CHAPTER 5

EXPERIMENTAL STUDY OF RANK 1 CHAOS IN CHUA’S OSCILLATOR

Chua’s oscillator is one of the simplest electronic circuits that are capable of producing chaos. It can exhibit a wide array of behaviour including a great variety of attractors, bifurcations and routes to chaos. In this chapter, the experimental results of rank 1 chaos in the switch-controlled Chua oscillator is provided by following a step-by-step procedure given by the theory of rank 1 maps. The design and use of this circuit was motivated by a recent mathematical theory of rank one attractors developed by Wang and Young [20]. The strange attractors are created by periodically kicking a weakly stable limit cycle emerging from the centre of a supercritical Hopf bifurcation. However, for this scheme of creating rank one attractors to work, the applied periodic pulses must have short pulse widths and long relaxation periods. This is one of the key components in creating this new class of chaotic attractors. Both the piecewise linear and the smooth nonlinear cases are considered in this investigation. The experimental results are found to be in concordance with the conclusions of the theory.

5.1 Introduction

Chaos as a dynamical phenomenon can be characterized in descriptive terms by sensitive dependence on initial conditions and unpredictability of evolutions of a generic orbit in the phase space. As was first observed by H. Poincaré [1921], the
occurrences of chaos are closely related to homoclinic tangles created by transversal intersection of the stable and unstable manifolds. Nonlinear systems may exhibit many types of complex behaviour such as chaos, and this complexity has attracted scientists from various fields (e.g. physics, mathematics, engineering etc.) to study such systems. Since it has been shown that simple electronic circuits may exhibit chaotic behaviour, the study of chaos in nonlinear electronic circuits received a great deal of attention in the last few years. An electronic circuit that has come to be known as Chua’s oscillator [Chua, 1994] is to be perceived as one among such endeavours. This circuit has undoubtedly become a standard benchmark in the study of nonlinear dynamics and chaos.

This chapter aims at introducing a new chaos theory, namely the theory of rank one chaos [Wang & Oksasoglu, 2005; Oksasoglu & Wang, 2006], based on a recent mathematical theory of rank-one maps [Wang & Young, 2001; Wang & Oksasoglu, 2008; Chen & Han, 2009] which originated from the principles of the Hénon family \( x_{t+1} = 1-a x_t^2 + y \), \( y_{t+1} = b x_t \). The important mathematical breakthroughs along the way of developments of rank one chaos are (i) Benedicks’ and Carleson’s theory on the Hénon maps [Benedicks & Carleson, 1991] (ii) the proof of Benedicks and Young on the existence of Sinai–Ruelle–Bowen (SRB) measures for good Hénon maps [Bowen, 1975; Benedicks & Young, 1993; Young, 2002] and (iii) the theories of rank one chaos for maps of various forms and dimensions [Benedicks & Carleson, 1991; Jakobson, 1981]. The theory of rank one maps helped to fill the gap between the chaos observed in simulation and its mathematical justifications.

The rank 1 attractors presented in this chapter are generated by small disturbances that are periodically applied to a Hopf limit cycle in the smooth case and a non-Hopf limit cycle in the piecewise linear (PWL) case. The Hopf limit cycle required for the smooth case in this study is created by employing the Chua oscillator with a nonlinear resistor of cubic nonlinearity [Zhong, 1994] and the non-Hopf (arbitrary) limit cycle for the PWL case by employing the Chua oscillator with a three-segment PWL resistor. In both cases, once a limit cycle is created, periodic pulses are applied to the circuit so as to modulate its state variables, namely, the capacitor voltages and the inductor current. The periodic pulses are applied in intervals controlled by analog switches to generate the kicking effect as suggested by
Wang and Young [2002, 2003]. When small disturbances in the form of periodic kicks are applied by using externally controlled switches, the shape of the weakly stable limit cycle is slightly deformed. Then the shearing force created by the nonlinearity of the original circuit exaggerates the initial deformation. Consequently, strange attractors appeared supporting the theory of rank one maps. It is worth noting that the use of various kicking schemes in the study of chaotic dynamics is not uncommon [Venkatesan et al., 2003; Schuster & Just, 2005]. However, the study of the strange attractors in this investigation differs from others in that it is well supported by a comprehensive theory of dynamics that has a long history. The design, implementation and experimental results obtained for the switch-controlled Chua oscillator [Gopakumar et al., 2011] is presented in subsequent sections.

5.2 Implementation of a Cubic-polynomial Chua’s Diode

For the investigation, the well-known Chua’s oscillator [Chua, 1994] shown in Fig. 5.1 is specifically chosen. This circuit contains five linear elements (two capacitors, one inductor and two resistors) and a nonlinear resistor $N_R$ called Chua’s diode. Chua’s diode (sec. 2.2) is described by a piecewise linear function given in Eqn. (2.2). But all features of a real circuit are not captured correctly by the simple piecewise circuit. Therefore, it is needed to realize a smooth nonlinearity by a cubic polynomial described by the following equation,

$$g(V_R) = a_0 + a V_R + b V_R^2 + c V_R^3 \quad (5.1)$$

The physical implementation of a cubic nonlinearity can be summarized as follows: Since the desired V-I characteristics of the nonlinear resistor $N_R$ in Chua’s oscillator is an odd symmetric function with respect to origin, the cubic polynomial of Eqn. (5.1) is used with the coefficient values $a_0 = 0, a < 0, b = 0$ and $c > 0$.

$$i_R = g(V_R) = a V_R + c V_R^3 \quad (5.2)$$

The basic circuit to realize the cubic polynomial of Eqn. (5.2) is a multiplier with a feedback loop as shown in Fig. 5.2.
The output of the multiplier is given by  
\[ W = \frac{V_1 V_2}{10V} + V_3 \]

where 10V is an inherent scaling voltage in the multiplier and V_3 is a dc voltage.

From the circuit, \( i_R = \frac{(V_1 - W)}{R} \)

If \( V_1 = V_R \) and \( V_2 = \frac{V^2_R}{10V} \), then

\[ i_R = \frac{V_R}{R} - \frac{1}{R} \frac{1}{10V} \frac{1}{10V} - \frac{V_3}{R} \quad (5.3) \]

Comparing with Eqn. (5.2) it can be seen that the specified condition is met only if \( V_3 \) is zero and \( R \) is negative. Based on this, the required circuit can be realized as shown in Fig. 5.3. AD 711 op amp is used to realize linear negative resistance. The input resistance of the op amp is given by

\[ R = \frac{R_a}{1 - (1 + \frac{R_b}{R_c})} \]

If \( R_a = R_b \), then \( R = - R_c \) and Eqn. (5.3) becomes

\[ i_R = \frac{V_R}{R_c} + \frac{1}{R_c} \frac{1}{10V} \frac{1}{10V} V^3_R \quad (5.4) \]

If some gain factor is required for the system, then the circuit shown in Fig. 5.3 can be modified by including two more resistors as shown in Fig. 5.4. For the modified circuit

\[ W_1 = \frac{V_R^2}{10V} \quad \text{and} \quad W_2 = \frac{V^3_R}{10V \cdot 10V} + W_2 \cdot \frac{R_c}{R_d + R_c} \]

\[ \therefore W_2 = \frac{V^3_R}{10V \cdot 10V} \cdot \frac{R_d + R_c}{R_d} \quad (5.5) \]

From the circuit,

\[ i_R = \frac{-(V_R - W_2)}{R_c} \]
Fig. 5.1 Chua’s oscillator.

Fig. 5.2 The multiplier with a feedback loop.

Fig. 5.3 Implementation of a cubic-polynomial Chua’s diode.

Fig. 5.4 Practical circuit.
Comparing with Eqn. (5.2), the coefficients are obtained as;

\[
\begin{align*}
    a &= -\frac{1}{R_c}, \\
    c &= \frac{R_d + R_c}{R_d R_c} \frac{1}{10V} \frac{1}{10V} \quad \text{and gain factor} = \frac{R_d + R_c}{R_d} 
\end{align*}
\]  

(5.7)

5.3 Design Criteria of Switch-controlled Chua’s Oscillator

5.3.1 Smooth case

To investigate rank 1 chaos in Chua’s oscillator, consider an autonomous system given by

\[
\frac{du}{dt} = f(u) 
\]  

(5.8)

where \( u \in \mathbb{R}^m, m \geq 2 \) represents the system state variables. Assume that this system is capable of generating a Hopf limit cycle. To obtain rank one chaos in such a system, a forcing term is added to it as shown below:

\[
\frac{du}{dt} = f(u) + \Phi P_{T,p}(t) 
\]  

(5.9)

where \( P_{T,p}(t) \) is a periodic pulse train of period \( T \) and pulse width \( p \). The parameter \( \Phi \) decides the shape and magnitude of the forcing. If \( \Phi \) in Eqn. (5.9) contains only first-order terms, then the forcing term can be realized by modulating the system’s state variables by a periodic pulse train. For electrical circuits with capacitor voltages and inductor currents as system state variables, externally controlled switches can be used to achieve the desired modulation scheme. Such a scheme is depicted in Fig. 5.5. For the switch-controlled Chua’s oscillator shown in Fig. 5.5, if \( nT_0 \leq t < nT_0 + p_0 \), the state equations are
\[ C_1 \dot{V}_1 = \frac{(V_2 - V_1)}{R} - \frac{V_1}{R_1} - g(V_1) \]
\[ C_2 \dot{V}_2 = \frac{(V_1 - V_2)}{R} + i_L - \frac{V_2}{R_2} \]
\[ L \dot{i}_L = -V_2 - i_L R_0 \]  

and if \( nT_0 + p_0 < t < (n+1)T_0 \), with \( n = 0, 1, 2 \) etc., then the state equations are

\[ C_1 \dot{V}_1 = \frac{(V_2 - V_1)}{R} - g(V_1) \]
\[ C_2 \dot{V}_2 = \frac{(V_1 - V_2)}{R} + i_L \]
\[ L \dot{i}_L = -V_2 - i_L R_0 \]  

where \( g(V_1) = aV_1 + cV_3 \)

Combining the above two equations, we obtain

\[ C_1 \dot{V}_1 = \frac{(V_2 - V_1)}{R} - g(V_1) - \frac{V_1}{R_1} F_n(t) \]
\[ C_2 \dot{V}_2 = \frac{(V_1 - V_2)}{R} + i_L - \frac{V_2}{R_2} F_n(t) \]
\[ L \dot{i}_L = -V_2 - i_L R_0 \]  

where \( F_n(t) = 1, nT_0 \leq t < nT_0 + p_0 \)

\[ = 0, \text{ elsewhere} \]

By assuming \( x = \frac{V_1}{V_0}; \ y = \frac{V_2}{V_0}; \ z = \frac{i_L}{I_0} \) and \( t \rightarrow \frac{t}{\omega_n} \)

The following dimensionless set of equations is obtained:
\[
\begin{align*}
\dot{x} &= -\alpha h(x) + \alpha y - \varepsilon_1 x P_{T,p}(t) \\
\dot{y} &= \gamma x - \gamma y + \gamma \eta z - \varepsilon_2 y P_{T,p}(t) \\
\dot{z} &= -\beta y - \beta \xi z
\end{align*}
\]  

where  
\[ P_{T,p}(t) = \frac{1}{p} F_n(t) \]  
\[ p = p_0 \omega_n, \quad T = T_0 \omega_n \]

\[ h(x) = b_1 x + b_3 x^3, \quad b_1 = 1 + aR \]
\[ b_3 = cRV_0^2 \]
\[ \alpha = \frac{1}{RC_1 \omega_n}, \quad \beta = \frac{R_n}{L \omega_n}, \quad \gamma = \frac{1}{RC_2 \omega_n} = 1 \]  

\begin{align*}
\varepsilon_1 &= \frac{\alpha RP}{R_1}, \quad \eta = \frac{R}{R_n}, \quad R_n = \frac{V_0}{I_0} \\
\varepsilon_2 &= \frac{\gamma RP}{R_2}, \quad \xi = \frac{R_0}{R_n}
\end{align*}  

The numerical results of the smooth Chua's oscillator is obtained by directly solving Eqn. (5.14), with parameter values \( \alpha = 2.0, \beta = 1.04, \gamma = 1.0, \eta = 1.23, b_1 = 0.35, \) and \( b_3 = -1.0. \) The rank 1 attractor can be numerically obtained for different parameter values and kicking schemes. Computations are performed using fourth order Runge-Kutta algorithm. The weakly stable limit cycle, obtained by setting all \( \varepsilon_i = 0, \) is depicted in Fig. 5.6(a). Here the value of \( T \) is irrelevant. This limit cycle is then kicked periodically to create various pictures of chaos. In the numerical analysis this is achieved by increasing either \( \varepsilon_1 \) or \( \varepsilon_2. \) Here \( \varepsilon_1 \) is used as the control parameter and \( \varepsilon_2 \) is kept at zero. Fig. 5.6(b) shows as observable strange attractor obtained with \( \varepsilon_1 = 0.3. \) The frequency spectrum of the \( x \)-coordinate for the orbit versus discrete time \( k \) is plotted in Fig. 5.6(c). The broadband nature of the frequency spectrum shown illustrates the chaotic behaviour of the phase portrait.
Fig. 5.5 Switch-controlled Chua’s oscillator.

Fig. 5.6 (a) A Hopf limit cycle from smooth Chua’s oscillator for $\varepsilon = 0$, (b) strange attractor from smooth Chua’s oscillator for $T = 107 \& \varepsilon_1 = 0.3$ and (c) frequency spectrum of $x_k$. 
5.3.2 Piecewise linear case

The state equations are the same as in the previous case with

\[
g(V_i) = m_0 V_1 + 0.5 (m_1 - m_0) \left[ |V_1 + B_p| - |V_1 - B_p| \right]
\]

(5.16)

The circuit (Fig. 2.3) implementation of this function had been discussed in section 2.2. Using the change of variables given in Eqn. (5.13), the same set of dimensionless equations as given in (5.14) is obtained. i.e.,

\[
\begin{align*}
\dot{x} &= -\alpha h(x) + \alpha y - \varepsilon_1 x P_{T_p}(t) \\
\dot{y} &= \gamma x - \gamma y + \gamma \eta z - \varepsilon_2 y P_{T_p}(t) \\
\dot{z} &= -\beta y - \beta \xi z
\end{align*}
\]

(5.17)

where

\[
\begin{align*}
P_{T_p}(t) &= \frac{1}{p} F_p(t) \\
p &= p_0 \omega_n, \quad T = T_0 \omega_n
\end{align*}
\]

and

\[
\begin{align*}
\alpha &= \frac{1}{RC_1 \omega_n}, \quad \beta = \frac{R_n}{L \omega_n}, \quad \gamma = \frac{1}{RC_2 \omega_n} = 1 \\
\varepsilon_1 &= \frac{\alpha R_p}{R_1}, \quad \eta = \frac{R}{R_n}, \quad R_n = \frac{V_0}{I_0} \\
\varepsilon_2 &= \frac{\gamma R_p}{R_2}, \quad \xi = \frac{R_0}{R_n}
\end{align*}
\]

\[
h(x) = M_1 x + 0.5 \left( M_0 - M_1 \right) \left( |x + B_q| - |x - B_q| \right)
\]

(5.18)

The numerical results of the circuit is obtained by directly solving Eqn. (5.17), with parameter values \( \alpha = 2.0, \beta = 1.04, \gamma = 1.0, \eta = 0.78, M_0 = -0.81, M_1 = -0.52 \) and \( B_q = 1.0 \). Computations are performed using fourth order Runge-Kutta algorithm. The weakly stable limit cycle, obtained by setting all \( \varepsilon_i = 0 \), is depicted in Fig. 5.7(a). Here also the value of \( T \) is irrelevant. This limit cycle is then kicked periodically to create various pictures of chaos. The attractor depicted in Fig. 5.7(b) is for the single switch.
scheme with a value of $T = 150$ and $\epsilon_1 = 0.5$. The frequency spectrum of the $x$-coordinate for the orbit versus discrete time $k$ is plotted in part (c).

Fig. 5.7 A non-Hopf limit cycle from numerical simulations for (a) $\epsilon = 0$, (b) A single-switch case rank 1 attractor with $T = 150$ & $\epsilon_1 = 0.5$ and (c) frequency spectrum of $x_k$. 
5.4 Experimental Results

5.4.1 Smooth case

The circuit schematic of smooth case switch-controlled Chua’s oscillator is realized by combining the basic switch-controlled Chua’s oscillator (Fig. 5.5) and the cubic-polynomial Chua’s diode (Fig. 5.4) as shown in Fig. 5.8. The circuit is then constructed on a breadboard with component values corresponding to the parameters chosen for the numerical analysis. The values are then obtained as, \( R_0 = 0.015 \), \( R = 1.5 \), \( R_a = 2 \), \( R_b = 2 \), \( R_c = 1.8 \), \( R_d = 3.3 \), and \( R_e = 7.8 \) kΩ resistors; \( R_2 = 2.5 \) and \( R_1 = 4.7 \) kΩ potentiometers; \( C_1 = 2.2 \) and \( C_2 = 4.4 \) nF capacitors; and \( L = 10 \) mH inductor. AD711 BiFET operational amplifier and AD633 analog multipliers are used with supply voltage ±15V. CD 4016 quad CMOS switch is used to provide the required kick. In the experimental study, a two-dimensional projection of the attractor is obtained by connecting \( V_1 \) and \( V_2 \) to the X and Y channels of an oscilloscope in X-Y mode.

With these component values, and with the CMOS switch CD 4016 opened, the Hopf limit cycle obtained experimentally is depicted in Fig. 5.9(a). This limit cycle is then kicked periodically by controlling the switch with a rectangular pulse. The experiment is then carried out by closing the switch connected to \( R_2 \) and by keeping the one connected to \( R_1 \) opened. This is achieved by applying rectangular pulses of long duration to pin number 13 of CD4016. The results obtained for different duty cycles are shown in Figs. 5.9(b) to (d). The experiment is also carried out for the single switch case for resistor \( R_1 \) by connecting the pulse to pin number 5 of CD4016 and by keeping the other switch opened. The various attractors obtained are depicted in Figs. 5.10(a) to (d).
Fig. 5.8 Switch-controlled Chua’s oscillator in smooth case.

Fig. 5.9 Strange attractors from smooth Chua’s oscillator; (a) a Hopf limit cycle with switches $S_1$ and $S_2$ opened, (b) with $S_2$ closed and $T = 40$, (c) Time response plot with $T = 40$ and (d) the attractor obtained with $T = 85$. 
Fig. 5.10 Strange attractors from smooth Chua’s oscillator; (a) Hopf limit cycle with switches $S_1$ and $S_2$ opened and (b) with $S_1$ closed and $T = 40$, (c) $T = 75$ and (d) $T = 93$. 
5.4.2 Piecewise linear case

The circuit schematic of PWL switch-controlled Chua’s oscillator is realized by combining the basic switch-controlled Chua’s oscillator (Fig. 5.5) and the three-segment PWL Chua’s diode (Fig. 2.3) as shown in Fig. 5.11. It is then constructed on a breadboard with the same parameter values used in the numerical simulation. The component values are then obtained as 

- \( R_0 = 0.012 \) k\( \Omega \), \( R = 1.5 \), \( R_p = 47 \), \( R_q = 3.3 \), \( R_r = 3.3 \), \( R_s = 47 \), \( R_t = 1.2 \)
- \( R_u = 0.29 \) k\( \Omega \) resistors; \( R_2 = 2 \) and \( R_1 = 5 \) k\( \Omega \) potentiometers; \( C_1 = 2.2 \) nF capacitors; and \( L = 10 \) mH inductor.
- A741C operational amplifier is used with supply voltage \( \pm 15V \) and 4016, the quad CMOS switch is used to provide the required kick. With these parameter and component values, the circuit is closed to non-Hopf bifurcations.

With these component values, and with the CMOS switch 4016 opened, the weakly stable limit cycle obtained experimentally is depicted in Fig. 5.12(a). This limit cycle is then kicked periodically by controlling the switch with a rectangular pulse. The experiment is then carried out by closing the switch connected to \( R_2 \) and by keeping the one connected to \( R_1 \) opened. This is achieved by applying rectangular pulses of long duration to pin number 13 of CD4016. The results obtained for different duty cycles are shown in Figs. 5.12(b) to (d). The experiment is also carried out for the single switch case for resistor \( R_1 \) by connecting the pulse to pin number 5 of CD4016 and by keeping the other switch opened. The various attractors obtained are depicted in Figs. 5.13(a) to (d).

This section can be concluded by explaining how strange attractors are created by the application of pulses controlled by external switches. In the absence of external forcing, the phase portrait appears like a circle. This closed loop represents the limit cycle where the time domain representation is periodic. When the switch is closed, this closed loop is deformed to become the solid curve and two factors are put into action, the shearing and the attraction. The shearing exaggerates the initial deformation brought in by the kicking force and the attraction acts against this deformation. If the shearing is weak, and the stability of the original limit cycle is
Fig. 5.11 Switch-controlled Chua's oscillator in piecewise linear case.

Fig. 5.12 Strange attractors from PWL Chua's oscillator; (a) a non–Hopf limit cycle with switches S1 and S2 opened and (b) with S2 closed and T = 22, (c) T = 55 (d) and T = 105.
Fig. 5.13 Strange attractors from PWL Chua's oscillator; (a) a non–Hopf limit cycle with switches S₁ and S₂ opened; with S₁ closed and with (b) $T = 22$, (c) $T = 55$ and (d) $T = 105$. 
strong, the attraction overcomes the initial deformation. In this case, all points nearby are attracted to a simple closed curve. On the other hand, if the shearing is strong and the stability of the original limit cycle is weak, then the deformation overcomes the attraction. Now, the attracting sets get disintegrated into a finite collection of periodic saddles and sinks. As shearing gets further strengthened, the deformation is exaggerated further into the dominance of sinks as the only observable set of attraction. Simultaneously, strange attractors with complicated fractal structure occur and eventually, chaotic attractors dominate.

5.5 Conclusions

This chapter elucidates the experimental proof of rank 1 chaos in the switch-controlled Chua oscillator. Both the smooth and the PWL cases have been investigated under different kicking schemes. The basic Chua's oscillator circuit is modified in such a way to modulate the capacitor voltages and the inductor current with a periodic pulse train. The modulation scheme is accomplished by using externally controlled switches. This study is useful in obtaining chaotic attractors from any given nonlinear circuit of weakly stable oscillations by a general switch-controlled scheme. The experimental results are found to be in perfect agreement with the predictions of the theory. Multiple kicks of different magnitudes can also be employed in one forcing period to generate strange attractors of various geometric and dynamical structures. An important point to be considered here is that the width of the applied pulses is not crucial (other than practical concerns for the physical switches used) as long as it is followed by a much longer relaxation interval.