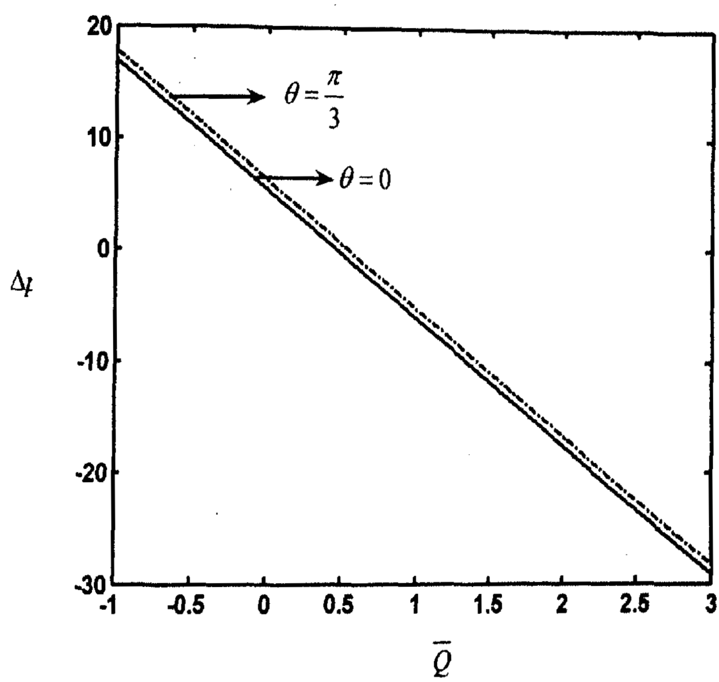

Fig 5.5. The variation of pressure rise Δp with average volume flow rate $\overline{Q}$ for different values of θ with $\phi = 0.6, y_0 = 0.01, M = 0.5$, and $\eta_1 = 1$.

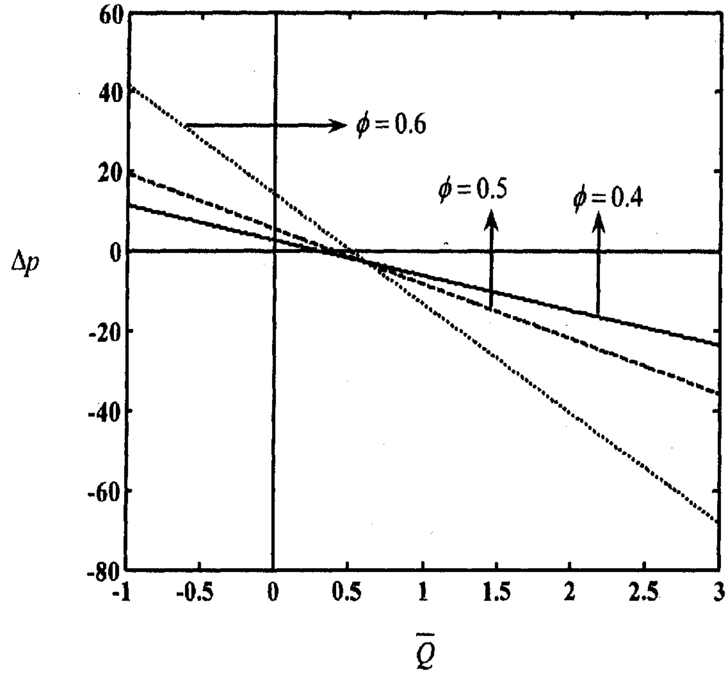

Fig 5.6. The variation of pressure rise Δp with average volume flow rate $\bar{Q}$ for different values of ϕ with $\theta = \pi/4, y_0 = 0.2, M = 0.5$, and $\eta_1 = 0.5$.

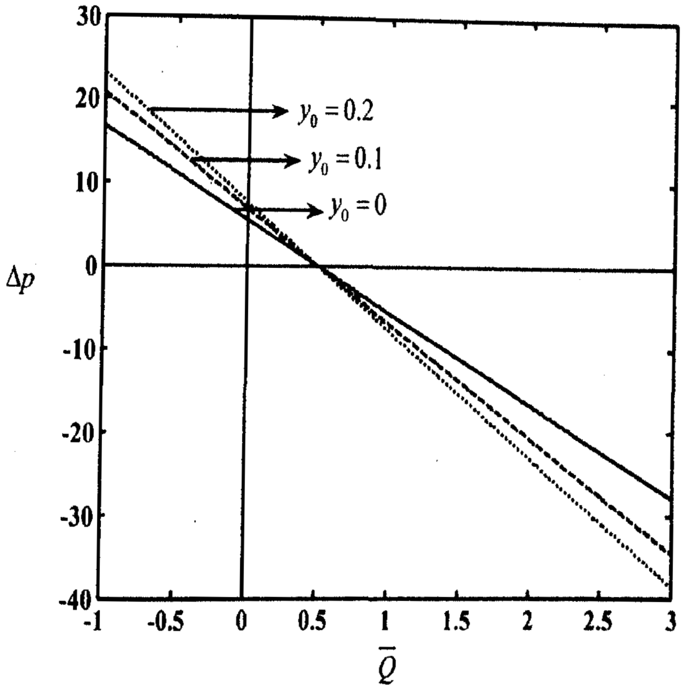

Fig 5.7. The variation of pressure rise Δp with average volume flow rate $\overline{Q}$ for different values of y_0 with $\theta = \pi/4, \phi = 0.6, M = 0.5$, and $\eta_1 = 0.5$.