CHAPTER 4

SYSTEM REDUCTION USING
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4.1 INTRODUCTION

In this chapter, a new method of system reduction using Sturm's sequence is discussed and illustrated with numerical examples. Before presenting this method, it is but appropriate to review an approximation method based on Routh stability table. Its merits and demerits are discussed and improvements suggested to overcome the limitations are also presented. The method discussed is straightforward and simple when compared to the other methods [14-42].

Consider a linear time-invariant single-input single-output system having the transfer-function

\[ H(s) = \frac{b_n s^{n-1} + \ldots + b_0}{a_n s^n + \ldots + a_0} \]  \hspace{1cm} \ldots (1)

The objective is to compute from the given \( H(s) \) a transfer-function \( H_k(s) \) of reduced order which approximates \( H(s) \).

Hutton [24-25] has proposed a method to achieve low-order models from high-order systems using Routh Stability Table. It consists of four steps:
To apply the reciprocal transformation to $K(s)$ leading to $\hat{H}(s)$.

To compute 'α-β' expansion of $\hat{H}(s)$ (akin to continued fraction expansion).

To truncate and construct the required model to $\hat{H}_k(s)$ and

To apply again the reciprocal transformation to $\hat{H}_k(s)$ to get $H_k(s)$.

But the above method has got the following limitations [24-25].

* It requires the application of two reciprocal transformations.

* It assumes that $H(s)$ has no eigen-values at the origin.

* When the original system is unstable, then the above method does not yield a unique reduced-order transfer-function.

Later a more simplified procedure is evolved to construct the low-order models from high-order systems without resorting to the reciprocal transformations by a new set of algorithms [26-27]. In both the methods,
the $\alpha$-$\beta$ parameters are calculated from the Routh-table. But the calculation of $\beta$-parameters is not straight-forward.

In this chapter, a procedure is outlined to calculate the $\alpha$-$\beta$ parameters directly. It eliminates one of the limitations of the Routh-array, i.e. the appearance of a zero leading element in the first column of any array. It also eliminates the limitations of the above methods [24-27]. This new method of obtaining a low-order system from a high-order system is based on the Sturm's sequence which ensures the stability of low-order model, provided the original system is stable [13].

4.2 THE PROCEDURE FOR CALCULATING $\alpha$-$\beta$ PARAMETERS USING STURM'S SEQUENCE FUNCTIONS

\[ H(s) = \frac{N(s)}{D(s)} = \frac{b_0 + \ldots + b_{n-1}s^{n-1}}{a_0 + \ldots + a_n s^n} \quad \ldots(2) \]

To obtain without ambiguity and with ease the $\alpha$-$\beta$ parameters, the following table is constructed.

\[ D(s) = f_1(s) = a_0 + a_1s + a_2s^2 + a_3s^3 + a_4s^4 + a_5s^5 + \ldots \quad \ldots(3) \]

\[ f_1(0) > 0 \]

Choose \[ f_2(s) = a_1 + a_3s^2 + a_5s^4 + \ldots \quad \ldots(4) \]

Using \( f_1(s) \) and \( f_2(s) \) construct the Sturm's sequence function \( f_i(s) \), \( i = 3, 4, \ldots, 2n+1 \) shown below. This we identify as 'a' table. Because of the nature of construction
of the table the number of right-hand side poles are equal
to the number of sign changes in the elements $a_5$, $b_5$, $c_4$, $d_3$

**TABLE 1 : 'a' TABLE**

<table>
<thead>
<tr>
<th>$a_1=a_0/a_1$</th>
<th>$a_0$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_5$</th>
<th>$\dots$</th>
<th>$f_1(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>0</td>
<td>$a_3$</td>
<td>0</td>
<td>$a_5$</td>
<td>0</td>
<td>$\dots$</td>
<td>$f_2(s)$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$a_2=b_1/b_2$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
<th>$b_4$</th>
<th>$b_5$</th>
<th>$\dots$</th>
<th>$f_3(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_2$</td>
<td>0</td>
<td>$b_4$</td>
<td>0</td>
<td>$\dots$</td>
<td>$f_4(s)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$a_3=c_1/c_2$</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
<th>$c_4$</th>
<th>$\dots$</th>
<th>$f_5(s)$</th>
<th>$\dots(5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_2$</td>
<td>0</td>
<td>$c_4$</td>
<td>0</td>
<td>$\dots$</td>
<td>$f_6(s)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$a_4=d_1/d_2$</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$\dots$</th>
<th>$f_7(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_2$</td>
<td>0</td>
<td>0</td>
<td>$\dots$</td>
<td>$f_8(s)$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$a_5=e_1/e_2$</th>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$\dots$</th>
<th>$f_9(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_2$</td>
<td>0</td>
<td>$\dots$</td>
<td>$f_10(s)$</td>
<td></td>
</tr>
</tbody>
</table>

| $f_{2n+1}(s)$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $f_{2n+1}(s)$ |

where

\[
\begin{align*}
    b_1 &= a_1 \cdot a_1 - a_0 \cdot 0 \\
    b_2 &= a_1 \cdot a_2 - a_0 \cdot a_3 \\
    b_3 &= a_1 \cdot a_3 - a_0 \cdot a_5 \\
    b_4 &= a_1 \cdot a_4 - a_0 \cdot a_5 \\
\end{align*}
\]
The above coefficients can be computed by the generalized algorithms given below:

For $i$ odd

$$b_i = a_i a_i$$
$$c_i = b_i b_{i+1}$$

......

......

......

...(6)

For $i$ even

$$b_i = a_i a_i - a_0 a_{i+1}$$
$$c_i = b_i b_{i+1} - b_1 b_{i+2}$$

......
Next consider the numerator

\[ N(s) = g_1(s) = b_0 + b_1 s + b_2 s^2 + b_3 s^3 + b_4 s^4 + \ldots \] ...

Choose

\[ g_i(s) = f_1(s) ; i = 2, 4, 6, \ldots \] ...

Using the above functions construct the Sturm's sequence functions as shown below. The table is completed with similar operations as in \( \alpha \)-table. This we identify as \( \beta \)-table.

**TABLE 2: '8' TABLE**

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\beta_1 = b_0/a_1 & b_0 & b_1 & b_2 & b_3 & b_4 & \cdots \rightarrow g_1(s) \\
\hline
a_1 & 0 & a_3 & 0 & a_5 & \cdots \rightarrow g_2(s) \\
\hline
\beta_2 = G_1/b_2 & C_1 & G_2 & C_3 & G_4 & \cdots & \rightarrow g_3(s) \\
\hline
b_2 & 0 & b_4 & 0 & \cdots & \rightarrow g_4(s) \\
\hline
\beta_3 = D_1/c_2 & D_1 & D_2 & D_3 & \cdots & \rightarrow g_5(s) \\
\hline
c_2 & 0 & c_4 & \cdots & \rightarrow g_6(s) \\
\hline
\beta_4 = E_1/d_2 & E_1 & E_2 & \cdots & \rightarrow g_7(s) \\
\hline
d_2 & 0 & \cdots & \rightarrow g_8(s) \\
\hline
\vdots & F_1 & \cdots & \rightarrow g_9(s) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \rightarrow g_{2n+1}(s) \\
\hline
\end{array}
\]
4.3 TRANSFER-FUNCTION OF REDUCED-ORDER SYSTEM

In this section an algorithm is developed to obtain reduced-order models of a given original system.

Let the transfer-function of a fourth order system be

$$H(s) = \frac{b_0 + b_1s + b_2s^2 + b_3s^3}{a_0 + a_1s + a_2s^2 + a_3s^3 + a_4s^4} \quad \ldots (10)$$

Dividing the numerator and the denominator by \(a_1s + a_3s^3\) we have

$$H(s) = \frac{b_0 + b_1s + b_2s^2 + b_3s^3}{a_0 + a_1s + a_2s^2 + a_3s^3 + a_4s^4} \quad \ldots (11)$$

Simplifying and rearranging we have

$$H(s) = \frac{\beta_1 s + E_1s^2 + E_3s^3}{1 + \frac{a_1s + a_3s^3}{a_1s + a_3s^3}} \quad \ldots (12)$$

where

\[
\begin{align*}
\beta_1 &= \frac{b_0}{a_1} \\
E_1 &= \frac{b_1}{a_1} \\
E_2 &= \frac{a_1b_2 - b_0b_3}{a_1} \\
E_3 &= \frac{b_3}{a_1}
\end{align*}
\]
and

\[ a_1 = \frac{a_0/a_1}{(a_1a_2 - a_0a_3)} \]

\[ F_1 = \frac{a_1}{(a_1a_2 - a_0a_3)} \]  \hspace{1cm} \ldots(14)

\[ F_2 = a_4 \]

Consider the second term of the numerator: multiplying and dividing by \((F_1s^2 + F_2s^4)\) we have,

\[ \frac{E_1s + E_2s^2 + E_3s^3}{a_1s + a_3s^3} = \frac{F_1s^2 + F_2s^4}{a_1s + a_3s^3} \cdot \frac{E_1s + E_2s^2 + E_3s^3}{F_1s^2 + F_2s^4} \]

\[ = \frac{1}{a_1s + a_3s^3} \times \frac{E_1s + E_2s^2 + E_3s^3}{F_1s^2 + F_2s^4} \]  \hspace{1cm} \ldots(15)

Substituting the values of \(F_1, F_4, E_1, E_2, E_3\) from Equations (13) and (14)

\[ = \frac{1}{a_1s + a_3s^3} \]

This can be written as:

\[ = \frac{1}{a_2^2} \left[ \frac{\beta}{s} + \ldots \right] \]  \hspace{1cm} \ldots(16)
where
\[ \alpha_2 = \frac{a_1}{(a_1a_2 - a_0a_3)} \]
\[ \beta_2 = \frac{a_1b_1}{(a_1a_2 - a_0a_3)} \]

Now the transfer function
\[ H(s) = \frac{\beta_1}{s + \frac{1}{\alpha_2}} \left[ \frac{\beta_2}{s} + \ldots \right] \]
\[ = \frac{\beta_1}{s + \frac{1}{\alpha_2}} \frac{\beta_2}{s} \ldots (17) \]

The values of \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) obtained here are same as obtained from \( \alpha-\beta \) tables in Equations (5) and (9). The second-order reduced model can be obtained by truncation as shown below.
\[ H(s) = \frac{\beta_1}{s + \frac{1}{\alpha_2}} \left[ \frac{\beta_2}{s} \right] \]
\[ = \frac{\beta_1}{s + \frac{1}{\alpha_2}} \frac{\beta_2}{s} \ldots (18) \]
\[ = \frac{\beta_2s + \alpha_2 \beta_1}{s^2 + \alpha_2s + \alpha_1 \alpha_2} \ldots (19) \]

Therefore the transfer function of the second-order model can be written as
\[ H_2(s) = \frac{N_2(s)}{D_2(s)} = \frac{\beta_2s + \alpha_2 \beta_1}{s^2 + \alpha_2s + \alpha_1 \alpha_2} \ldots (20) \]
Extending this procedure a general algorithm can be formulated as shown below.

Let 

\[ H(s) = \frac{b_0 + \ldots + b_n s^{n-1}}{a_0 + \ldots + a_n s^n} = \frac{N(s)}{D(s)} \]

where the subscripts 'e' and 'o' denotes the even and odd terms of the denominator respectively.

\[ \frac{N(s)}{D_o(s)} = \frac{\beta_1 + \frac{N'(s)}{D_o'(s)}}{1 + \frac{\alpha_1}{D_o'(s) / D_o'(s)}} \]

\[ \frac{\beta_1 + \frac{N'(s)}{D_o'(s)}}{1 + \frac{\alpha_1}{D_o'(s) / D_o'(s)}} \]

\[ = \frac{\beta_2 + \frac{N''(s)}{D_o''(s)}}{1 + \frac{\alpha_2}{D_o''(s) / D_o''(s)}} \]

\[ = \ldots \]

\[ \ldots \]

\[ \ldots \]
\[
\begin{align*}
[\frac{\beta_1}{s} + \left(\frac{1}{s} + \frac{\alpha_1}{s} + \frac{\alpha_2}{s} + \frac{\alpha_3}{s} + \frac{\alpha_4}{s} + \cdots\right)]^2 + 1
&= \frac{\beta_2}{s} + \left(\frac{1}{s} + \frac{\alpha_3}{s} + \frac{\alpha_4}{s} + \cdots\right) + \frac{\beta_3}{s} + (\frac{1}{\alpha_4})(\frac{\beta_4}{s}) + \cdots \\
&= \frac{\beta_4}{s} + \cdots
\end{align*}
\]

where
\[
\begin{align*}
N_1(s) &= \beta_1/s \\
D_1(s) &= 1 + \frac{\alpha_1}{s} \\
D_2(s) &= 1 + \frac{\alpha_1}{s} + \frac{1}{\alpha_2/s} \\
N_2(s) &= \frac{\beta_1}{s} + \frac{\beta_2}{s} + \frac{1}{\alpha_2/s} \\
N_3(s) &= \frac{\beta_1}{s} + \frac{1}{\alpha_2/s} + \frac{\beta_2}{s} + \frac{\beta_3}{\alpha_3} + \frac{1}{\alpha_3/s} + \cdots \\
D_3(s) &= 1 + \frac{\alpha_1}{s} + \frac{1}{\alpha_2/s} + \frac{1}{\alpha_3/s} + \cdots
\end{align*}
\]

\(\cdots(24)\)

\[
\begin{align*}
N_1(s) &= \beta_1/s \\
D_1(s) &= 1 + \frac{\alpha_1}{s} \\
D_2(s) &= 1 + \frac{\alpha_1}{s} + \frac{1}{\alpha_2/s} \\
N_2(s) &= \frac{\beta_1}{s} + \frac{\beta_2}{s} + \frac{1}{\alpha_2/s} \\
N_3(s) &= \frac{\beta_1}{s} + \frac{1}{\alpha_2/s} + \frac{\beta_2}{s} + \frac{\beta_3}{s} + \cdots \\
D_3(s) &= 1 + \frac{\alpha_1}{s} + \frac{1}{\alpha_2/s} + \frac{1}{\alpha_3/s} + \cdots
\end{align*}
\]

\(\cdots(25)\)
The reduced-order models are given by

\[ H_k(s) = \frac{s N_k(s)}{D_k(s)} = \frac{N_k(s)}{D_k(s)}. \] \hfill (26)

The algorithm can therefore be written as (since \( s \) in the numerator cancels with that in the denominator)

\[ N_1(s) = \beta_1 \]
\[ D_1(s) = s + \alpha_1 \]
\[ N_2(s) = \beta_2 s + \alpha_2 \beta_1 \]
\[ D_2(s) = s^2 + \alpha_2 s + \alpha_1 \alpha_2 \] \hfill (27)

In general,

\[ N_k(s) = \beta_k s^{k-1} + s^2 N_{k-2}(s) + \alpha_k N_{k-1}(s) \] \hfill (28)
\[ D_k(s) = s^2 D_{k-2}(s) + \alpha_k D_{k-1}(s) \]

with \( N_{-1}(s) = 0 \)
\[ N_0(s) = 0 \quad \text{and} \]
\[ D_{-1}(s) = 1/s \]
\[ D_0(s) = 1 \]

Therefore

\[ H_k(s) = \frac{N_k(s)}{D_k(s)} \] \hfill (30)
4.4 **ILLUSTRATIVE EXAMPLES**

**Example 1**

Let us consider a stable system with transfer-function $H(s)$ given by [20-21].

$$
H(s) = \frac{1441.53s^3 + 78319s^2 + 525286.125s + 607693.25}{s^7 + 112.04s^6 + 3755.92s^5 + 39736.75s^4 + 363650.56s^3 + 759894.19s^2 + 683656.25s + 617497.375}
$$

The values of the $\alpha-\beta$ parameters obtained as computer outputs are:

- $\alpha_1 = 0.903227864$
- $\alpha_2 = 1.584610559$
- $\alpha_3 = 1.404645959$
- $\alpha_4 = 9.778864742$
- $\alpha_5 = 12.4852225$
- $\alpha_6 = 25.77875914$
- $\alpha_7 = 97.25297954$
- $\beta_1 = 0.888887141$
- $\beta_2 = 1.21753296$
- $\beta_3 = -0.800255066$
- $\beta_4 = -1.367780928$

For a second-order model

$$
H_2(s) = \frac{N_2(s)}{D_2(s)} = \frac{\beta_2 s + \alpha_2 \beta_1}{s^2 + \alpha_2 s + \alpha_1 \alpha_2}
$$
For a third-order model

\[ H_3(s) = \frac{N_3(s)}{D_3(s)} = \frac{(\beta_1+\beta_2)s^2 + \beta_2a_3s + \beta_1a_2a_3}{s^3 + (a_1+a_2)s^2 + a_2a_3s + a_1a_2a_3} \]

\[ = \frac{0.088632075s^2 + 1.716290276s + 1.985542648}{s^3 + 2.312973823s^2 + 2.233734871s + 2.017576092} \]

The time-moments of the original, second and third-order systems are:

\[ H(s) = 0.984122807 - 0.2388926476s - 1.021047s^2 + 0.84719877s^3 + 0.2411789s^4 - 0.717666259s^5 + 0.06326s^6 + ... \]

\[ H_2(s) = 0.984122807 - 0.238892638s - 0.423102122s^2 + 0.6353436s^3 + ... \]

\[ H_3(s) = 0.984122807 - 0.238892639s - 0.81974394s^2 + 0.693654366s^3 + ... \]

Here the first two time-moments of the second and third order models are matching up to 8th decimal point, and therefore these models will very nearly follow the response of the original system. The responses of the original system and the simplified system are shown in Fig.4.1.
Example 2: Modeling of Bio-systems [45-46]

The behaviour of most of the Bio-systems is very much complicated, and to understand them it is necessary to have reduced-order models which can explain the physical behaviour of the original bio-system. The necessary modifications if any can easily be incorporated in the reduced-order model rather than in the complicated Bio-system.

Pupillary-Retinal System

The eye exhibits the inherent feedback characteristics. The light reaching the retina controls the diameter of the pupil, which controls the amount of light that reaches the retina.

Let us consider the open-loop transfer function \( G(s) \) of the pupillary-retinal system based on gain consideration and under the assumption of minimum phase [45].

\[
G(s) = \frac{160}{s^3 + 30s^2 + 300s + 1000}
\]

To obtain a second-order model the values of \( \alpha-\beta \) parameters are:

\( c_1 = 3.33 \)

\( \omega_n = 90/8 \)
\[ P_1 = \frac{8}{15} \]
\[ P_2 = 0 \]

The second-order model based on the algorithm is

\[ G_2(s) = \frac{\beta_2 s + \alpha_2 \beta_1}{s^2 + \alpha_2 s + \alpha_1 \alpha_2} \]

\[ = \frac{6}{s^2 + 11.25s + 37.5} \]

Time moments of the original and the second-order systems are

\[ G(s) = 0.16 - 0.048s + 0.0096s^2 \]
\[ G_2(s) = 0.16 - 0.048s + 0.01013s^2 \]

It can be observed that the time-moments of \( G(s) \) and \( G_2(s) \) are very close, indicating that the responses of the original system and the reduced-order system will nearly be the same. The step responses of the original system and the simplified system are shown in Fig.4.2.

4.5 **EXTENSION TO UNSTABLE SYSTEMS**

So far in the discussion, it has been assumed that the system to be approximated is asymptotically stable. In many applications, in particular, in the design of control systems, this may not be the case. In this section, the application of the new method to unstable systems is dealt with. The resulting reduced model retains the
unstable eigen values and is unique. The two cases considered here are:

- asymptotically stable systems except for poles at the origin,
- systems having poles with positive real parts.

* Case (a): System with Poles at the Origin

Consider a system transfer-function $G(s)$ with poles at the origin, given by

$$G(s) = \frac{c_{n+2}s^{n+1} + \cdots + c_1}{s^b(a_n s^n + \cdots + a_0)}$$

...(31)

The first step is to separate the asymptotically stable part and rewrite $G(s)$ as follows:

$$s^p G(s) = d_2 s + d_1 + \frac{b_n s^{n-1} + \cdots + b_1}{a_n s^n + \cdots + a_0}$$

...(32)

The strictly proper rational function on the right side of the above equation is asymptotically stable and hence, can be approximated by the method. This approach is illustrated by an example in Section 4.6.

* Case (b): System Having Poles with Positive Real Parts

Consider an unstable system with the transfer-function

$$G(s) = \sum_{i=1}^{p} \frac{A_i}{s - \lambda_i} + \frac{a}{s + \lambda_a} + \frac{b}{s + \lambda_b} + \frac{c}{s + \lambda_c}$$

...(33)
where $\lambda_i$ are the right hand side poles. This is first frequency-shifted by an amount '$a$' given by

$$a = \max \text{ Re} \left( \lambda_i \right) \quad i = 1, ..., p$$

The frequency-shifted transfer-function $G(s)$ is given by

$$G(s) = G(s+a) \quad \ldots (34)$$

This transforms the unstable system to an asymptotically stable one with pole or poles on the imaginary axis. $G(s)$ is then used to reduce the order of the system by the proposed method. If the frequency-shifted $K$-th order model is given by $G_k(s)$, then the $K$-th order approximant is given by

$$G_k(s) = G_k(s-a) \quad \ldots (35)$$

which is the reduced-order transfer-function after a frequency-shift of '$a$' to the left. The method is illustrated by an example in Section 4.6.

4.6 **ILLUSTRATIVE EXAMPLE OF UNSTABLE SYSTEMS**

* **Case (a): Poles at the Origin** [24]

Consider the following transfer-function

$$G(s) = \frac{2s^3 + 60s^2 + 432s + 640}{s^2(s^2 + 11s + 10)}$$
which can be written as
\[ s^2 G(s) = 2s + 38 + \frac{-6s + 260}{s^2 + 11s + 10} \]

The strictly proper rational function on the right side is reduced to a first order model by the application of the new method:
\[ a_1 = \frac{10}{11} \]
\[ \beta_1 = \frac{260}{11} \]

This yields
\[ s^2 G_3(s) = 2s + 38 + \frac{260}{11s + 10} \]
\[ = \frac{22s^2 + 438s + 640}{11s + 10} \]

Therefore
\[ G_3(s) = \frac{22s^2 + 438s + 640}{s^2 (11s + 10)} \]

The two unstable poles at the origin are retained in the above reduced-model.

* Case (b) : Poles with Positive Real Parts [25]

Consider the following transfer-function
\[ G(s) = \frac{3}{s-2} + \frac{5}{s+1} + \frac{2}{s+10} - \frac{1}{s+12} \]
\[ = \frac{9s^3 + 192s^2 + 790s - 868}{s^4 + 21s^3 + 96s^2 - 164s - 240} \]
Shifting with \( s = s + 2 \):

\[
G(s+2) = \frac{3}{s} + \frac{5}{s + 3} + \frac{2}{s + 12} - \frac{1}{s + 14}
\]

\[
= \frac{9s^3 + 2365s^2 + 1626s + 1512}{s(s^2 + 29s^2 + 240s + 504)}
\]

To separate the asymptotically stable part of the system, it can be rewritten as:

\[
sG(s+2) = 9 - \frac{25s^2 + 588s + 3024}{s^3 + 29s^2 + 246s + 504}
\]

Reducing the strictly proper rational function to a second order model and rearranging:

\[
G_3(s+2) = \frac{9s^2 + 60.35s + 56.2}{s(s^2 + 9.128s + 18.7)}
\]

Substituting \( s = s - 2 \):

\[
G_3(s) = \frac{9s^2 + 24.35s - 28.5}{s^3 + 3.128s^2 - 5.81s - 8.9}
\]

\[
= \frac{3}{s - 2} + \frac{5.465}{s + 1.109} + \frac{1.148}{s + 4.044}
\]

The partial fraction expansion above shows that the unstable pole and the dominant poles are retained in the reduced model. Since the value of shift is unique, the reduced-order model is also unique for the given unstable system. The optimum value of the shift can be determined from the knowledge of the right hand side poles.
4.7 SINGULAR INPUT MULTI-OUTPUT SYSTEM

Let us consider a single-input-multi-output system with the transfer-function [25],

\[ H(s) = \frac{\begin{bmatrix} 28 \\ 12 \\ 12 \end{bmatrix} s^3 + \begin{bmatrix} 496 \\ 528 \\ 1400 \end{bmatrix} s^2 + \begin{bmatrix} 1800 \\ 1400 \\ 4320 \end{bmatrix} s + \begin{bmatrix} 2400 \\ 6000 \\ 4200 \end{bmatrix}}{2s^4 + 36s^3 + 204s^2 + 360s + 240} \]

This represents the transfer-function of a system of the fourth order with two outputs and a single input.

From the \( \alpha \)-table

\[ \begin{align*}
\alpha_1 &= \frac{2}{3} \\
\alpha_2 &= 2 \\
\alpha_3 &= \frac{45}{8} \\
\alpha_4 &= 16
\end{align*} \]

For the first output,

\[ \begin{align*}
\beta_1 &= \frac{20}{3} \\
\beta_2 &= 10 \\
\beta_3 &= 8 \\
\beta_4 &= 4
\end{align*} \]

For the second output

\[ \begin{align*}
\beta_1 &= 12 \\
\beta_2 &= 8 \\
\beta_3 &= 3 \\
\beta_4 &= -2
\end{align*} \]

The models using the algorithms are:
The procedure explained above gives rise to reduce order systems whose time-moments at \( s = 0 \) and \( s = \infty \) are closer to those of the original systems. Whereas the time-moments of the reduced-order models obtained by Padé approximation method and Chen method do not agree with those of the original system around \( s = \infty \). This is illustrated with the following example.

*Example 3:*

Given \( G(s) = \frac{3s^3+13s^2+3s+10}{0.5s^4+1.5s^2+s^2+s+1} \) \( \quad (1) \)

Expanding around \( s = 0 \), \( G(s) = 10-7s+(0)s^2+2s^3+... \) \( \quad (2) \)

Expanding around \( s = \infty \), \( G(s) = \frac{6}{s} + \frac{8}{s^2} - \frac{42}{s^3} \) \( \quad (3) \)
Chen and Shieh Second-order approximant

\[ G_{2c}(s) = \frac{-10.1893s + 34.966}{s^2 + 1.42857s + 3.4966} \]  \hspace{1cm} (4)

Expanding around \( s=0 \), \( G_{2c}(s) = 10 - 6.995 + (0)s^2 + (2.0019s^3) \ldots \)  \hspace{1cm} (5)

Expanding around \( s = \infty \), \( G_{2c}(s) = -10.1893s^{-1} + 49.522s^{-2} - 35.1176s^{-3} \ldots \)  \hspace{1cm} (6)

Padé second order approximant

\[ G_{2p}(s) = \frac{-2.9184s + 10}{0.2857s^2 + 0.4082s + 1} \]  \hspace{1cm} (7)

Expanding around \( s=0 \), \( G_{2p}(s) = 10 - 7s + (0)s^2 + 2s^3 \)  \hspace{1cm} (8)

Expanding around \( s = \infty \), \( G_{2p}(s) = -10.214s^{-1} + 49.592s^{-2} - 35.103s^{-3} \ldots \)  \hspace{1cm} (9)

By the Proposed method

\[ G_2(s) = \frac{6s + 20}{s^2 + 2s + 2} \]  \hspace{1cm} (10)

Expanding around \( s=0 \), \( G(s) = 10 - 7s + 2s^2 + 1.5s^3 \)  \hspace{1cm} (11)

Expanding around \( s = \infty \), \( G(s) = 6s^{-1} + 8s^{-2} - 28s^{-3} \ldots \)  \hspace{1cm} (12)

When the step responses are plotted it is observed that model (10) gives a better response over model (4) or model (7). And the reason for this is the first two terms of the expansion (12) are completely matched with the corresponding terms of the expansion (3) while either the expansion (9) or the expansion (6) differs significantly from the expansion (3). It is because of this effect the model
given by equation (10) produces a better time-response as compared to the model (4) or the model (7). The time-response is shown in Fig.4.3.

4.9 COMPUTER PROGRAM

A computer program has been developed to compute the α-β tables and parameters and also to find out the number of poles lying in the Right-Half-Plane and give the stability of the system. The program also computes the reduced-order models from \( K = K_1 \) to \( K = K_2 \). When the original system is of order \( n \) and if \( K_1 = 1 \) and \( K_2 = n \), then all the possible models including the original is computed. The computer output is presented in Table 4.1.

4.10 CONCLUSION

The procedure outlined in this chapter gives rise to reduced-order models with improved responses. It eliminates the possibility of getting a zero leading element in the Routh-array, so that the stability array is simple. Calculation of α-β parameters is straightforward. It ensures the stability of the low-order model, provided the original system is stable. The versatility of this method has been illustrated by examples. The approach
has been illustrated for the case of single-input single output systems (SISO) and single-input multiple-output system (SIMO) with suitable examples.

This approach can also be applied to Bio-systems to obtain lower-order models. A computer program has been developed and its efficacy is illustrated by numerical examples.
FIG. 4.1. TIME RESPONSE EXAMPLE 1

1 — ORIGINAL SYSTEM \( H(A) \)
2 — SIMPLIFIED SYSTEM \( H_2(A) \)
FIG. 4.2. TIME RESPONSE. EXAMPLE 2

STEP INPUT 0.01

1 ORIGINAL SYSTEM $G_A$

2 SIMPLIFIED SYSTEM $G_{1,2}$
FIG. 4.3 TIME RESPONSE. EXAMPLE 3
**ALPHA AND BETA TABLES**

**Table 4.3 (a) Alpha and Beta Parameters And Reduced Order Models for the 14th-order System given by**

\[ H(s) = \frac{1}{s^{14} + 15.6s^{13} + 126.68s^{12} + 693.98s^{11} + 2808.63s^{10} + 8612.78s^9 + 20130.27s^8 + 36027.06s^7 + 49275.42s^6 + 50802.42s^5 + 37419.64s^4 + 18522.28s^3 + 5574.29s^2 + 865.8s + 46.08} \]

\[ s^{13} + 17.1s^{12} + 143.6s^{11} + 798.37s^{10} + 3257.18s^9 + 10073.1s^8 + 23815.83s^7 + 43165.96s^6 + 59697.67s^5 + 61878.41s^4 + 46394.93s^3 + 23669.7s^2 + 7327.5s + 1034.65 \]