CHAPTER 4

DESIGN OF WAVE DIGITAL OPTIMAL RESONANT LOWPASS DIFFERENTIATOR FOR ELECTROCARDIOGRAM PROCESSING

4.1 INTRODUCTION

The advantages of digital processing of ECG have generated interest in digital lowpass differentiators, which are algorithms combining the two operations of differentiation and lowpass filtering. A survey of digital lowpass differentiators has been presented in Chapter 1. ECGs are in general contaminated by several noise sources, and a good R wave detection implies the need for preprocessors that reduce the influence of these noises. Common noise types are powerline hum, artifacts due to electrode motion, and muscle noise. Interference of these noises is observed in the form of presence of 50 HZ sinusoid, base line drift, and additive white noise. Usually several cascaded stages are employed to reduce these effects. Inclusion of digital lowpass differentiator as one of the stages generally improves the detection capability of the ECG instrument. For ambulatory patients, a portable arrhythmia monitor forms an important component of the ECG instrument. Portable monitors must be small in size and, therefore, their register length is restricted to be small. These monitors are designed to consume less power and, therefore, the number of memory chips must be minimum possible. Small register length gives rise to appreciable level of finite word length effects (Jerome Cox, et al, 1972) (Antoniou, 1979) and one needs to pay attention to selection of algorithms least sensitive to these effects. Minimum memory and hardware means that one should look for algorithms with minimum number of coefficients, and which can be implemented in smaller number of stages. A design procedure for spectral amplitude matched UEWDF optimal resonant lowpass
differentiator for QRS complex detection is developed in this Chapter. An algorithm for R wave detection and baseline drift suppression is derived. The algorithm achieves a number of ECG processing requirements simultaneously: lowpass filtering, baseline drift suppression, identification of QRS complex abnormalities, low sensitivity to finite word length implementation, good stability under linear and finite arithmetic conditions, and a particularly simple form of hardware implementation. The design procedure is based on the concept of approximation in compact Hausdorff space presented in Chapter 3.

4.2 DIGITAL RESONATOR - DIFFERENTIATOR - DETECTION CONCEPT

A digital lowpass filter is obtained from an analog passive distributed parameter resonator type filter through spectral amplitude matching. The resonator will be of recursive type but the stability under linear condition is ensured since the reference filter is a passive structure. For the same reason, the digital resonator will also exhibit excellent sensitivity property under finite arithmetic conditions. Next, the digital resonator is implemented as a differentiator through a novel technique. The resonance effect will enhance certain characteristic features of the input ECG wave form, and the differentiation effect will sharpen undulations along the contour of the wave form. Detectors based on this concept can be made to display the rhythm accurately, thereby facilitating easy diagnosis of arrhythmias. They can provide accurate detection of R wave amplitudes leading to a better context for automated diagnosis of diseases, such as Left Ventricle hypertrophy, constrictive pericarditis, and cardiac failure. Resonance - differentiation - detection effect can bring out even slightest abnormalities of QRS complex. The presence and absence of P, R and T waves are registered more clearly. This could be extremely helpful for automated diagnosis of a number of myocardial infarctions with better accuracy.
A fundamental requirement in the design of digital filters is that the entire frequency response characteristic \( F(z) \) of the filter be reproduced within the principal range \( 0 < \omega T < \pi/2 \). A commensurate transmission line filter produces the frequency response characteristic \( H(\lambda) \) within the principal range \( 0 < \omega r < \pi/2 \) where \( r \) is the one way commensurate delay. However, the approximation problem is solved in terms of rational function of \( \lambda \) in the infinite interval \( 0 < \lambda < \infty \). This suggests that the spectrum of the signal to be detected may be completely described in the principal range \( 0 < \omega t < \pi/2 \) and then be transformed to the infinite interval \( 0 < \Omega < \infty \) where \( \Omega = \lambda/\omega \). This is guaranteed by the homeomorphic transform \( x_1 = \tan \frac{\pi}{2} x \) which transforms a function \( X(x)e^{i\Omega} \) to \( X(x_1)e^{i\Omega} \). Let such a transform be \( X(j\Omega) \). Under the conditions mentioned below, a rational function \( H(j\Omega) \) approximating \( X(j\Omega) \) can be generated which in turn can be synthesized as a cascade of unit elements (Karl Renner et al, 1974).

Let the matched wave digital filter be a two port network characterized by its impulse response \( h(n) \) and system function \( H(j\Omega) \) such that

\[
   h(n) = x(n-k) \quad (4.1)
\]

\[
   H(j\Omega) = kX^*(j\Omega) \exp(-j\Omega_0) \quad (4.2)
\]

where

- \( x(n-k) \) is the time shifted input signal
- \( X^*(j\Omega) \) is the complex conjugate of homomorphically transformed spectrum of \( x(n) \)
- \( k \) is the scale factor
- \( j\Omega \) imaginary part of Richards' variable
- \( \Omega_0 = \tan \omega t_0 \) is the homeomorphically transformed time delay

if \( H(j\Omega) \) is assumed to be the forward transmission coefficient of a unit element wave digital filter (UEWDF), one can write
The condition for optimum signal to noise ratio detection in the presence of additive white noise is stated as

\[ | S_{21}(j\Omega) | = k | X(j\Omega) | \] (4.4)

Equation (4.4) cannot be realized unless \( X(j\Omega) \) and \( S_{21}(j\Omega) \) are compatible, in the sense that both satisfy unitary conditions (Herbert Carlin, 1971). This suggests that the following conditions may be prescribed on \( X(j\Omega) \):

\( X(j\Omega) \) is continuous in \( \Omega \) and periodic in the interval \( 0 < \Omega < \infty \) (4.5)

\[ | X(j\Omega) |^2 = | X(j\Omega) X(-j\Omega) | \leq 1 \] (4.6)

\[ | X(j\Omega) | \leq 1 \] (4.7)

\( X(j\Omega) \) may possess a finite zero at \( \Omega = \Omega_c \) such that 90% of the signal energy is contained at \( \Omega = \Omega_c \) (4.8)

\[ | X(j\Omega) | \] has a unique supremum over the interval \( 0 \leq \Omega \leq \Omega_c \) (4.9)

The complete statement of a linear optimum WDF can now be stated as follows:

Let \( j\Omega = jx_1 \) where \( x_1 = \tan \pi x/2 \)

If \( |S_{21}(x)| \epsilon [0,1], |x(x)| \epsilon [0,1] \) can be assigned to \( C[0,1] \), and \( |S_{21}(x)|, |X(x)| \) possess the supremum metric

\[ d(f,g) = \text{Sup} \{ |f(x)| - |g(x)| : x \epsilon [0,1] \} \] (4.10)
where $C[0,1]$ is a class of functions defined over compact Hausdorff space, and $|d(f,g)|_{\text{max}} = 1$, then, the optimal WDF is defined by

$$|S_{21}(x_1)| \leq x_1 \in [0, \infty] = k \frac{X(x_1)}{1 + L_a(x^2)} x \in [0,1]$$

(4.11)

Let $|S_{21}(x)|^2 = \frac{1}{1 + L_a(x^2)} x \in [0,1]$ (4.12)

such that $S_{21}(x)$ and $X(x)$ can be chosen from $C[0,1]$ with the metric (4.10); in fact, $S_{21}(x)$ is so chosen as to match $X(x)$ in terms of the metric (4.10). Upon using the homomorphic transform $x_1 = \tan \frac{\pi x}{2}$, we get $S_{21}(x_1)$ and $X(x_1)$ where $jx_1 = j\Omega = \lambda$. $S_{21}(\lambda)$ can now be synthesized as a cascade of $n$ unit elements in which case

$$S_{21}(\lambda) = \frac{(1-\lambda^2)^{n/2}}{A_0 + A_1 \lambda + A_2 \lambda^2 + \ldots + A_n \lambda^n}$$

(4.13)

In the $Z$ domain, Equation (4.13) becomes

$$S_{21}(z) = \frac{N(z)}{D(z)} = \frac{k z^{-n/2}}{A_0 + A_1 z^{-1} + \ldots + A_n z^{-n}}$$

(4.14)

The difference equation corresponding to (4.14), after removing $z^{-n/2}$, is

$$y(n) = k \cdot x(n) \pm \sum_{k=1}^{N} A_k y(n-k)$$

(4.15)

which is a unit element wave digital matched resonator algorithm.

Let $Y_1(z) = \frac{k x(z)}{(1 + \sum_{k=1}^{N} A_k z^{-k})}$

(4.16)
\[ Y_2(z) = \frac{k x'(z) z^{-1}}{N (1 \pm \sum_{k=1}^{\infty} A_k z^{-k})} \]  

(4.17)

\[ W(z) = Y_1(z) + Y_2(z) \]  

(4.18)

From equations (4.16) and (4.17), it is obvious that

\[ W(z) = Y_1(z) - z^{-1} Y_1(z) \]  

(4.19)

The difference equation corresponding to (4.19) can be shown to be

\[ w(n) = k [x(n) - x(n-1)] + 2 \sum_{k=1}^{N} \pm A_k y(n-k) \]  

(4.20)

Equation (4.20) is the UEWDF resonant lowpass differentiator algorithm.

### 4.4 DECISION THRESHOLD

A basic issue in the design of optimal detector based on maximization of the output signal-to-noise ratio, as described in the previous sections, is the choice of a decision threshold. The matched detector is basically a correlation receiver. Based on the correlation receiver model (Simon Haykin, 1978), a decision rule is formulated for the UEWDF optimal resonant lowpass differentiator. Let the input ECG signal be the observed signal denoted as \( x(t) \). Assigning \( H_0 \) to \( x(t) \) if the observed signal is due solely to white Gaussian noise \( w(t) \) of zero-mean and spectral density \( N_0/2 \), or \( H_1 \) to \( x(t) \) if the observed signal is due to signal \( x(t) \) and noise \( w(t) \), one may write

\[ H_0: \quad x(t) = w(t) \]
\[ H_1: \quad x(t) = s(t) + w(t) \]
The problem is to observe the received signal over an interval from zero to $T$ seconds and decide whether $H_0$ or $H_1$ is true, according to some criterion. Let the noise component be a band limited white Gaussian noise with spectral density

$$S_w(f) = \begin{cases} \frac{N_0}{2} & |f| < B \\ 0 & |f| > B \end{cases}$$

If the signal $x(t)$ containing an additive noise is sampled at intervals $T_s = 1/2B$, the samples uncorrelated, and being Gaussian, they are statistically independent. A total of $m = T/T_s = 2BT$ statistically independent samples can be collected over the observation interval from zero to $T$. The joint probability density function $P_{X/H_0}(X/H_0)$ or $P_{X/H_1}(X/H_1)$ is, therefore, the product of the probability density functions of the individual components of a random vector $X$, assuming that $H_0$ or $H_1$ is true. Consider the receiver model shown in Figures 4.1 (a) and 4.1(b); One may write the following decision rule:

$$\int_{t_0}^{t} s(t) x(t) \, dt > D_T \quad (4.21)$$

where $x(t)$ is the input signal, $s(t)$ is the stored replica, and $D_T$ is the decision threshold given by

$$D_T = \frac{1}{2} N_0 \ln \left( \frac{P}{q} \right) + \frac{1}{2} \frac{d}{dt} \int_{0}^{t} s^2(t) \, dt \quad (4.22)$$

It is assumed that the two-point differentiator stage does not introduce any high frequency noise component. In Equation (4.22), $P$ and $q$ are the a priori probabilities of the signal occurrence. Assigning the a priori probabilities $p$ and $q$, however, is very difficult, because there is a variety of QRS complex morphologies due to the presence of artifacts and interferences. Generally, a
Figure 4.1(a) Correlation receiver model of an integrator dump detector
Figure 4.1(b) Correlation receiver model of an integrator-differentiator - dump detector
variable threshold is added to a minimum fixed threshold. The variable threshold is numerically calculated in relation to the type of arrhythmia such as Bradycardia, Tachycardia and Asystole etc. Therefore, these thresholds can be changed to agree with an individual physician’s analysis of ECG in general, or even as required for a specific patient (Willis Tompkins et al, 1981). If the variable threshold is denoted as $V_T$, then the decision rule can be expressed as

$$\frac{d}{dt} \int_0^t s(t) x(t) \, dt \left[ \frac{H_1}{H_0} \right] V_T + \frac{1}{2} \frac{d}{dt} \int_0^t s^2(t) \, dt$$

(4.23)

For the topological model of an ECG as shown in Figure 3.8 of Chapter 3, the QRS complex can be characterized by

$$x(t) = 1 - \frac{|t|}{T}, \quad |t| < t_1 \text{secs}, \quad T = t_1 \text{secs}$$

(4.24)

where $t_1$ is the duration of the QRS complex. The decision rule for the topological model of the ECG can be written as

$$\frac{d}{dt} \int_0^t (1 - \frac{|t|}{T}) (1 - \frac{|t|}{T}) dt \left[ \frac{H_1}{H_0} \right] V_T + \frac{1}{2} \frac{d}{dt} \int_0^t (1 - \frac{|t|}{T})^2 \, dt$$

(4.25)

The $Z$-transform of the UEWDF resonant lowpass differentiator characterized by Equation (4.20) can be shown to be

$$Y(z) = \frac{(1 - z^{-1})}{1 \pm \sum_{k=1}^{N} A_k z^{-k}}$$

(4.26)
Equation (4.26) can be expressed as

\[
Y(z) = \left[ \frac{1}{1 \pm \sum_{k=1}^{N} A_k z^{-k}} \right] (1-z^{-1})
\] (4.27)

The first term on the RHS is a lowpass Legendre function, and the second term is a two point difference function. Therefore, one can model the UEWDF matched resonant lowpass differentiator as a cascade of a lowpass filter (Integrator) and a differentiator. The digital equivalent of correlation receiver corresponding to Figure 4.1(b) can be modelled as shown in Figures 4.2(a) and 4.2(b). The receiver equation can be formulated as

\[
y(n) = \sum_{n=0}^{N-1} x(n) + \begin{bmatrix} \sum_{n=0}^{N-1} x(n) - \sum_{n=0}^{N-2} x(n-1) \end{bmatrix}
\] (4.28)

\[
y(n) = \sum_{n=0}^{N-1} x(n) + \begin{bmatrix} \sum_{n=0}^{N-1} x(n) - \sum_{n=0}^{N-1} x(n-1) - x(n) \end{bmatrix}
\] (4.29)

Equation (4.29) can be simplified as

\[
y(n) = 2 \sum_{n=0}^{N-1} x(n) - \sum_{n=0}^{N-1} x(n-1) + x(n)
\] (4.30)

Taking Z-transform and simplifying, Equation (4.30) becomes

\[
Y(z) = \sum_{n=0}^{N-1} X(z) \left[ 1 \cdot z^{-1} \right]
\] (4.31)
Figure 4.2(a) Correlation receiver model of a digital integrator - differentiator - dump detector
Figure 4.2(b) Implementation of correlation receiver model of digital integrator-differentiator - dump detector.
It can be concluded, therefore, that the proposed digital lowpass differentiator corresponding to Equation (4.27) is, indeed, an equivalent of correlation receiver. In accordance with decision rule (4.25), and discrete model given by Equation (4.28), the decision rule for the digital lowpass differentiator can be expressed as

$$
N-1 \sum_{n=0}^{N-1} x(n) s(n) + \left[ N-1 \sum_{n=0}^{N-1} x(n) s(n) - N-1 \sum_{n=0}^{N-1} x(n-1) s(n-1) \right]
$$

(4.32)

where

$$D_T = V_T + 1/2 \left[ N-1 \sum_{n=0}^{N-1} x(n) s(n) + 1/2 \sum_{n=0}^{N-1} x(n) s(n) - N-2 \sum_{n=0}^{N-2} x(n-1) s(n-1) \right]
$$

(4.33)

4.5 NUMERICAL STABILITY

It has been shown that WDFs derived from passive reference filters are stable under linear, finite arithmetic, and looped conditions (Alfred Fettweis, 1983). The possibility of numerical stability of UEWDFs have not been considered in the literature on WDFs. As in the case of conventional digital filters, WDFs are finally implemented as difference equations. In principle, the solution of difference equation characterizing a WDF should be identically equal to the solution of differential equation characterizing the reference analog filter. In addition, a numerical solution of the differential equation should not differ from the difference solution. Let the analytical solution of the differential equation
characterizing a passive reference filter be \( y(t_k) \), the difference solution of corresponding WDF be \( y_k \), and the numerical solution of the differential equation be \( y_k^1 \). The three different solutions can be related by the equation (Jain et al, 1985)

\[
| y(t_k) - y_k^1 | \leq | y(t_k) - y_k | + | y_k - y_k^1 | \quad (4.34)
\]

The value \( | y(t_k) - y_k | \) is the truncation error, and the value \( | y_k - y_k^1 | \) is the numerical error. If the numerical error does not remain bounded as \( k \to \infty \) with a fixed step size \( 'h' \), the solution is not P-stable. Numerical instability is indirectly present in the design of UEWDFs. This is investigated in this section.

The transfer function of a UEWDF is of the form

\[
H(z) = \frac{B}{1 \pm \sum_{k=1}^{N} A_k z^{-k}} \quad (4.35)
\]

It has been observed by the author that alternate coefficients (either odd or even) of the denominator polynomial of Equation (4.35) become zero and, hence, Equation (4.35), assuming that odd coefficients become zero, can be express as

\[
H(z) = \frac{B}{1 - A_2 z^{-2} + A_4 z^{-4} + \ldots + A_{2n} z^{-2n}} \quad (4.36)
\]

Furthermore, the denominator polynomial consists of coefficients with alternating positive and negative coefficients. Therefore, transfer function (4.35) can be expressed as

\[
H(z) = \frac{B}{\sum_{i=1}^{N} \pi \left( 1 \pm C_i z^{-2} \right)} \quad (4.37)
\]
It has been established in Chapter 2 that Equation (4.37) corresponds to a differential equation of the form

\[ a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + \lambda_E y(t) = 0 \]  

(4.38)

A second order approximation of the 'n' the order Equation (4.38) in terms of the dominant roots can be expressed as

\[ \frac{d^2 y(t)}{dt^2} + \lambda_E y(t) = 0, \lambda_E = \omega_n^2 \]  

(4.39)

The solution of Equation (4.39) is P-Stable, depending on the type of the coefficients and the type of the numerical method employed; for example, applying fourth order Runge-Kutta method to Equation (4.39), it can be shown that this method is P-Stable if \( h^2 \lambda_E \leq 8 \). For higher order systems, the problem could become more serious, and P-Stability is not always guaranteed. From Equations (4.35) --- (4.39), it is obvious the UEWDF resonator is stable if

\[ A_2 < 1, A_4 < A_2, A_6 < A_4, \ldots, A_{2N} < A_{2N-2} \]  

(4.40)

Stability condition (4.40) includes P-Stability. If this condition is not satisfied, the coefficients need to be adjusted according to some criterion. Since a bandlimited lossless system has a solution of the form

\[ h_d(k) = p \frac{\sin q(k+1)}{q(k+1)}, 0 \leq k \leq N \]  

(4.41)

the impulse response coefficients \( h(k) \) of the UEWDF resonator can be compared with impulse response coefficients \( h_d(k) \), and coefficients \( h(k) \) that violate the stability conditions can be replaced by corresponding coefficients \( h_d(k) \). Such adjustments of impulse response coefficients to ensure P-Stability, however, cause
distortion in the frequency response of the UEWDF. Presence of P-Instability in the design of UEWDFs is demonstrated in section 4.6.

4.6 RESULTS AND DISCUSSION

A homemorphic spectrum $x(\lambda)$ for the topological model of the QRS complex, as characterized by Equation (4.24), is generated in accordance with Equations (4.5) --- (4.9). The spectrum is of the form

$$x(\lambda) = \kappa \sin^2(\lambda T/2) / \lambda^2 T$$

(4.42)

and corresponds to the Equation

$$f(0 \ldots \infty) = \frac{\sin m(0 \ldots \infty)}{m(0 \ldots \infty)} = f(x_1)$$

with a finite zero at $\Omega_c$ rads. An 'n' th order Legendre Polynomial $L_n(x^2)$ is chosen in accordance with Equations (4.11) and (4.12) to provide a cut off frequency of $f_c$ HZ so that

$$|S_{21}(\lambda)| \leq \frac{1}{1 + L_n(\lambda^2)}$$

(4.43)

Equation (4.43) is synthesized as a cascade of 'n' unit elements (see Appendix 1) which yields recursive Equation (4.15). The UEWDF matched resonant lowpass differentiator algorithm is obtained by substituting the coefficients of Equation (4.15) in Equation (4.20). The results for a three cycle single Lead ECG test data are shown in Figures 4.3-4.12 and Table 4.1. The design specifications are : QRS complex duration $T = 0.02$ Secs, cut off frequency $f_c = 40$ Hz, attenuation at stopband $X_{dB} = 40$ dB at 60 Hz, response type to be Legendre, $\Omega_c = 100$ rads, $k<1$, sampling rate = 12 fc. Figure 4.3 shows the impulse response of matched digital resonator corresponding to Equation (4.15). Figure 4.4 is the magnitude
Table 4.1 Data describing the design of Legendre UEWDF resonant digital lowpass differentiator

<table>
<thead>
<tr>
<th>1. Specifications</th>
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<tbody>
<tr>
<td>Cut off frequency</td>
</tr>
<tr>
<td>Attenuation at stopband X dB</td>
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<tr>
<td>Response type</td>
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<td>Sampling rate</td>
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<tr>
<th>2. Order = 7</th>
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<tbody>
<tr>
<td>[ S_{11}(z) = \frac{1123.4589z^7 + 2869.4713z^6 + 3118.0381z^5 + 1876.8054z^4 + 680.1083z^3 + 149.5660z^2 + 18.6213z + 1.005}{1123.4589z^7 + 2477.811z^6 + 2203.4287z^5 + 1017.7077z^4 + 261.4962z^3 + 36.0312z^2 + 2.0096z + 0.01} ]</td>
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<tr>
<th>3. Characteristic Impedances</th>
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<tbody>
<tr>
<td>( r_0 = 1, r_1 = 6.25, r_2 = 0.353, r_3 = 1.54, r_4 = 0.596, )</td>
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<tr>
<td>( r_5 = 0.587, r_6 = 1.52, r_7 = 0.99 )</td>
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<tr>
<th>4. Adaptor Coefficients</th>
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<tr>
<td>( a_1 = -0.724, a_2 = 0.893, a_3 = -0.627, a_4 = 0.442, a_5 = 0.007, )</td>
</tr>
<tr>
<td>( a_6 = 0.587, a_7 = 0.0106, a_8 = 0.206 )</td>
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<th>5. Transfer Function of the Differentiator</th>
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<tbody>
<tr>
<td>( S_{21}(z) = \frac{K(1 - z^{-1})}{(1 + A_1z^{-1} + \ldots + A_nz^{-n})} )</td>
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<tr>
<td>( K ) can be conveniently chosen</td>
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<tr>
<td>( A_1 = A_3 = A_5 = A_7 = A_9 = A_{11} = A_{13} = 0 )</td>
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<tr>
<td>( A_2 = -0.234, A_4 = 0.243, A_6 = -0.011, A_8 = 0.016, )</td>
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<tr>
<td>( A_{10} = -0.012, A_{12} = 0.025, A_{14} = -0.010 )</td>
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<tr>
<th>6. UEWDF Resonant Lowpass Differentiator</th>
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<tbody>
<tr>
<td>( w(n) = k[x(n) - x(n-1)] + 2 \sum_{k=1}^{N} A_k \eta(n-k) )</td>
</tr>
<tr>
<td>Values of ( A ) are as shown in the previous item</td>
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<tr>
<th>7. Results</th>
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<tbody>
<tr>
<td>Illustrated in Figures 4.3 - 4.12</td>
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</table>
Figure 4.3 Impulse response of the wave digital resonator
Figure 4.4 Magnitude response of the wave digital resonator
response of the matched digital resonator. As can be seen, Equation (4.15) is a high frequency enhanced lowpass filter and, therefore, exhibits differentiation property at the higher end of the passband. Baseline drift is a very low frequency component and, hence, is not removed. This can be observed from Figure 4.5 and 4.6. Figure 4.5 shows a locally generated three cycle ECG test data with baseline drift. The forced response of matched digital resonator corresponding to Equation (4.15) and Figure 4.3 is shown in Figure 4.6. Waveform sharpening due to differentiation of high frequency components that does not affect baseline drift can be noticed. Equation (4.20) is the UEWDF resonant lowpass differentiator algorithm. Presence of the first difference term \( x(n) - x(n-1) \) in equation (4.20) corresponds to differentiation of low frequency components within the passband. This leads to elimination of baseline drift as illustrated in Figure 4.8. The magnitude and phase response of UEWDF resonant lowpass differentiator corresponding to Equation (4.20) is shown in Figure 4.7. The error behaviour of UEWDF resonant lowpass differentiator is shown in Figures 4.9 and 4.10. The output of UEWDF resonant lowpass differentiator with decision threshold applied is shown in Figure 4.11. Excellent detection of R-waves can be observed. The application of UEWDF resonant filter characterized by Equations (4.15) to (4.20) for ECG processing is summarized in Figure 4.12. The distortion in magnitude response shown in Figure 4.4 is due to coefficient adjustment to ensure P-stability. This is necessitated by the fact that the coefficients of the seventh order Legendre UEWDF resonator described in Table 4.1 violate the stability criterion. The distortion can also be noticed in the magnitude and phase responses of UEWDF resonant lowpass differentiator shown in Figure 4.7. Table 4.2 presents the data describing the design of a typical Chebyshev UEWDF. The coefficients have not been adjusted for P - Stability. The magnitude response, impulse response, and step response are shown in Figures 4.13, 4.14, and 4.15 respectively. The filter has been designed using BIERGE - VIETA method. Figures 4.14 and 4.15 demonstrate the possibility of P-instability in the design of UEWDFs. The noise performance of the proposed differentiator has also been studied:
Figure 4.5: Three cycle ECG test data.
Figure 4.6 Forced response of the wave digital resonator
Figure 4.7 Magnitude and phase characteristics of the unit element wave digital lowpass differentiator
Figure 4.8 Forced response of the unit element wave digital lowpass differentiator
**Figure 4.9** Error behaviour of unit element wave digital lowpass differentiator as compared to a typical lowpass differentiator

- **WDRD**: Amplitude response of UEWD resonant lowpass differentiator (order 7)
- **RD**: Amplitude response of optimal resonant lowpass differentiator
- **ID**: Amplitude response of ideal differentiator
- **SPD**: Amplitude response of seven point third order digital low pass differentiator
- **EWDRD**: Error curve of WDRD
- **ERD**: Error curve of RD
- **ESPD**: Error curve of SPD
Figure 4.10 Error behaviour of unit element wave digital lowpass differentiator as compared to a typical wideband differentiator

WBD Amplitude response of wideband digital differentiator
EWBD Error curve of WBD
Figure 4.11 Output of UEWDF resonant lowpass differentiator with decision threshold
Figure 4.12 ECG processing by UEWD resonant differentiator
### Table 4.2 Data Describing the Design of Fifth Order Chebychev UEWDF

<p>| | |</p>
<table>
<thead>
<tr>
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<th></th>
</tr>
</thead>
</table>
| **1.** Specification | Ripple factor in pass band = 0.41dB  
Cut of frequency = 50Hz  
Attenuation at 150Hz (stop band) not less than 40 dB  
Low pass filter with chebychev performance. |
| **2.** $S_{11}(z)$ | \[
\begin{align*}
114.5z^5 + 44.27z^3 + 3.16z \\
114.5z^5 + 83.21z^4 + 74.48z^3 + 28.89z^2 + 8.53z + 1
\end{align*}
\] |
| **3.** Characteristic Impedances | $r_0 = 1$  
r$^1 = 3.18$  
r$^2 = 0.443$  
r$^3 = 4.37$  
r$^4 = 0.443$  
r$^5 = 3.18$  
r$^6 = 1$ |
| **4.** Adaptor Co-efficients | $\alpha_1 = -0.521$  
$\alpha_2 = 0.755$  
$\alpha_3 = -0.8186$  
$\alpha_4 = 0.8186$  
$\alpha_5 = -0.755$  
$\alpha_6 = 0.521$ |
| **5.** Transfer function in Z domain | $H(z) = \frac{9.676z^5}{-0.217z^4 + 1.304z^3 - 2.887z^2 + 3.64z + 2.693}$ |
| **6.** Impulse response | $y(n) = 9.676 \left[ (1.101768 - 1.0946251)^n (-24.824^o + n \times 32.588^o) \
+ 1.101768 \times (1.0946251)^n \cos(24.824^o - n \times 32.588^o) \right]$ |
| **7.** Step Response | $y(n) = 0.500086 \left[ (1.094621)^n \cos(85.893^o + n \times 32.588^o) + 0.044663(1.094621)^n \cos(70.36^o + n \times 32.588^o) + 0.044663(1.094621)^n \cos(70.36^o + n \times 32.588^o) + 0.7817749 \times (-1)^n \right]$ |
Figure 4.13 Magnitude response of fifth order Chebyshev UEWD lowpass filter

Cutoff frequency = 49.4 Hz
Attenuation at 150 Hz = -41.35 dB
Figure 4.14 Impulse response of the UEWD chebyshev filter with P-instability.
Figure 4.15 Step response of the UEWD Chebyshev filter with P-instability
The total signal power at the input $= (2.87 \text{mv})^2$
The total noise power $= 1.02 \times 10^{-19}$
The total power at the output $= (0.970 \text{mv})^2$

\[
\frac{(S/N) \text{ input}}{} = \frac{(2.87 \times 10^{-3})^2}{(1.02 \times 10^{-19})} = 139.15 \text{ dB}
\]

\[
\frac{(S/N) \text{ output}}{} = \frac{(0.970 \times 10^{-3})^2}{(1.02 \times 10^{-19})} = 129.73 \text{ dB}
\]

In certain applications, such as ECG processing, there is a need for differentiators that can eliminate very low frequency interferences, at the same time providing reasonable differentiation of high frequency signal components. Wideband differentiators are not generally used on account of their complexity in terms of order and implementation. Lowpass differentiators that are employed for ECG processing do not provide accurate differentiation of high frequency components, as evidenced by their error behaviour shown in Figure 4.9. Furthermore, lowpass differentiators are subject to finite wordlength errors. In this Chapter, a wave digital optimal resonant differentiator has been proposed, which consists of a wave digital resonant lowpass section, and a two point difference function connected in cascade. The WDF differentiator possesses excellent sensitivity properties with respect to finite wordlength errors. The improvement in error behaviour compared to a typical lowpass differentiator has also been demonstrated. UEWDF algorithms have the added advantages of minimal number of multipliers (Karl Renner et al, 1974), short coefficient wordlengths (Karl Renner et al, 1973), and a particularly simple form of hardware (Stuart Lawson, 1981). The proposed digital differentiator algorithm can be an attractive choice for real time ECG processing where errors due to finite storage limitation cannot be ignored.

A comparison of wave digital resonant lowpass differentiator with wideband and lowpass digital differentiators is presented in Table 4.3.
### Table 4.3: Comparison of UEWDF resonant lowpass digital differentiator with wideband and lowpass digital differentiators

<table>
<thead>
<tr>
<th>Sl. No.</th>
<th>Type</th>
<th>Order</th>
<th>Optimality</th>
<th>Accuracy</th>
<th>Bandwidth</th>
<th>Noise Performance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Type</td>
<td>FIR, IIR</td>
<td>Low order Typically ≤ 10</td>
<td>Better than lowpass type, less accurate compared to wideband type</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>Order</td>
<td>Generally very high Typically 15</td>
<td>Low order Typically ≤ 10</td>
<td>Low order Typically ≤ 10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>Optimality</td>
<td>Standard optimality criteria in frequency domain</td>
<td>Standard optimality criteria in frequency domain</td>
<td>Maximum S/N ratio</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>Accuracy</td>
<td>Very high No definite value Depends on the order and type</td>
<td>Poor No definite value Depends on the order</td>
<td>Better than lowpass type, less accurate compared to wideband type</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td>Bandwidth</td>
<td>Full band $0 \leq \omega T \leq \pi/2$</td>
<td>Partial band $0 \leq \omega T \leq \omega \pi/2 (a&lt;1)$</td>
<td>Partial band in two regions over $0 \leq \omega T \leq \omega \pi/2, a&lt;1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6.</td>
<td>Noise Performance</td>
<td>No definite value available</td>
<td>No definite value available but generally $(S/N)<em>{out} &lt; (S/N)</em>{in}$</td>
<td>$(S/N)<em>{in} = 139.15$ dB $(S/N)</em>{out} = 129.73$ dB</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>