3.1 INTRODUCTION

A less widely known lossless structure, which in some respect is the simplest one, is the chain connection of unit elements usually referred to as Richards' structure (Richards, 1948) (Fraiture et al, 1969) (Matsumoto, 1970), (Karl Renner et al, 1973), (Karl Renner et al, 1974), (Thiran, 1977), (Hyder Ali et al, 1978). The digital equivalent of Richards' reactance structure is known as unit element wave digital filter (UEWDF). The author has observed that certain properties of Richards' structure can make UEWDF extremely useful in low frequency signal detection applications. Besides, it is possible to design digital resonators satisfying prescribed frequency domain specifications from Richards' reactance structures. The need for such a design, however, is dictated by the nature of the application in mind. Generally, one requires a mathematical model that incorporates not only the system features but also specific features of the signals to be processed. This chapter proposes an abstract mathematical model for linear optimum digital processing of electrocardiograms by unit element wave digital resonators. A number of abstract concepts, namely, structurewise equivalent sets, tan like numerical transforms, selectivity metric and topological twisting of well behaved functions in time and frequency domains are introduced. Functions simulating ECG, white noise, arbitrary distortions of ECG, and transfer characteristic of an optimal detector are defined in such a manner that these functions possess topological, metrical, and algebraic properties. This is achieved by selecting several well behaved sinusoidal and rational functions bounded in domain and range and assigning them to a topological space formed by the
intersection of a metric, Hausdorff, and Banach spaces. A spectral amplitude matched wave digital filter for optimum signal detection is defined by making use of Tychonoff’s theorem from general topology (George Simmons, 1963). The various advantages of the proposed model are discussed. This is followed by a separate chapter demonstrating practical implementation and uses of the abstract model.

3.2 AN OPTIMAL WAVE DIGITAL FILTER FOR ELECTROCARDIOGRAM PROCESSING

Wave digital filters represent a class of particular interest to filter design engineers (Alfred Fettweis, 1986). One application area where wave digital filters have not received considerable attention is biomedical signal processing. Several reasons can be quoted for this. It is known that the fundamental requirement in biosignal processing is the detection of a signal of known waveform embedded in noise. In the development of a suitable algorithm, one is faced with a number of problems; firstly, a mathematical representation of specific features of signals either in time domain or frequency domain is to be obtained a priori; secondly, the algorithm should satisfy some optimality criterion, since signal is normally embedded in noise; thirdly, the algorithm must be causal, stable and physically realizable; finally, the algorithm must perform best in finite wordlength environment (Jerome Cox et al, 1972). It is quite obvious that, though wave digital filters are inherently stable, causal, and least sensitive to finite wordlength effects, formulation of wave digital filter problems for optimal detection of signals possessing specific a priori features has not been possible, since wave digital filters are translations of reference passive structures (Alfred Fettweis, 1983). Naturally, there is a doubt as to the existence of reference structures that can be related in some way to a priori features of signals to be detected; furthermore, there is no way a wave quantity (Herbert Carlin, 1971) characterizing a Richards’ reference filter can be defined possessing features chosen a priori and capable of extracting certain a priori information of signals to be processed. Such a wave quantity, even if we are able to define one, must finally satisfy physical realizability conditions.
(Van Valken Burg, 1984). Presently available wave digital filter design methods do not show any evidence of dealing with a situation of this type. This chapter presents a model that enables a designer to overcome these particular difficulties, and succeed in developing a class of wave digital filters for possible applications as optimum linear ECG processors for best performance in finite arithmetic conditions.

3.3 STRUCTURE-WISE EQUIVALENT SETS AND TAN LIKE NUMERICAL TRANSFORMS

In this section, the concepts of structure-wise equivalent sets and tanlike numerical transform are introduced.

Let \( \tan x \triangleq \frac{1}{n} \triangleq f(n) \) \hspace{1cm} (3.1)

where \( n \in [0,1] \)
\( x \in [0,1] \)

and \([0,1]\) is a closed interval on the real line. It is further assumed that \( \forall x \exists n \) such that \( f(n) = K \) where \( K \) is some number.

Therefore,

\( \tan x \triangleq f(n) \triangleq \{1/n : n \in [0,1]\} \) \hspace{1cm} (3.2)

Definition (3.2) suggests that \( \tan x \) can be replaced by another set of numbers \( x_1 \)

where

\( x_1 \triangleq 1/[0,1] \triangleq \{1/n : n \in [0,1]\} \) \hspace{1cm} (3.3)

Definition (3.3) is an abstract representation of the idea that \( \tan x = x_1 \) can be considered to be a set which contains elements of the form unity divided into infinite parts so that
\[ \tan x = x_1 = \{0, \ldots, \infty\}, \ x \in [0, 1] \]  

(3.4)

Let \( x \in [0, 1] \) and \( x_1 \in [0, \infty] \)

be called structure-wise equivalent sets but not value-wise. This is indeed so, because \( x_1 \) contains infinite number of divisions of a finite interval \( x \in [0, 1] \).

Consider the quantity \( \pi x/2 \) where \( x \in [0, 1] \). As \( x \) takes values between 0 and 1, \( \pi x/2 \) takes values between 0 and \( \pi/2 \) rad. Therefore, one can define that an angular distance \( 0 \leq \Theta \leq \pi/2 \) is numerically equivalent to a linear distance \( 0 \leq x \leq 1 \).

Let \( \tan x \), \( \tan \pi x/2 \), and \( x \) be defined a structure-wise equivalent sets and a function of \( x \) be denoted as \( f[0, 1] \), and a function of \( x_1 \) as \( f[0, \infty] \) where \( x_1 = \tan x \triangleq \tan \pi x/2 \). Here, a topological equivalence is meant, and 'tan' is an operator and is called tan like numerical transformer. As an example, consider the function

\[
\sin \frac{m}{m x} \quad x \in [0, 1] \]

\[
f[0, 1] \triangleq \frac{\sin m[0, 1]}{m[0, 1]} \triangleq f(x) \]  

(3.5)

\[
f[0, \infty] \triangleq \frac{\sin m[0, \infty]}{m[0, \infty]} \triangleq f(x_1) \]  

(3.6)

Where \( x_1 = \tan \pi x/2 \)

\[
f(x_1) \triangleq \frac{\sin m x_1}{m x_1} \]  

(3.7)

\( f(x) \) and \( f(x_1) \) are topologically equivalent functions.
3.4 REAL LINE TO REAL LINE NUMERICAL TRANSFORMS

Let $R \rightarrow R$ denote a transformation of a set of real numbers to another set of real numbers. A variable over $R$ can be expressed as $X \epsilon [R]$ and a numerical homeomorphism $f(n)$ can be defined as $f: R \rightarrow R$ which transforms $x$ to $x_1$ as

$$x^1 = f(n)x \quad x_1 \epsilon [R]$$

(3.8)

In the definition $x_1 = \tan \frac{\pi x}{2}$, the transform is one to one, one-one onto and continuous; the inverse function $f^{-1}(n)$ is also continuous. Hence $x_1$ is homeomorphic to $x$ (Seymour Lipschutz, 1981).

In accordance with equations (3.5), (3.6) and (3.7), two functions $f(x)$, $f(x_1)$ are $f[0,1]$, $f[0,\infty]$ respectively, and are topologically equivalent. In the same manner, one can show that the transforms

$$x_2 = \sin^2 \frac{\pi x}{2} \quad x \epsilon [0,1]$$

(3.9)

$$x_3 = \frac{\tan^2 \frac{\pi x}{2}}{1 + \tan^2 \frac{\pi x}{2}} \quad x \epsilon [0,1]$$

(3.10)

where $x_3 = \sin^2 \frac{\pi x}{2} \Delta x_2$

are homeomorphic transforms of $x$. The sets $x_1$, $x_2$ and $x_3$ have same cardinality and satisfy order and completeness axioms. In otherwords, $x$, $x_1$, $x_2$ and $x_3$ all have the usual topology of the real line and are $R \rightarrow R$ transforms which preserve topological properties of functions defined over $R$ under appropriate conditions.
Therefore, one can show that

\[
\begin{align*}
    f(x) &\quad x \in [0,1] \\
    f(x_1) &\quad x_1 \in [0,\infty] \\
    f(x_2) &\quad x_2 \in [0,1] \\
    f(x_3) &\quad x_3 \in [0,\infty] 
\end{align*}
\]

are equivalent functions in topological spaces.

### 3.5 A PHYSICAL REALIZABILITY THEOREM

The abstract numerically equivalent functions over real line just introduced have very specific objectives. In the first place, band limited signals have spectrums defined over finite range of frequencies; for example, a normalized lowpass signal function is defined over an interval \([0,1]\). In the second place, physically realizable system functions of lowpass type have zeroes at infinity and, hence, are defined over an interval \([0,\infty]\). How are we to approximate a signal function \(f[0,1]\) by a system function \(f[0,\infty]\) over a finite interval \([0,1]\)? In the preceding section, a homeomorphic transform of \(f[0,1]\) to \(f[0,1]\) and to \(f[0,\infty]\) was introduced so that system function and signal function could be defined over the same interval, at the same time preserving the topological properties under such transforms. Indeed, physical realizability requires such transforms as expressed by the following theorem:

**Theorem:** A function \(f(x)\) defined over the real axis \(x \in \mathbb{R}\) is physically realizable, only if \(x \in [0,\infty]\) is an infinite set bounded below.

**Proof:** Let \(x \in [0,a]\) where \(a\) is a finite least upper bound and \(f(x)\) is a function whose domain is \(x\). Physically realizable functions are of the form \(n(x)/d(x)\) where \(n(x)\) and \(d(x)\) are numerator and denominator polynomials of rational function \(f(x)\). Firstly, the domain of rational functions of the form \(n(x)/d(x)\) is the entire real axis. Secondly, elements of physical networks can be extracted from \(f(x)\) at the poles \(x = \infty\). Therefore, it is required that \(f(x)\) be defined over the entire real
axis. Later this theorem will be used for synthesizing a reference network whose transfer characteristic is matched in some optimum manner to spectrums of a class of bandlimited signals.

3.6 RESONANCE TYPE FUNCTIONS AND SELECTIVITY METRIC

Consider a class of continuous functions $C[0,1]$ defined on the closed interval $[0,1]$. A metric on $C[0,1]$ can be defined as follows:

$$d(f,g) = \text{Sup} \{ | f(x) | - | g(x) | : x \in [0,1] \} \tag{3.15}$$

where $d(f,g)$ represents the greatest vertical gap between two functions $f$ and $g$. If $X$ is a set containing functions $f \in [0,1]$, the class of open spheres in set $X$ with the metric $d$ is a base for a topology on $X$. The topology generated by $d$ is called metric topology, and $(X,T)$ is a metric topological space. The metric space is also a normal space with the norm defined by

$$| f | = \{ \text{Sup} | f(x) | : x \in [0,1] \} \tag{3.16}$$

The space $C[0,1]$ of all continuous functions from $I = [0,1]$ into $R$ with the above norm is a complete normed vector space. The supremum norm denotes a maximum attainable value of a function $f(x)$, and this occurs in relation to one of the elements of the set $[0,1]$. Let a function $f(x) \in C[0,1]$ be called resonant type function resonating at $K$ where $K$ is contained in $x \in [0,1]$. The metric supremum may now be called selectivity metric.

3.7 APPROXIMATION IN TOPOLOGICAL SPACES

Consider the function $f(x) \epsilon (x,T)$ and the homeomorphic transform $x_1 = \tan \pi x/2$. Under this transform, we have

$$f(x_1) \epsilon (x_1, T) \tag{3.17}$$
Consider a function of the form

\[ f(x) = \frac{n(x)}{d(x)} = \frac{1}{1 + L(x^2)} \quad x \in [0, \infty] \]  

(3.19)

Under the transform (3.9), (3.19) becomes

\[ f(x^2) = \frac{1}{1 + L(x^2)} \quad x^2 \in [0, 1] \]  

(3.20)

If \( f(x^2) \in (x^2, T) \) where \( (x^2, T) \) is a compact Hausdorff space, than \( f(x^2) \) possesses a supremum. Such functions in compact Hausdorff space can be chosen as approximation functions for signal spectrums represented in the same compact Hausdorff space. The transforms (3.4), (3.9) and (3.10) translate \( f(x_1) \) of signal spectrum, and \( f(x_2) \) of approximating function \( f(x_2) \), with selectivity metric preserved. If \( f(x_3) \) is of the form (3.19), it satisfies the physical realizability theorem and can be synthesized as a wave digital filter. Further, \( f(x_3) \) exhibits selectivity character since supremum norm is preserved under topological transformations. In this manner, a selective realizable function can be generated in topological spaces for defining an optimum detection problem.

3.8 A SYNTHESIS PHILOSOPHY

Let \( (X, T) \) be a compact Hausdorff space with a supremum metric \( d(f, g) \). Let \( S(x) \) be a function \( \epsilon(X, T) \) where \( x \in [0, 1] \) or \( x \in [a, b] \) possessing the usual topologies \( \epsilon \). A function \( S(x_1) \) isometric and homeomorphic to \( S(x) \) can be generated under the transform \( x_1 = \tan^\pi x / 2 \). Let \( f(x) \) be a function of the form (3.19). Under transform (3.9), \( f(x) \) transforms to \( f(x_2) \) and is isometric and homeomorphic to \( f(x) \). Furthermore, \( f(x_2) \in (X, T) \). Let \( S(x) \) be subjected to transform \( X_{x^*} = \tan^\pi x / 2 \) and \( f(x_2) \) to transform (3.10) at one and the same time. The resulting functions are \( S(x_1) \) and \( f(x_3) \) in compact Hausdorff space possessing
supremum metric. An optimum detection of $S(x_1)$ by $f(x_3)$ can now be formulated. Supremum norm ensures that a resonant band of $f(x_3)$ may overlap a resonant band of $S(x_1)$. If $f(x_2)$ is a transfer characteristic, then $f(x_3)$ can be synthesized as a reference optimum network. This optimum network will resonate to a band of spectral components of $S(x_1)$.

3.9 MODELLING OF ECG WAVEFORM IN TOPOLOGICAL SPACES

It is well known in general topology that geometrical shapes can be stretched, twisted, pulled and bent according to certain rules in such a way that resulting shapes are topologically equivalent to original shape. Certain metrical properties of geometrical shapes are invariant under such operations. Consider the triangle shown in Figures 3.1, 3.2, 3.3. It is possible to define a metric $m(f,g)$ such that it remains invariant under twisting of the triangle into different shapes. The same property can be attributed to the spectrum of a triangle as illustrated in Figures 3.4 and 3.5. The QRS complex of an ECG can be looked upon as a triangle or a triangle twisted in an appropriate manner. Diseases associated with the contour of QRS complex can be modelled as topologically distorted triangles. Triangle is a well behaved function over an interval and can be taken as a basic model of QRS complex. Appropriate topological twisting of the spectrum of QRS complex would ensure simulations of a specified number of heart diseases associated with QRS complex.

3.10 A MODEL FOR LINEAR OPTIMUM FILTERING OF ECG SIGNALS BY WAVE DIGITAL FILTERS

A linear, time invariant, causal, and stable digital filter for optimal detection of ECG signals is described in this last section. The formulation of the optimum detection in Hausdorff space has been made possible by the well known Tychonoff's theorem (George Simmons, 1963). This enables us to write the causal equation in topological space as
Figure 3.1  A triangle in time domain

Figure 3.2  A metric for the triangle

Figure 3.3  Topological twisting of a triangle in time domain
Figure 3.4  The triangle in frequency domain

Figure 3.5  Topological twisting of the triangle in frequency domain
One can rewrite equation (3.21) as

\[
S_{\text{out}} (x_1) \in (X_1, T) = S_{\text{in}} (x_1) \in (X_1, T) f_{\text{system}} (x_3) \in (X_3, T)
\]  

(3.22)

\[
f_{\text{system}} (x_3, T) = \frac{S_{\text{out}} (x_1) \in (X_1, T)}{S_{\text{in}} (x_1) \in (X_1, T)}
\]  

(3.23)

For maximum signal to noise ratio performance, one must choose a priori a system function such that \( f^{\ast} = S^{\ast} \) where the asterisk denotes complex conjugate. In the simplest case, this equation defines a spectral amplitude matching since \( S(x) \) and \( f(x) \) are real functions of real variable \( x \). Optimum filtering in compact Hausdorff space is illustrated in Figures 3.6 and 3.7. In Figure 3.7, \( f, g \) and \( h \) are bounded continuous (discrete) functions defined on compact Hausdorff space. This space is a metric space with supremum norm and pointwise addition and multiplication can be carried out on this space. The output of optimum filter is also a compact Hausdorff space and is shown in Figure 3.6.

3.11 ADVANTAGES OF PROPOSED DESIGN PHILOSOPHY

1. Models of signals and systems can be generated on computer in terms of well behaved function.

2. Processor characteristics can be chosen a priori in correspondence with specific features of input signals.
Figure 3.6 Filtering in compact Hausdorff space
Figure 3.7 Model for linear optimum detection of ECG signal by wave digital filter in compact Hausdorff space
3. A number of readily available rational functions and spectrums can be chosen.

4. Desired features of signals can be defined a priori in terms of metrics in Hausdorff space, and this remain invariant at the output, thereby facilitating a better interpretation of signal in the background of noise.

5. Signals can be multiplied (amplified), differentiated, and linearly combined without affecting the metrical properties.

6. The model can be extended to complex topological spaces, n-dimensional spaces, and time domain designs.

3.12 RESULTS AND DISCUSSION

The topological model of an ECG has been generated by graphically combining two discrete raised cosine functions and a discrete triangular pulse as shown in Figure 3.8. The spectrum of the function can clearly be assigned to a metric space with supremum norm and pointwise addition of a slur on the contour does not affect the metrical and topological properties of the spectrum. The spectrum is shown in Figure 3.9. A wave digital matched filter has been designed whose transfer function is shown in Figure 3.10. The spectrum of the output of the filter is shown in Figure 3.11. The application of Tychonoff's theorem is illustrated in Figure 3.12. Figures 3.8-3.12 illustrate the concept of approximation in topological space. The design, implementation, and application of a wave digital resonant differentiator are presented in Chapter 4.
Figure 3.8  
An ECG simulated in Topological space
Figure 3.9  Spectrum of the ECG signal in topological space
Figure 3.10 Transfer function of a UEWDF matched lowpass filter in topological space
Figure 3.11   Spectrum of the output signal of UEWD matched filter in topological space
Tychonoff's theorem: Product of two compact Hausdorff spaces is an Hausdorff space

\[
\mathbb{S} = \mathbb{S} \times \mathbb{S}
\]

\[
\mathbb{S} = \tan \frac{\pi x}{2}
\]

Figure 3.12 Approximation in topological space using Tyhonoff's theorem