CHAPTER 2

AN APPROACH TO THE DESIGN OF RESONANT LOWPASS DIGITAL DIFFERENTIATORS

2.1 INTRODUCTION

Reactance networks or lossless networks play a fundamental role in passive network synthesis. Evidently, their most important property is losslessness. Losslessness has two important consequences: Lossless networks exhibit excellent resonance effect and they have excellent sensitivity properties. There has been considerable interest in developing digital filter equivalent of lossless networks and resonators (Bruton et al., 1976), (Deprettere et al., 1980), (Davis, 1982) (PP Vaidyanathan, 1985), (Alfred Fettweis, 1986a) (Gabor Peceli, 1988), (Erfani et al., 1991), (Larsen et al., 1991), (Janos Sztipanovits, 1992), (Mohammed Ismail et al., 1992). Investigation of the properties of Z domain reactance functions continues to be an interesting research problem. Recently, a methodology for synthesis of discrete-time reactance functions similar to the one known for the synthesis of continuous - time reactance functions in S-domain has been proposed by Mohammed Ismail et al (1992). However, synthesis of Z-domain lossless transfer functions satisfying prescribed time domain specifications has not yet been reported. A novel approach to design a digital resonator satisfying prescribed time domain specifications is proposed in this chapter. Resonance is treated as a characteristic value problem governed by Sturm-Liouville type equation (George Simmons, 1972). Consequently, a resonator is modelled as a dynamic system whose solution corresponds to the solution of a suitably formed Sturm-Liouville type equation. A corresponding digital resonator is obtained by translating the Sturm-Liouville type differential equation into a difference equation. The transfer function corresponding to the difference equation is obtained, and cascaded with
a two point difference. It is shown that the digital resonator - two point difference combination behaves as a resonant lowpass differentiator. The performance, advantages, and use of the resonant digital lowpass differentiator are described.

2.2 MODELS FOR LOSSLESS DIFFERENTIATORS

In the literature, the magnitude response of analog differentiators are commonly characterized by the equation

\[ | H(j\omega) | = \omega, \quad 0 \leq \omega \leq \infty \] (2.1)

There is a basic difference between analog differentiators and digital differentiators. Digital differentiator is a bandlimited system and, therefore, it's frequency response characteristic is defined between two finite values of frequency \( \omega \) i.e. \( 0 \leq \omega \leq \omega_p \) where \( \omega_p \) is a finite upper bound. Consequently, the magnitude characteristic of a digital differentiator can be characterized by

\[ | H(e^{j\omega T}) | = \tan(\omega T), \quad 0 \leq \omega T \leq \omega_p T \] (2.2)

where \( T \) is the sampling interval and

\[ | H(e^{j\omega T}) | = 0 \quad \text{for} \quad \omega T = 0 \]

\[ | H(e^{j\omega T}) | = \infty \quad \text{for} \quad \omega T = \pi/2 \]

Equations (2.1) and (2.2) are illustrated in Figures 2.1 and 2.2 respectively. Since \( | H(e^{j\omega T}) | \) can never be negative in physical situations, Figure 2.2 is modified as shown in Figure 2.3. An important specification in digital differentiation is differentiation band. Usually differentiation band is defined as the frequency range \( 0 \leq \omega T \leq \pi/2 \). Over this differentiation band, generally no distinction is made between analog and digital differentiation and, hence, one can characterize a digital differentiator magnitude response by
Figure 2.1 Magnitude characteristic of an ideal analog differentiator
Figure 2.2  Magnitude characteristic of an ideal digital differentiator
Consider the resonant system shown in Figure 2.4. The system equations are:

\[
H(s) = \frac{S^2}{S^2 + 1/LC} \quad (2.4)
\]

\[
\frac{d^2 x(t)}{dt^2} + \frac{1}{LC} x(t) = \frac{d^2 u(t)}{dt^2} \quad (2.5)
\]

As can be observed, Equation (2.5) is a characteristic value problem whose magnitude response is shown in Figure 2.5. Consider a lossless system shown in Figure 2.6. The equations of this system are:

\[
H(s) = \frac{1/LC}{S^2 + 1/LC} \quad (2.6)
\]

\[
\frac{d^2 x(t)}{dt^2} + \frac{1}{LC} x(t) = \frac{1}{LC} u(t) \quad (2.7)
\]

The magnitude response of the system given by Equations (2.6) and (2.7) are shown in Figure 2.7. It is obvious from Equations (2.1) - (2.7) and Figures 2.1 - 2.7 that lossless systems generally behave as differentiators over a narrow range of frequencies in the neighbourhood of resonant frequency. Equation (2.6), indeed, represents a lowpass filter with resonance at \( \omega = \omega_p \). It can be inferred from Equations (2.4) - (2.7) and Figures 2.5 - 2.7 that a resonant lowpass filter acts as a differentiator for frequencies near the resonant frequency. Equation (2.7) suggests that a differentiator can be modelled as a harmonic oscillator as follows:

\[
\frac{d^2 x(t)}{dt^2} + \lambda_{E} x(t) = \frac{d u(t)}{dt} \quad (2.8)
\]

\[
\frac{d^2 x(t)}{dt^2} + \lambda_{E} x(t) = \frac{d^2 u(t)}{dt^2} \quad (2.9)
\]
Figure 2.3  Modified magnitude characteristic of the ideal digital differentiator

Figure 2.4  A resonant highpass system
Figure 2.5  
Magnitude characteristic of the resonant highpass system

Figure 2.6  
A resonant lowpass system
where $x(t)$ is the output, $u(t)$ is the input, and $\lambda_E$ is the characteristic value. Equation (2.8) corresponds to a second order system approximating a first order derivative and Equation (2.9) corresponds to a second order system approximating a second order derivative. It is possible to define an $'n'$ th order system given by

$$\frac{d^nx(t)}{dt^n} + \lambda_E x(t) = \frac{d^n u(t)}{dt^n} \quad (2.10)$$

where $'n'$ is even so that Equation (2.10) represents a lossless system having purely imaginary roots.

The homogeneous equation corresponding to Equation (2.10) is

$$\frac{d^nx(t)}{dt^n} + \lambda_E x(t) = 0 \quad (2.11)$$

The $'n'$ roots of Equation (2.11) are given by

$$S^n = \lambda_E \quad (2.12)$$

The $'n'$ roots can be calculated from the formula

$$S_k = \frac{1}{n} e^{(2\pi k+1)/n}, k = 0,1,2,..,n-1 \quad (2.13)$$

In general, all the roots will be complex except for $n=2$. For $n=2$, there is a pair of purely imaginary axis roots, and the system is loss less. Selecting only LHS poles, one can characterize the system given by Equation (2.11) as

$$a_{n/2} \frac{d^{n/2}x(t)}{dt^{n/2}} + a_{n/2-1} \frac{d^{n/2-1}x(t)}{dt^{n/2-1}} + ... + a_0 x(t) = 0 \quad (2.14)$$
The system given by Equation (2.14) behaves as an insertion loss network.

A higher order lossless system can be considered to be a cascade of 'n' second order systems as follows:

\[ H(s) = \frac{1}{(s^2 + \lambda_{E1})(s^2 + \lambda_{E2}) \cdots (s^2 + \lambda_{En})} \quad (2.15) \]

In differential equation form, the system given by Equation (2.15) can be expressed as

\[
\frac{d^{2n}x(t)}{dt^{2n}} \pm (\lambda_{E1} + \lambda_{E2} + \cdots + \lambda_{En}) \frac{d^{2n-2}x(t)}{dt^{2n-2}} \pm (\lambda_{E1}\lambda_{E2} + \lambda_{E2}\lambda_{E3} + \cdots) \frac{d^{2n-4}x(t)}{dt^{2n-4}} \\
\pm (\lambda_{E1}\lambda_{E2}\lambda_{E3} + \cdots) \frac{d^{2n-6}x(t)}{dt^{2n-6}} \pm \cdots \pm (\lambda_{E1}\lambda_{E2}\lambda_{E3}\cdots \lambda_{En}) = 0 
\]

(2.16)

The system described by Equation (2.16) behaves as a circuit made up of tightly coupled 'n' circuits with peaks at natural frequencies \( \omega_{n1}, \omega_{n2}, \ldots, \omega_{nn} \)

An 'n' th order system may also be approximated by the dominant roots of a second order system, and one may express the 'n' th order system in terms of the second order dominant roots as

\[
\frac{d^n x(t)}{dt^n} + \lambda_E x(t) = 0, \quad \lambda_E = \omega_n^2 
\]

(2.17)

The corresponding differential equation of the system becomes

\[
an \frac{d^n x(t)}{dt^n} + a_{n-1} \frac{d^{n-1} x(t)}{dt^{n-1}} + \cdots + \lambda_E x(t) = 0 
\]

(2.18)
Out of the three models corresponding to Equations (2.14), (2.16) and (2.18), a digital resonator algorithm will be developed from Equation (2.18).

2.3 A DYNAMIC SYSTEM MODEL FOR RESONANT LOWPASS DIFFERENTIATOR

Let a second order differentiator be modelled as shown in Figure 2.8. The model can be described by

\[ A \frac{d^2x(t)}{dt^2} + \lambda_E x(t) = B \frac{du(t)}{dt} \quad t \in [-\infty, \infty] \tag{2.19} \]

where \( u(t) \) is the input, \( x(t) \) is the output, \( \lambda_E \) is the eigen value, and \( A, B \) are constants. The homogeneous equation corresponding to Equation (2.19) is

\[ A \frac{d^2x(t)}{dt^2} + \lambda_E x(t) = 0 \tag{2.20} \]

Equations (2.19) and (2.20) characterize a lossless system. For physical realizability, the solution of equation (2.20) must satisfy the physical realizability conditions

\[ x(t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{and} \quad \int_{-\infty}^{\infty} |x(t)| \, dt \to \alpha \quad \text{as} \quad t \to \infty \tag{2.21} \]

where \( \alpha \) is a finite constant. A digital differentiator is a band limited system and, therefore, let Equations (2.19) and (2.20) characterize a bandlimited lossless system. For a physically realizable band limited lossless differentiator, it is reasonable to assume a solution of the form

\[ x(t) = \frac{1}{t}(e \sin \frac{\pi rt}{\nu rt}) \tag{2.22} \]
Figure 2.7  
Magnitude characteristic of the resonant lowpass system.

Figure 2.8  
A dynamic system model of a second order two point differentiator
where $c$ and $p$ are appropriately chosen constants. However, such a choice requires that Equation (2.20) be modified as

$$\frac{d}{dt} \left[ \frac{d^2x(t)}{dt^2} \right] + t^2 \lambda E x(t) = 0 \quad (2.23)$$

Equation (2.23) can be expanded as

$$A t^2 \frac{d^2x(t)}{dt^2} + A t \frac{dx(t)}{dt} + \lambda E t^2 x(t) = 0 \quad (2.24)$$

For large values of time 't', Equation (2.24) reduces to the time invariant form

$$A \frac{d^2x(t)}{dt^2} + \lambda E x(t) = 0$$

which is the same as Equation (2.20). Assume that the higher order system dynamics of Figure 2.8 can be approximated by the dominant roots of equation (2.23). The 'n' th order time invariant difference equation for Figure 2.8 can now be written as

$$A \Delta^nx_k + \lambda E x_k = B \Delta u_k, \ k \in [0, \infty] \quad (2.25)$$

where $\Delta x_k$ denotes backward difference. Expanding the first term of Equation (2.25)

$$\frac{A}{T^n} \sum_{j=1}^{N} Q x_{k-j} + \frac{(A + T^n \lambda E)}{T^n} x_k = \frac{B}{T} \Delta u_k$$

where 'n' is the order of the equation and $N$ is the number of terms.

$$\quad \frac{A}{T^n} \sum_{j=1}^{N} x_{k-j} + \frac{(A + T^n \lambda E)}{T^n} x_k = \frac{B}{T} \Delta u_k \quad (2.26)$$

where $T$ is the sampling interval. For impulse input $\Delta u_k = \delta_k - \delta_{k-1}$.
where \( \delta_k = \begin{bmatrix} 1 & k = 0 \\ 0 & k \neq 0 \end{bmatrix} \quad \delta_{k-1} = \begin{bmatrix} 1 & k = 1 \\ 0 & k \neq 1 \end{bmatrix} \)

The impulse response corresponding to Equation (2.26) can be shown to be

\[
h_k = \frac{T^{n-1}B}{A+T^n\lambda_E} (\delta_k - \delta_{k-1}) - \sum_{j=1}^{N} Q_j h_{k-j} \tag{2.27}
\]

Equation (2.27) characterizes a differentiator as a dynamic system as depicted in Figure 2.8. A simple time domain design of an optimal resonant digital differentiator can now be developed based on the model shown in Figure 2.8 and characterized by Equation (2.27).

Let the impulse response of an ideal bandlimited lossless differentiator corresponding to Equation (2.27) be of the form

\[
h_d(k) = \frac{p \sin \frac{q(k+1)}{q}}{q(k+1)}, \quad 0 \leq k \leq N \tag{2.28}
\]

In Equation (2.28), \( p \) and \( q \) can be determined by appropriate specifications. The coefficients \( Q_j \) can be computed by equating Equations (2.27) and (2.28) at \( N \) points. The computation will yield the values of \( Q_j \) as well as \( h_{k-j} \) of Equation (2.27). Let the approximation error be given by

\[
E = \sum_{k=0}^{N} [ | h(k) | - | h_d(k) | ]^2 \tag{2.29}
\]

where \( h(k) = h_k \) of Equation (2.27). Let \( E \) be minimum when \( Q_j^1 = Q_j + \Delta Q_j \).

Rewriting Equation (2.27) in terms of \( Q_j^1 \) and solving for \( Q_j^1 \)
Substituting the optimum values $Q_j^1$ in Equation (2.27)

$$h_k = \frac{T^{n-1} B}{(A+T^nE)} \left( \delta_k - \delta_{k-1} \right) - \frac{A}{(A+T^nE)} \sum_{j=1}^{N} \pm Q_j^1 h_{k-j} \quad (2.31)$$

A frequency domain design can also be proposed based on Equation (2.27). The response $x_k$ corresponding to an input $u_k$ can be expressed by

$$x_k = \frac{T^{n-1} B}{(A+T^nE)} \left( u_k - u_{k-1} \right) - \frac{A}{(A+T^nE)} \sum_{j=1}^{N} \pm Q_j^1 x_{k-j} \quad (2.32)$$

The transfer function corresponding to equation (2.32) can be written as

$$H(z) = \frac{T^{n-1} B (1-z^{-1})}{(A+T^nE)} \quad (2.33)$$

\[ h_d(k) = (A+T^nE)^{-1} (h(k) - d(k-1)) \]
Equation (2.33) can be rewritten as

\[
H(z) = H_1(z) H_2(z) = \left[ \frac{T^{n-1}B}{(A+T^n A_E)} \right] \left[ \frac{A}{\sum_{j=1}^{N} \pm Q_j z^{-j}} \right] (1-z^{-1}) \quad (2.34)
\]

It is obvious from Equation (2.34) that \(H_1(z)\) is an all-pole function and, hence, exhibits lowpass characteristic. The design consists in computing values of \(Q_j\) such that \(H(z)\) is a lowpass filter with a specified cut off frequency, and cascading with a two point difference algorithm \(H_2(z)\).

2.4 TIME DOMAIN DESIGN OF RESONANT LOWPASS DIFFERENTIATOR

Equations (2.18) and (2.32) suggest that a resonant lowpass digital differentiator can be approximated by an 'n' th order difference equation of the form

\[
x(n) = B' u(n) \pm Q_1 x(n-1) \pm Q_2 x(n-2) \pm ... \pm C_{2N} x(n-2N) \quad (2.35)
\]

It's impulse response is given by

\[
h(n) = B' u(n) \pm Q_1 h(n-1) \pm Q_2 h(n-2) \pm ... \pm C_{2N} h(n-2N) \quad (2.36)
\]

Let the solution (2.22) corresponding to Equation (2.23) satisfy the boundary conditions

\[
x(t) = 1 \quad \text{for} \ t=0
\]

\[
x(t) = 0 \quad \text{for} \ t=1 \quad (2.37)
\]
One possible set of values of \( p \) and \( q \) in Equation (2.22) are \( p=q=\pi/2 \) and, therefore, Equation (2.22) becomes

\[
h(t) = \frac{\pi/2 \sin \pi/2 (t+1)}{\pi/2 (t+1)}
\]

(2.38)

For discrete-time system, Equation (2.38) becomes,

\[
h(n) = \frac{\pi/2 \sin \pi/2 (n+1)}{\pi/2 (n+1)}
\]

(2.39)

If Equation (2.39) is the desired response, then

\[
h_d(n) = \frac{\pi/2 \sin \pi/2 (n+1)}{\pi/2 (n+1)}, \quad 0<n\leq N-1
\]

(2.40)

In Equation (2.40), \( h_d(n)=0 \) for all odd values of \( n \). Equivalence of Equations (2.36) and (2.40) requires that all odd terms in Equation (2.36) be zeroes. Equation (2.36) can now be rewritten as

\[
h(n) = B^d(n) - Q_2 n(n-2) + Q_4 h(n-4) - \ldots + Q_{2N} h(n-2N)
\]

(2.41)

The corresponding Equation in Z domain becomes

\[
H(z) = \frac{B^z}{1 \pm \sum_{n=1}^{N} Q_{2n} z^{-2n}}
\]

(2.42)
For $n=0,1,..10$ the values of $h_d(n)$ can be calculated from Equation (2.40) and are given below:

$$
\begin{align*}
    h_d(0) &= 1 & h_d(6) &= -0.143 \\
    h_d(1) &= 0 & h_d(7) &= 0 \\
    h_d(2) &= -0.383 & h_d(8) &= 0.11 \\
    h_d(3) &= 0 & h_d(9) &= 0 \\
    h_d(4) &= 0.20 & h_d(10) &= -0.091 \\
    h_d(5) &= 0 & 
\end{align*}
$$

The coefficients $Q_j$ of the digital differentiator can be calculated by using the impulse response coefficients in Equation (2.41). Using Equations (2.29) and (2.30), the optimal coefficients $Q^1_j$ can be calculated and are listed below:

$$
\begin{align*}
    Q^1_1 &= 0 & Q^1_6 &= 0.0468 \\
    Q^1_2 &= 0.333 & Q^1_7 &= 0 \\
    Q^1_3 &= 0 & Q^1_8 &= 0.037 \\
    Q^1_4 &= 0.089 & Q^1_9 &= 0 \\
    Q^1_5 &= 0 & Q^1_{10} &= 0.0255 \\
\end{align*}
$$

The resonant differentiator given by Equation (2.35), approximated to 10th order difference equation, becomes

$$
\begin{align*}
    x(n) &= B' u(n) - 0.0333 x(n-2) + 0.089 x(n-4) - 0.0468 x(n-6) + 0.037 \\
         & \quad x(n-8) - 0.0255 x(n-10) \\
\end{align*}
$$

For convenience, choose $B'=1$ so that the magnitude response can be expressed as

$$
\begin{align*}
    |H(e^{j\Theta})| &= \frac{1}{[1+0.333 \cos 2\Theta - 0.089 \cos 4\Theta + 0.0468 \cos 6\Theta \\
            -0.037 \cos 8\Theta + 0.0255 \cos 10\Theta] + j [0.089 \sin 4\Theta \\
            -0.333 \sin 2\Theta - 0.0468 \sin 6\Theta + 0.037 \sin 8\Theta \\
            -0.0255 \sin 10 \Theta]} \\
\end{align*}
$$
2.5 STABILITY OF RESONANT LOWPASS DIFFERENTIATOR

Let Equation (2.42) be expressed in the form

\[ H_1(z) = \frac{B^1}{\prod (1 \pm C_i z^{-2})} \]  

(2.45)

consider a simple differentiator of the form

\[ x(n) = \frac{1}{T} [u(n) - u(n-1)] \]  

(2.46)

In Z domain, Equation (2.46) can be expressed as

\[ H_2(z) = \frac{1}{T} [1-z^{-4}] \]  

(2.47)

A cascade of systems \( H_1(z) \) and \( H_2(z) \) yields

\[ H(z) = H_1(z) H_2(z) = \frac{B^{1/T} (1-z^{-1})}{\prod (1 \pm C_i z^{-2})} \]  

(2.48)

The denominator factors give poles of the form \( z^2 = C_i \) where

\[ C_i = r_i e^{i\Theta_i} \]  

(2.49)
Therefore, Equation (2.48) becomes

\[ H(z) = \frac{B^1/T (1-z^{-1})}{\prod_{i=1}^{N} (1 \pm r_i e^{i\Theta_i} z^{-2})} \quad (2.50) \]

Solving Equation (2.50)

\[ z = r_1^{1/2} e^{i(\Theta/2 + 2\pi N/2)} = r_1^{1/2} e^{i(\Theta/2 + m\pi)} \quad (2.51) \]

i.e. \[ z = r_1^{1/2} e^{i(\Theta/2)} r_1^{1/2} e^{i(\Theta/2 + \pi)} \]

For a typical second order system

\[ H(z) = \frac{B^1/T (1-z^{-1})}{(1-C_1 z^{-2})} \]

with poles

\[ z = r_1^{1/2} e^{i(\Theta/2)} r_1^{1/2} e^{i(\Theta/2 + \pi)} \]

and zeroes \[ z = 1 \]

For stability, one requires that \( r_1, r_2, \ldots, r_N < 1 \). This implies that one has to establish a condition for stability in terms of the coefficients of the system function given by Equation (2.48). By expanding the denominator of Equation (2.48), one can show that
\[
N \sum_{i=1}^{i} (1-C_i z^{-2}) = 1 - [\Sigma N_{c,i}] z^{-2}
\]

\[
+ \Sigma [N_{c2}] z^{-4} - \ldots + [\Sigma N_{cN}] z^{-2N}
\]

\[
= 1 - (c_1 + c_2 + \ldots + c_N) z^{-2} + (C_1 C_2 + C_2 C_3 + \ldots + C_1 C_N + C_2 C_N)
\]

\[
+ C_3 C_4 z^{-4} - (C_1 C_2 C_5 + C_2 C_3 C_5 + C_3 C_4 C_5 + C_3 C_1 C_5)
\]

\[
+ \ldots + C_1 C_{N-1} C_{N+1} + C_2 C_N C_{N+1} + C_3 C_N C_{N+1}) z^{-6}
\]

\[
+ \ldots (C_1 C_2 C_3 \ldots C_{N}) z^{-2N}
\]

(2.52)

Let \( B^4 = 1 \) for convenience and \( \delta(n) \) be a unit impulse in Equation (2.41). Equation (2.41) can be rewritten as

\[
h(n) = \delta(n) - Q_2 h(n-2) + Q_4 h(n-4) - Q_6 h(n-6)
\]

\[
+ Q_8 h(n-8) + \ldots + Q_{2N} h(n-2N)
\]

(2.54)

For \( n = 0, 1, 2 \ldots \) it can be shown that

\( h(0) = 1, \ h(1) = 0, \ h(2) = -Q_2, \ h(3) = 0, \)

\( h(4) = Q_2^2 + Q_4, \ h(5) = 0, \ h(6) = -Q_2 (Q_2^2 - 2Q_4) - Q_6, \)

\( h(7) = 0 \) and \( h(8) = Q_2^2 (Q_2^2 - Q_4) + Q_4^2 + Q_8 \)

For stability

\( h(0) > h(2) > h(4) > h(6) > \ldots \) Selection of

\( Q_2, Q_4, Q_6 \) and \( Q_8 \) such that \( Q_2 < 1, Q_2^2 + Q_4 < | Q_2 |, \)

\( | -Q_6 (Q_2^2 - 2Q_4) - Q_6 | < Q_2^2 + Q_4 \)

satisfies the stability requirement \( h(0) > h(2) > h(4) > h(6) \)

Let \( Q_2 < 1 \) (2.55)

\( Q_2^2 + Q_4 < Q_2 \) (2.56)

Condition (2.56) implies that \( Q_4 < Q_2 \). Comparing Equations (2.53) and (2.54).
\[ Q_2 = Nc_1 = C_1 + C_2 + \ldots C_N \]
\[ Q_4 = Nc_2 = C_1C_2 + C_2 C_3 + C_3C_1 + \ldots C_1C_N + C_2 C_N + C_3 C_N \]

From Equations (2.55) and (2.56), it is obvious that
\[(C_1 + C_2 + \ldots C_N) < 1, \Sigma Nc_2 < \Sigma Nc_1\]

Since \(Nc_1 > Nc_t\) for any given \(r > 2\), it follows that
\[\Sigma Nc_1 > \Sigma Nc_2, \Sigma Nc_2 > \Sigma Nc_3 \ldots \]
\[\Sigma Nc_{N-1} > \Sigma Nc_N\]

This implies that if \(Q_2 < 1, Q_4 < Q_2, Q_6 < Q_4, \ldots Q_{2N} < Q_{2N-2}\), the system remains stable; for \(Q_2 < 1, Q_4, Q_6, \ldots Q_{2N}\) are all less than 1.

### 2.6 A NOTE ON THE OPTIMALITY CRITERION

Let \(h_d(n)\) be the desired impulse response and \(h_r(d)\) be the approximated filter impulse response. Equation (2.41) can be written as

\[
d(n) = \sum_{k=1}^{N} Q_{2k} h_r(n-2k) = h_d(n), 0 \leq n \leq \infty
\]

\[
h_d(n) = \frac{\pi/2 \sin (n+1) \pi/2}{(n+1) \pi/2}
\]

\[
\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \end{cases}
\]

The differentiator Equation (2.27) is obtained by selecting \(n = 10\) (any desired order can be chosen). Therefore, over a specified interval \(0 \leq n \leq M\), Equation (2.27) will deviate from \(h_d(n)\) by a certain amount for every value of \(n\). A simple
procedure to minimize this error can be developed. The problem is to select best values of \(Q_{2k}\) that minimize error \(|E(n)|\) for a given order \(n\) where

\[
|E(n)|^2 = \sum_{n=0}^{M} [ | h_r(n) | - | h_d(n) | ]^2
\] (2.57)

Though optimization techniques are available, a simpler procedure can be adopted when order of the filter is reasonably small. Let \(|E(n)|\) be minimum when \(Q_{2k}\) are chosen as \(Q_{2k}^1\) where \(Q_{2k}^1 = Q_{2k} + \Delta Q_{2k}\). Let the impulse response of the filter whose coefficients are \(Q_{2k}^1\) be \(h_r(n)\). Equation (2.41) can be rewritten as

\[
\delta(n) - \sum_{k=1}^{N} Q_{12k}^1 h_r(n-2k) = h_d(n), 0 \leq n \leq M \] (2.58)

Upon transposing \(\delta(n)\) to the RHS, Equation (2.58) becomes

\[
\sum_{k=1}^{N} Q_{12k}^1 h_r(n-2k) = \delta(n) - h_d(n), 0 \leq n \leq M \] (2.59)

Equation (2.59) can be expressed in vector form as

\[
[h_r(n-2K)] [Q_{12k}^1]^T = [\delta(n)] - [h_d(n)]
\] (2.60)

\[
[Q_{2k}^1]^T = [\delta(n)] [h_r(n-2k)]^{-1} - [h_d(n)] [h_r(n-2k)]^{-1}
\] (2.61)

Since \(h_r(n-2K)\) and \(h_d(n)\) are already computed for a specified value of \(n\), \(Q_{12k}^1\) can be computed using Equation (2.61).

2.7 RESULTS AND DISCUSSION

In Equation (2.28), \(p\) and \(q\) have been chosen to satisfy boundary conditions (2.37), which implies a normalised resonant differentiator whose natural frequency is \(\pi/2\) rad. The constants \(A\) and \(B\) in Equation (2.31) have been suitably chosen so that the values of \(Q_{1}^1\) in Equation (2.31), or the coefficients in Equation (2.43) are the smallest possible. A suitably normalized three cycle digitized ECG data
Numerical computations and results are presented in Table 2.1 and Figures 2.9 - 2.17. Figure 2.9 is the impulse response of the resonator characterized by Equation (2.45). Figure 2.10 is the corresponding step response. The magnitude response of the basic resonator is shown in Figure 2.11. Figure 2.12 shows a noisy three cycle ECG test data with baseline drift. Figure 2.13 exhibits forced response of the resonator. Differentiation of high frequencies can be noticed in Figure 2.13. However, the resonator does not act as a differentiator for low frequencies. This can be observed from the presence of baseline drift in Figure 2.13. When coefficients of Equation (2.43) are substituted in Equation (2.31), the desired resonant lowpass differentiator is obtained. The resonant lowpass differentiator is effective in the lower end of the differentiation band as well higher end of the differentiation band. Figure 2.14 shows magnitude and phase characteristics of the optimal resonant digital lowpass differentiator. Figure 2.15 shows the output of resonant digital lowpass differentiator characterized by Equation 2.31. It can be noticed in Figure 2.15 that baseline drift has been eliminated. Excellent detection of positive and negative going slopes, and removal of baseline drift can be observed in Figure 2.15. Figure 2.16 shows gain and error behaviour of the resonant differentiator (RD) with respect to frequency. The gain of an ideal differentiator (ID) and a seven point third order differentiator (SPD) are also shown. The error behaviour of the proposed differentiator (ERD) has been compared with that of SPD (ESPD) and good ERD over entire differentiation band is demonstrated. The proposed resonant differentiator has also been compared with a typical wideband digital differentiator in Figure 2.17. One of the advantages of the proposed differentiator is that it is frequency transformable. The basic resonator given by Equations (2.43), (2.44) and Figure 2.11 has a cut off frequency 47.75 HZ (\(\omega_c = 300 \text{ rad/s}\)) when denormalized. It can be transformed to another resonator with cut off frequency, say, 67.75 HZ (\(\omega_d = 425.69 \text{ rad/s}\)). The transformation required is

\[
\begin{align*}
z^{-1} &\rightarrow \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}} \\
(2.62)
\end{align*}
\]
Table 2.1 Data Describing the design of optimal resonant digital lowpass differentiator

1. Model of Analog Resonant Differentiator

\[ a_n \frac{d^n x(t)}{dt^n} + a_{n-1} \frac{d^{n-1} x(t)}{dt^{n-1}} + \ldots + \lambda_Ex(t) = 0 \]

2. Model of Digital Resonator

\[ h(n) = B^1\delta(n) - Q^1_2 h(n-1) + Q^4_4 h(n-4) + \ldots + Q^{2N}_2 h(n-2N) \]

\[ H(z) = \frac{B^1}{1 \pm \sum_{n=1}^{N} Q_{2n}^1 z^{-2n}} \]

3. Model of Resonant Optimal Lowpass Differentiator

\[ h_k = \frac{T^n B}{A + T^n \lambda_E^1} (\delta_k - \delta_{k-1}) - \frac{A}{A + T^n \lambda_E^1} (\sum_{j=1}^{N} Q^1_j h_{k-j}) \]

\[ H(z) = \frac{T^n B}{A + R^n \lambda_E^1} (1 - z^{-1}) \]

\[ \frac{1 + \frac{A}{A + T^n \lambda_E^1} \sum_{j=1}^{N} Q^1_j z^{-j}}{A + T^n \lambda_E^1} \]
4. Specifications

Normalized resonant frequency $\omega_n = \pi/2$ rad
Differentiation band required $0 \leq \omega \leq \pi/2$
Resonant band is $0 \leq \omega \leq a\pi/2$ where 'a' can be chosen such that $0 < a \leq 1$

Desired impulse response $h_d(k) = \frac{\pi/2 \sin \pi/2 (k+1)}{\pi/2 (k+1)}$

Order of the differentiator is $n = N$ so that Error $E < 1$.
Choose $A = T$, $B = 2T$, $B^1 = 1$, $\lambda_E = 1$.

5. Computed Values

$N = 10$, $Q_1^1 = Q_3^1 = Q_5^1 = Q_7^1 = Q_9^1 = 0$, $Q_2^1 = -0.333$, $Q_4^1 = 0.089$, $Q_6^1 = -0.0468$, $Q_8^1 = 0.037$, $Q_{10}^1 = -0.0255$

6. Results

Illustrated in Figures 2.9 - 2.17
Figure 2.9 Impulse response of the resonator
Figure 2.10 Step response of the digital resonator
Figure 2.11  Magnitude response of the digital resonator
Figure 2.12  Three cycle ECG test data with base line drift
Figure 2.13 Forced response of the digital resonator
Figure 2.14 Magnitude and phase response of the optimal resonant digital lowpass differentiator.
Forced response of the optimal resonant digital lowpass differentiator

Figure 2.15
Figure 2.16 Gain and error behaviour of the optimal resonant digital lowpass differentiator as compared to a typical lowpass digital differentiator

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>ID</td>
<td>Ideal differentiator</td>
</tr>
<tr>
<td>SPD</td>
<td>Seven point third order digital differentiator</td>
</tr>
<tr>
<td>RD</td>
<td>Resonant digital lowpass differentiator (two point tenth order)</td>
</tr>
<tr>
<td>ESPD</td>
<td>Error behaviour of SPD</td>
</tr>
<tr>
<td>ERD</td>
<td>Error behaviour of RD</td>
</tr>
</tbody>
</table>
Figure 2.17  Gain and error behaviour of the optimal resonant digital lowpass differentiator as compared to a typical wideband digital differentiator

RD  Resonant digital lowpass differentiator
WBD  Wideband digital differentiator
ERD  Error behaviour of RD
EWBD  Error behaviour of WBD
Substituting Equation (2.65) in Equation (2.44) yields

\[
H(z) = \frac{1}{(1+0.333 z^{-2} - 0.089 z^{-4} + 0.0468 z^{-6} -0.037 z^{-8} + 0.0255 z^{-10})}
\]

(2.66)

The amplitude response of Equation (2.66) is shown in Figure 2.18. From the plot we observe the following:

when \( \Theta = \omega_d T = 425.69 \times 0.008 = 3.168639 \) rads

\[
f_d = \frac{\Theta}{2\pi T} = \frac{3.168639}{2\pi \times 0.008} = 63.0386 \text{ HZ}
\]

The frequency transformed filter has a cut off frequency 63.0386 HZ. The noise performance of the proposed differentiator can also be demonstrated. Wideband differentiators as well as lowpass differentiators generally have a lower output signal-to-noise ratio than input signal-to-noise ratio. A similar result has been observed in the case of resonant lowpass digital differentiator introduced in this chapter. A white Gaussian noise of power spectral density \( N_0/2 \) is assumed for computation of signal-to-noise ratio. Let

\[
\frac{N_0}{2} = \frac{K T_0}{2} = \frac{1.37 \times 10^{-23} \times 300}{2}
\]
Figure 2.18  Amplitude response of frequency transformed optimal resonant digital lowpass differentiator
The total noise power for a bandwidth of 50 HZ is $KT_0/2 \times B = 1.02 \times 10^{-19}$. The total signal power at the input of the resonant lowpass differentiator is $(2.8 \text{ mV})^2$. The total signal power at the output of the resonant differentiator is $(0.924 \text{ mv})^2$.

\[
\begin{align*}
\text{(S/N)}_{\text{input}} &= 10 \log \frac{(2.8 \times 10^{-3})^2}{1.02 \times 10^{-19}} \quad 138.9 \text{ dB} \\
\text{(S/N)}_{\text{output}} &= 10 \log \frac{(0.924 \times 10^{-3})^2}{1.02 \times 10^{-19}} \quad 129.3 \text{ dB}
\end{align*}
\]

The results are presented in detail in Table 2.1.

The concept of resonant lowpass differentiation has been introduced in this chapter. A simple time domain design has been developed, and a normalized resonant optimal differentiator has been designed. Its use for signal detection has been demonstrated. A comparison with typical wide band and lowpass differentiators (Table 2.2) shows that the resonant differentiator requires minimum number of coefficients, relatively smaller computations, and acceptable error behaviour. They can be used for on-line signal differentiation applications where a compromise between order and accuracy is desirable.
Table 2.2  Comparison of Resonant Lowpass Digital Differentiator with Wideband and Lowpass Digital Differentiators

<table>
<thead>
<tr>
<th>Sl. No.</th>
<th>Type</th>
<th>Wideband Differentiators</th>
<th>Lowpass Differentiators</th>
<th>Resonant Lowpass Differentiator</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Type</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>Order</td>
<td>Generally very high</td>
<td>Low order Typically ≤ 10</td>
<td>Low order Typically ≤ 10</td>
</tr>
<tr>
<td>3.</td>
<td>Optimality</td>
<td>Standard optimality criteria in frequency domain</td>
<td>Standard optimality criteria in frequency domain</td>
<td>Minimum Mean Square Error in time domain</td>
</tr>
<tr>
<td>4.</td>
<td>Accuracy</td>
<td>Very high No definite value Depends on the order and type</td>
<td>Poor No definite value Depends on the order</td>
<td>Better than lowpass type, less accurate compared to wideband type</td>
</tr>
<tr>
<td>5.</td>
<td>Bandwidth</td>
<td>Full band $0 \leq \omega \leq \pi/2$</td>
<td>Partial band $0 \leq \omega \leq a\pi/2 (a&lt;1)$</td>
<td>Partial band in two regions over $[0,\pi/2]$</td>
</tr>
<tr>
<td>6.</td>
<td>Noise Performance</td>
<td>No definite value available but generally $(S/N)<em>{out} &lt; (S/N)</em>{in}$</td>
<td>No definite value available but generally $(S/N)<em>{out} &lt; (S/N)</em>{in}$</td>
<td>$(S/N)<em>{in}=138.9$ dB (order 10) $(S/N)</em>{out} = 129.3$ dB $(S/N)<em>{out} &lt; (S/N)</em>{in}$</td>
</tr>
</tbody>
</table>