CHAPTER 2

A SIMPLE STABILITY TEST FOR 2-D RECURSIVE DIGITAL FILTERS

2.1 INTRODUCTION

Two dimensional (2-D) recursive digital filters have applications in image processing, seismic record processing, remote sensing and medical electronics. When once a 2-D digital filter is designed satisfying certain magnitude characteristics, one has to test the filter transfer function for Bounded Input Bounded Output (BIBO) stability.

Consider a recursive first-quadrant 2-D filter transfer function whose denominator polynomial is $B(Z_1, Z_2)$. It is assumed that there are no non-essential singularities of the second kind in the literature that appeared during 1970's and 1980's most of which covered by Huang (1981) in a detailed way, several algebraic and mapping methods were suggested for 2-D first quadrant filters for stability testing. The advantages and disadvantages of these two types of methods were discussed. Both these types of methods are tedious to implement and enormous time is required to test filter polynomial $B(Z_1, Z_2)$ for stability even if the degree of $B(Z_1, Z_2)$ is only two. The method of testing presented in this chapter is very simple and requires less time. In this chapter a simple test for testing the stability of 2-D recursive digital filters is presented when compared to all the existing methods in the literature covered by Huang (1972).
In Section 2.2 certain fundamentals regarding stability of 1-D filters are dealt with and a method to test stability of such filters in a partial way is given. Section 2.3 contains some basics about 2-D polynomials including an existing theorem on stability. In Section 2.4, 2.5 and 2.6 the stability of 2-D first quadrant filters is dealt with and it is shown that testing for stability require testing of only a few 1-D polynomials. Section 2.7 deals with some important discussions which highlights the significance of the results, obtained in this chapter.

2.2 STABILITY FUNDAMENTALS

In this section 1-D digital filters are first dealt with and some of the basic concepts of stability are explained. An alternative technique of testing 1-D polynomials for stability using mathematical optimization, employing the concept of Lagrange multipliers is discussed by Rao (1987).

Consider a 1-D polynomial \( A(Z) \), which is the denominator polynomial of a recursive 1-D digital filter. Unlike the conventional method of defining Z-transform of a discrete signal (Antoniou et al 1979) positive powers of \( Z \) are assumed and used in the definition of Z-transform. With this assumption the following Theorem is stated.

*Theorem 1*

(i) The 1-D polynomial \( A(Z) \) is stable iff \( A(Z) \neq 0 \), for all \( |Z| < 1 \)

(ii) A stable \( A(Z) \) is said to be marginally stable if
A(Z) = 0, for some | Z | = 1

Auto correlation coefficients are defined as follows,

**Definition 1**

If \( A(Z) \) is an \( M^{th} \) degree polynomial, say
\[
A(Z) = a_0 + a_1Z + a_2Z^2 + \ldots + a_nZ^M,
\]
the co-efficient \( r_i \)'s of \( A(Z)A(Z^{-1}) \) written as
\[
A(Z)A(Z^{-1}) = r_0 + r_1(Z + Z^{-1}) + r_2(Z^2 + Z^{-2}) + \ldots + r_m(Z^M + Z^{-M})
\]
are called the autocorrelation coefficients of \( A(Z) \) (Robinson, 1967). They are \((M+1)\) in number. For a real polynomial \( A(Z) \), \( r_i \)'s are all-real.

Consider the following second-degree real polynomial,
\[
A_1(Z) = -35 - Z + 6Z^2 = (2Z - 5)(3Z + 7)
\]
From Theorem 1, \( A_1(Z) \) is a stable polynomial. The following polynomials that have the same auto correlation coefficients as \( A_1(Z) \), are all unstable polynomials.

\[
\begin{align*}
A_2(Z) &= 6Z - 35Z^2 = (-5Z + 2)(7Z + 3) \\
A_3(Z) &= 14Z^2 - 29Z - 15 = (-5Z + 2)(3Z + 7) \\
A_4(Z) &= 15Z^2 + 29Z + 14 = (2Z - 5)(7Z + 3)
\end{align*}
\]

For any polynomial \( A(Z) \) of degree \( M \), there are in general \( 2^M - 1 \) number of other polynomials having the same auto correlation coefficients as \( A(Z) \) not counting the negatives of these polynomials. All these polynomials which are totally \( 2^M \) in number are said to belong to a family. Out of this family only one polynomial
will be stable. This stable polynomial is the one that has the largest (magnitude wise) constant term. It can be observed that the polynomial, has to zero on the unit circle and so is marginally stable in view of Theorem 1. If the other three polynomials are constructed having the same autocorrelation coefficients as $C_1(Z)$ it is found that all these polynomials will have a zero on the unit circle and so are unstable. Thus while testing a 1-D polynomial for stability it is not certain that a particular polynomial in its family is stable simply because it has the largest (magnitude wise) constant term. One has to make sure that the polynomial has no zeros on the unit circle. Any method of testing the stability of a given 1-D polynomial simply based on the magnitude of the constant term can only be termed as partial test since the method tests only for marginal stability.

A method of testing 1-D polynomial for marginal stability is given now. This method is based on Lagrange multiplier technique in mathematical optimization (Rao, 1987). This method of testing 1-D polynomial for stability is several times cumbersome than merely finding out the zeros of the polynomial. But the theoretical procedure of this method when fully understood will be very useful in arriving at the results for 2-D filters in subsequent sections. Example 1 is given only to show that when the optimization procedure is employed, a stable or marginally stable unique 1-D polynomial is obtained. This also shows that maximizing the constant term yields a unique polynomial that is either stable or marginally stable. The given 1-D polynomial $A(Z)$ of degree $M$ has $M+1$ autocorrelation coefficients. They are

$$\sum_{r=0}^{M} a_{r+k} = \gamma, \quad s = 0, 1, \ldots, M \quad (2.1)$$
Let $A'(Z)$ be the stable version of $A(Z)$ whose coefficients are denoted by $a'_j$. The constraint equations are now the same as in (2.1) except that $a_j$'s are replaced by $a'_j$'s.

These can be written as

$$g_j = 0, \quad j = 1, 2, ..., M+1 \quad (2.2)$$

On maximizing $f = a'_0^2$ with (2.2) as constraints and forming the Lagrange function,

$$L(a'_0, \lambda_j) = a'_0^2 + \lambda_1 (a'_0^2 + a'_1^2 + ... + a'_m^2 - \gamma_0) + \lambda_2 (a'_0a'_1 + a'_1a'_2 + ...)$$

$$+ a'_{m+1} (a'_m - \gamma_1) + \lambda_{M+1} (a'_0a'_m - \gamma_M)$$

That is,

$$L(a'_0, \lambda_j) = a'_0^2 + \sum_{j=1}^{M+1} \lambda_j g_j \quad (2.3)$$

In (2.3), $\lambda_j$'s are called Lagrange multipliers and $g_j$'s are the constraints in (2.2). The necessary and sufficient conditions for the function $f$ to be global maximum and hence $a_0$ to be the greatest in magnitude are (Rao, 1987).

$$\frac{df}{da'_0} + \sum_{j=1}^{M+1} \lambda_j \frac{dg_j}{da'_0} = 0 \quad (2.4)$$

$$\lambda_j > 0, \quad j = 1, 2, ..., M+1 \quad (2.5)$$
For a given polynomial $A(Z)$, all the coefficients $a_i$'s ($i = 0, 1, \ldots, m$) are known and hence $y_i$'s are known. Then in order to test whether $A(Z)$ is marginally stable or not one has to do the following. Obtain the equation

$$\frac{\partial L(a'_0, \lambda)}{\partial a'_0} = 0$$

Equation (2.6) contains $\lambda_i$'s as unknowns and will be like the following:

$$2a'_0 + 2a'_0\lambda_1 + a'_1\lambda_2 + a'_2\lambda_3 + \ldots + a'_m\lambda_{m+1} = 0$$

That is,

$$a'_0 = \frac{-(a'_1\lambda_2 + a'_2\lambda_3 + \ldots + a'_m\lambda_{m+1})}{2(1 + \lambda)}$$

Substitute that $a'_0$, which is a linear function of all $\lambda_i$'s in (2.2). A total of $(M+1)$ equations involving $(M+1)$ number of $\lambda_i$'s and $M$ number of $a'_i$'s as unknowns. Let these equations be called (2.8). Solve equation (2.8) for $\lambda_i$'s and $a'_i$'s. All $\lambda_i$'s should turn out to be positive and $a'_i$'s real, satisfying (2.4) and (2.5). Then proceed to get maximum value of $a'_0$ from equation (2.7). If the maximum value of $a'_0$ is the same as $a_0$ of the given polynomial $A(Z)$, then $A(Z)$ is either marginally stable or stable.

This is illustrated by the Example 1

**Example 1**: Let $A(Z) = (2Z - 1)(Z + 3) = 2Z^2 + 5Z - 3$, where, $a_0 = -3$, $a_1 = 5$, $a_2 = 2$. 
Let $A'(Z) = a'_{2}Z^{2} + a'_{1}Z + a'_{0}$ represent the stable version having same autocorrelation coefficients as $A(Z)$. Now the autocorrelation constraint equation for $A'(Z)$ is as follows,

$$a'_{0}^{2} + a'_{1}^{2} + a'_{2}^{2} = 38$$
$$a'_{0}a'_{1} + a'_{1}a'_{2} = -5$$
$$a'_{0}a'_{2} = -6$$  \hspace{1cm} (2.9)

The equation corresponding to 2.5 is

$$2a'_{0} = 2\lambda_{1}a'_{0} + \lambda_{2}a'_{1} + \lambda_{3}a'_{2} = 0$$

Or

$$a'_{0} = \frac{-(a'_{1}\lambda_{2} + a'_{2}\lambda_{3})}{2(1 + \lambda_{1})}$$  \hspace{1cm} (2.10)

Substituting this $a'_{0}$ in equation (2.9)

$$\frac{(a'_{1}\lambda_{2} + a'_{2}\lambda_{3})^{2}}{4(1 + \lambda_{1})} + a'_{1}^{2} + a'_{2}^{2} = 38$$

$$\frac{-(a'_{1}\lambda_{2} + a'_{2}\lambda_{3})a'_{1}}{2(1 + \lambda_{1})} + a'_{1}a'_{2} = -5$$

$$\frac{-(a'_{1}\lambda_{2} + a'_{2}\lambda_{3})a'_{2}}{2(1 + \lambda_{1})} = -6$$  \hspace{1cm} (2.11)

Simplifying equations (2.11) the following equation is obtained
From (2.12) it is obtained

\[
\frac{a'_1 \lambda_2 + a'_2 \lambda_1}{2(1 + \lambda_1)} = \sqrt{38 - a'_1^2 - a'_2^2} = \frac{5}{a'_1} + a'_2 = \frac{6}{a'_2}
\]

(2.12)

Equation (2.13) as well as (2.9) can be satisfied only by \( a'_1 = -1 \), and \( a'_2 = -1 \).

From (2.12) it is also obtained

\[
\sqrt{38 - a'_1^2 - a'_2^2} = \frac{5}{a'_1} + a'_2 = \frac{6}{a'_2}
\]

(2.13)

where \( \lambda_2, \lambda_3 \) and \( \lambda_1 \) can be easily given positive values satisfying (2.14). The maximum value of \( a'_0 \) is obtained as follows,

\[
a'_0 = \frac{-(a'_1 \lambda_2 + a'_2 \lambda_3)}{2(1 + \lambda_1)} = -\frac{(-\lambda_2 - \lambda_3)}{2(1 + \lambda_1)} = \frac{\lambda_2 + \lambda_3}{2(1 + \lambda_1)}
\]

(2.15)

From (2.14) and (2.15),

\[ a'_0 = 6 \]
is obtained. So the polynomials, with $a'_1 = -1$, $a'_2 = -1$ and $a'_0 = a_0 = 6$, namely
\[ A'(Z) = -Z^2 - Z + 6 = (-Z + 2) (Z + 3) \]
is a stable polynomial belonging to the family of the given polynomial,
\[ A(Z) = 2Z^2 + 5Z - 3 \] of example (1).

It may be stressed that the above procedure to test 1-D polynomial for stability or marginal stability is applicable only if the polynomial is not lacunary, in the sense there are no missing terms between the highest degree term and the constant term. When the given polynomial $A(Z)$ is lacunary, $A'(Z)$ has to be assumed to be non-lacunary and follow the same procedure.

It may be noted that in the test procedure given above an attempt is made to maximize the magnitude of the constant term of $A'(Z)$ using optimization technique. After the test if it is concluded that the $A(Z)$ is not unstable $A(Z)$ may contain zeros on the unit circle. So this method of testing 1-D polynomials for stability will only reveal that the given polynomial is either stable or marginally stable.

2.3 SOME BASICS ABOUT 2-D POLYNOMIALS

In this Section some concepts already available in the literature about 2-D first quadrant polynomials is presented. The definition of form preserving 1-D polynomial with respect to a 2-D polynomial is restated. A stability theorem as applicable to 2-D first quadrant polynomials is also restated. The method of testing 2-D polynomials for stability based on Mersereau's One Projection Theorem and Projection Slice Theorem (Mersereau et al, 1974).
Definition 2

A 1-D polynomial $A_1(Z) = \sum_{k=0}^{N} a_k Z^k$ is a form preserving polynomial with respect to a 2-D polynomial $A(Z_1, Z_2) = \sum_{m=0}^{N_1} \sum_{n=0}^{N_2} a_{mn} Z_1^m Z_2^n$. If for every $(m, n)$ in $A(Z_1, Z_2)$ there exists a unique $k$ in $A_1(Z)$ such that $a_k = a_{mn}$ (Reddy et al, 1984)

As a consequence of the above definition, it can be said that the number of distinct terms in $A(Z_1, Z_2)$ is equal to the number of distinct terms in $A_1(Z)$. Let $A_1(Z)$ be the 1-D polynomial corresponding to the 2-D polynomial $A(Z_1, Z_2)$ obtained by transformation $Z_1 = Z^L$ and $Z_2 = Z$.

That is

$$A_1(Z) = A(Z_1, Z_2) \left| \begin{array}{c} Z_1 = Z^L \\ Z_2 = Z \end{array} \right.$$

(2.16)

The minimum value of $L$ for $A_1(Z)$ to be a form preserving polynomial of $A(Z_1, Z_2)$ is (Reddy et al, 1984)

$$L = N_2 + 1$$

(2.17)

It may be noted that for any positive integer value of $L$ greater than $(N_2+1)$ will result in a form preserving 1-D polynomial.

Theorem 2: If $A(Z_1, Z_2)$ does not have any zeros on the unit hypercircle $|Z_1| = 1$, $|Z_2| = 1$, then its form preserving 1-D polynomial $A_1(Z)$ also does not have zeros on the unit circle.
The above theorem has been proved by Reddy et al (1984) though it is trivial.

**Theorem 3:** The two-dimensional first quadrant polynomial,

\[ B(Z_1, Z_2) = \sum_{i=0}^{N} \sum_{j=0}^{N} b_{ij} Z_1^i Z_2^j \]

is stable if and only if (Huang, 1981)

(a) \( B(Z_1, Z_2) \neq 0 \), for \( |Z_1| = 1 \) and \( |Z_2| = 1 \) (that is on \( T^2 \))

(b) \( B(Z, Z) \neq 0 \) for \( |Z| < 1 \)

One of the methods to test condition (a) of Theorem 3 is based on Mersereau's One-Projection Theorem and Projection Slice Theorem (Mersereau et al 1974) and is called row and column concatenation algorithm. This method has been briefly explained by Huang (1981).

According to this a very large number of 1-D polynomials of the type \( B(Z_1^L, Z) \) and/or \( B(Z, Z_1^L) \) have to be obtained where \( L \) takes all the values starting from the minimum value \( L = N+1 \) and going upto infinity. Then all these infinite number of 1-D polynomials are checked for zeros on or inside the unit circle \( |Z| = 1 \). It may be noted that all these 1-D polynomials are form preserving polynomials of \( B(Z_1, Z_2) \). If all these 1-D polynomials have zeros outside unit circle \( |Z| = 1 \), then it can be made sure that \( B(Z_1, Z_2) \) has no zeros on \( |Z_1| = 1 \) and \( |Z_2| = 1 \) satisfying the condition (a) of Theorem 3. But testing infinite number of polynomials is impossible. Then the stability test can be completed by testing for condition (b) of Theorem 3 which is again a 1-D polynomial test.
In the next Section 2-D first-quadrant polynomials are dealt with and some interesting results are arrived at.

2.4 A STABILITY TEST FOR 2-D POLYNOMIALS

In this section a general first quadrant 2-D polynomials are considered and whether only a limited number of 1-D polynomials need be considered for testing is explored.

Let \( B(Z_1,Z_2) = \sum_{i=0}^{N} \sum_{j=0}^{N} b_{ij} Z_1^i Z_2^j \) \( (2.18) \)

be a given 2-D polynomial of degree \( N \), without any missing terms. To test for condition (a) of theorem 3, using Mersereau's (1974) One - Projection theorem and the Projection-Slice Theorem one has to form 1-D polynomials by the form preserving 2-D to 1-D transformation and get,

\( B(Z_1^L) \) or \( B(Z_2^L) \) \( (2.19) \)

Where the minimum value of \( L \) is

\[ L = N + 1 \] \( (2.20) \)

A specific example of a \( B(Z_1,Z_2) \) where \( N = 2 \) is considered.

\[
B(Z_1,Z_2) = b_{00} + b_{01}Z_2 + b_{02}Z_2^2 + b_{10}Z_1 + b_{11}Z_1Z_2 \\
+ b_{12}Z_1Z_2^2 + b_{20}Z_1^2 + b_{21}Z_1^2Z_2 + b_{22}Z_1^2Z_2^2
\]

\( (2.21) \)
The corresponding \( B(Z, Z) = B_t(Z) \) is given by

\[
B_1(Z) = b_{00} + b_{01}Z + b_{02}Z^2 + b_{10}Z^3 + b_{11}Z^4 + b_{12}Z^5 \\
+ b_{20}Z^6 + b_{21}Z^7 + b_{22}Z^8
\] (2.22)

The auto correlation coefficients \( \gamma \)'s of \( B_1(Z) \) are given by

\[
\begin{align*}
\gamma_0 &= b_{00}^2 + b_{01}^2 + b_{10}^2 + b_{11}^2 + b_{20}^2 + b_{21}^2 + b_{22}^2 \\
\gamma_1 &= b_{00}b_{01} + b_{01}b_{02} + b_{02}b_{10} + b_{10}b_{11} + b_{11}b_{12} + b_{12}b_{20} + b_{20}b_{21} + b_{21}b_{22} \\
\gamma_2 &= b_{00}b_{10} + b_{01}b_{11} + b_{02}b_{12} + b_{10}b_{20} + b_{11}b_{21} + b_{12}b_{22} \\
\gamma_3 &= b_{00}b_{11} + b_{01}b_{12} + b_{02}b_{20} + b_{10}b_{21} + b_{11}b_{22} \\
\gamma_4 &= b_{00}b_{12} + b_{01}b_{20} + b_{02}b_{21} + b_{10}b_{22} \\
\gamma_5 &= b_{00}b_{20} + b_{01}b_{21} + b_{02}b_{22} \\
\gamma_6 &= b_{00}b_{21} + b_{01}b_{22} \\
\gamma_7 &= b_{00}b_{22}
\end{align*}
\] (2.23)

Let \( B'_1(Z) \) be the stable version of \( B_1(Z) \) having the same autocorrelation coefficients as \( B_1(Z) \). Let the coefficients of \( B'_1(Z) \) be denoted by \( b'_{ij} \). To test \( B_1(Z) \) for stability, constraint equations have to be formed, of the same type as in (2.23), but \( b_{ij} \)'s on the left hand side of (2.23), being replaced by \( b'_{ij} \)'s. Let these equations be called (2.24). Then maximize \( b'_{00}^2 \) satisfying these constraint equations as mentioned in Section 2.2 by using Lagrange multiplier method. If the maximum value of \( b'_{00} \) namely \( b'_{00}^* \) happens to be equal to \( b_{00} \), then \( B'_1(Z) \) will be identically the same as \( B_1(Z) \) and \( B_1(Z) \) will be a stable (or marginally stable) polynomial. Otherwise \( B_1(Z) \) is unstable.
Let another 1-D polynomial be $B(Z^5, Z) = B_2(Z)$, where $B_2(Z)$ is given by

$$B_2(Z) = b_{00} + b_{01}Z + b_{02}Z^2 + 0 + 0 + b_{10}Z^5 + b_{11}Z^6 + b_{12}Z^7 + 0 + 0 + b_{20}Z^{10} + b_{21}Z^{11} + b_{22}Z^{12}$$  \hspace{1cm} (2.25)

It may be noted that $B_2(Z)$ is having missing term and so is lacunary. The polynomial $B_2(Z)$ has thirteen autocorrelation coefficients as follows, the asterisked equations have no $b_{00}$ in them.

\[
\begin{align*}
&b_{00}^2 + b_{01}^2 + b_{02}^2 + b_{10}^2 + b_{11}^2 + b_{12}^2 + b_{20}^2 + b_{21}^2 + b_{22}^2 = \gamma'_0 \\
&b_{00}b_{01} + b_{01}b_{02} + b_{10}b_{11} + b_{11}b_{12} + b_{20}b_{21} + b_{21}b_{22} = \gamma'_1 \\
&b_{00}b_{02} + b_{10}b_{12} + b_{20}b_{22} = \gamma'_2 \\
&* \ b_{00}b_{10} + b_{10}b_{20} = \gamma'_3 \\
&* \ b_{01}b_{10} + b_{02}b_{11} + b_{11}b_{20} + b_{12}b_{21} = \gamma'_4 \\
&b_{00}b_{10} + b_{01}b_{11} + b_{02}b_{12} + b_{10}b_{20} + b_{11}b_{21} + b_{12}b_{22} = \gamma'_5 \\
&b_{00}b_{11} + b_{01}b_{12} + b_{10}b_{21} + b_{11}b_{22} = \gamma'_6 \\
&b_{00}b_{12} + b_{10}b_{22} = \gamma'_7 \\
&* \ b_{00}b_{20} = \gamma'_8 \\
&* \ b_{01}b_{20} + b_{02}b_{21} = \gamma'_9 \\
&b_{00}b_{20} + b_{01}b_{21} + b_{02}b_{22} = \gamma'_{10} \\
&b_{00}b_{21} + b_{01}b_{22} = \gamma'_{11} \\
&b_{00}b_{22} = \gamma'_{12} \hspace{1cm} (2.26)
\end{align*}
\]

The autocorrelation coefficients $B_1(Z)$ given in (2.23) and the autocorrelation coefficients $B_2(Z)$ given in (2.26) are related as follows,
\[ \gamma_0 = \gamma_0 \]
\[ \gamma_1 = \gamma_1 + \gamma_3 \]
\[ \gamma_2 = \gamma_2 + \gamma_4 \]
\[ \gamma_3 = \gamma_5 \]
\[ \gamma_4 = \gamma_6 + \gamma_8 \]
\[ \gamma_5 = \gamma_5 + \gamma_9 \]
\[ \gamma_6 = \gamma_{10} \]
\[ \gamma_7 = \gamma_{11} \]
\[ \gamma_8 = \gamma_{12} \]

Let \( B'_2(Z) \) be another version of \( B_2(Z) \) having the same autocorrelation coefficient as \( B_2(Z) \) given in (2.26). The corresponding constraint equations are obtained by replacing \( b_9 \) by \( b'_9 \) in (2.26) on the left-hand side.

In view of the relations given in (2.27), they can now be written as follows,

\[ b'_{00}^2 + b'_{01}^2 + b'_{02}^2 + b'_{10}^2 + b'_{11}^2 + b'_{12}^2 + b'_{20}^2 + b'_{21}^2 + b'_{22}^2 = \gamma_0' = \gamma_0 \]
\[ b'_{00}b'_{01} + b'_{02}b'_{02} + b'_{02}b'_{10} + b'_{10}b'_{11} + b'_{11}b'_{12} + b'_{12}b'_{20} + b'_{20}b'_{21} + b'_{21}b'_{22} = \gamma_1' + \gamma_3' = \gamma_1 \]
\[ b'_{00}b'_{02} + b'_{01}b'_{10} + b'_{02}b'_{11} + b'_{10}b'_{12} + b'_{11}b'_{20} + b'_{12}b'_{21} + b'_{20}b'_{22} = \gamma_2' + \gamma_4' = \gamma_2 \]
\[ b'_{00}b'_{10} + b'_{01}b'_{11} + b'_{02}b'_{12} + b'_{10}b'_{20} + b'_{11}b'_{21} + b'_{12}b'_{22} = \gamma_5' = \gamma_3 \]
\[ b'_{00}b'_{01} + b'_{02}b'_{12} + b'_{10}b'_{20} + b'_{11}b'_{21} + b'_{12}b'_{22} = \gamma_6' + \gamma_8' = \gamma_4 \]
\[ b'_{00}b'_{12} + b'_{01}b'_{21} + b'_{02}b'_{22} = \gamma_7' + \gamma_9' = \gamma_5 \]
\[ b'_{00}b'_{20} + b'_{01}b'_{21} + b'_{02}b'_{22} = \gamma_{10}' = \gamma_6 \]
\[ b'_{00}b'_{21} + b'_{01}b'_{22} = \gamma_8' = \gamma_7 \]
\[ b'_{00}b'_{22} = \gamma_{12}' = \gamma_8 \]
As far as optimization using Lagrange multipliers is concerned the Lagrange function for $B_2(Z)$ and $B_1(Z)$ is the same and also constraint equations are the same except there are four additional equations listed at the end in (2.28). Since the equation (2.24) and Lagrange function equation corresponding to $B_1(Z)$ are only a proper subset of equations that correspond to $B_2(Z)$ and the variable being the same, the solution of equations of $B_2(Z)$ and the variable being the same, the solution of equations of $B_2'(Z)$ will also be the solution of equations (2.24) along with the Lagrange function equation. Hence the same optimum for $b'_{oo}$ is obtained and for the corresponding other $b'_{ij}$s and will also be the same. So if $B_1(Z)$ was found to be unstable, $B_2(Z)$ will also be unstable. But if $B_1(Z)$ is found to be stable it cannot be said that $B_2(Z)$ is also stable since $B_2(Z)$ has missing terms and as mentioned in Section 2.2, the testing method is not applicable to such lacunary polynomials. But it can be generalized for any value of $N$. It can be shown that any $B'(Z^L,Z)$, $L \geq 2N + 1$ will have exactly the same Lagrange function and identically the same autocorrelation constraint equations except a few additional equations not involving $b'_{oo}$.

So all these 1-D polynomials $B'(Z^L,Z)$, $L = 2N + 1, 2N + 2, \ldots, \infty$ will have the same maximum value $b'_{oo}^*$ for $b'_{oo}^2$. Since all these polynomials and the respective polynomials $B(Z^L,Z)$ are lacunary, they may or may not be stable. These polynomials may have zeros on the unit circle as described in Section 2.2. But it can be certainly said that if $B(Z^{N+1},Z)$ is unstable, then all the polynomials $B(Z^L,Z)$, $L \geq 2N+1$ will be unstable. Thus the following observations about the
given 2-D polynomial $B(Z_1, Z_2)$ are given. Let $B(Z_1, Z_2) = \sum_{i=0}^{N} \sum_{j=0}^{N} b_{ij} Z_1^i Z_2^j$, be the given 2-D polynomial.

(i) If $B(Z^{N+1}, Z)$ is unstable, then the polynomials $B(Z^L, Z)$, $L = (2N + 1)$, $(2N + 2) \ldots \infty$ will all be unstable.

(ii) If $B(Z^{N+1}, Z)$ is stable, or marginally stable then any polynomial $B(Z^L, Z)$, $L \geq 2N+1$ need not necessarily be stable. But they may be marginally stable, since they will have the same maximum $b_{00}'$ as that of $B(Z^{N+1}, Z)$.

It can be easily seen that the above discussion holds good even for $B(Z, Z^L)$.

**Example 2:** Consider the following third degree 2-D polynomial $B(Z_1, Z_2)$

$$B(Z_1, Z_2) = \begin{bmatrix} 1 & -0.661 & 0.438 & -0.245 \\ -0.661 & 0.119 & -0.110 & 0.241 \\ 0.438 & -0.110 & -0.023 & -0.120 \\ -0.245 & 0.241 & -0.120 & 0.085 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_1^2 \\ Z_2^3 \end{bmatrix}$$

$B(Z_1, Z_2)$ has zeros on unit bicircle. The 1-D polynomial $B(Z^4, Z)$ is unstable. Also all the polynomials $B(Z^L, Z)$, $L = 7, 8 \ldots \infty$ are unstable confirming observation (i) above.
Example 3: Consider the polynomial

\[ B(Z_1, Z_2) = Z_1^2 Z_2^2 + 2Z_1 Z_2 (Z_1 + Z_2) + 4(Z_1^2 + Z_2^2) - 32Z_1 Z_2 - 33 (Z_1 + Z_2) + 85 \]
It is found that \( B(Z^3, Z) \) is marginally stable, so too are the other polynomials.

\[ B(Z^{L}, Z), L = 2N+1, 2N+2, \ldots, \infty. \]

2.5 TESTING THE LACUNARY 1-D FORM PRESERVING POLYNOMIALS FOR STABILITY

In the previous section it is concluded that if the form preserving 1-D polynomial \( B(Z^{N+1}, Z) \) is unstable then all the form preserving 1-D polynomials \( B(Z^L, Z), L \geq 2N+1 \) will also be unstable. But if \( B(Z^{N+1}, Z) \) is found to be stable or marginally stable, then \( B(Z^L, Z) \) \( L \geq (2N+1) \) may or may not be stable or marginally stable. In this section for simplicity a first degree (\( N = 1 \)) 2-D polynomial \( B(Z_1, Z_2) = b_{00} + b_{01}Z_1 + b_{11}Z_1Z_2 \) is considered and then if \( B(Z^2, Z) \) is stable it is checked, whether all the 1-D polynomials \( B(Z^L, Z), L \geq 3 \) will be stable or not.

By hypothesis

\[ B(Z^2, Z) = b_{00} + b_{01}Z + b_{11}Z^2 + b_{11}Z^3 \]
is a stable polynomial and so \( b_{00} \) is the maximum in magnitude among the polynomials having the same autocorrelation coefficients as \( B(Z^2, Z) \).

\[ B(Z^3, Z) = B(Z) = b_{00} + b_{01}Z + 0 + b_{01}Z^3 + b_{11}Z^4 \]  \hspace{1cm} (2.29)
Let the autocorrelation coefficients of $B(Z^2, Z)$ be denoted by $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ and $\gamma_4$.

And they can be written as follows,

\[ b_{00}^2 + b_{01}^2 + b_{11}^2 = \gamma_0 \]
\[ b_{00}b_{01} + b_{01}b_{11} = \gamma_1 \]
\[ b_{01}^2 = \gamma_2 \]
\[ b_{00}b_{01} + b_{01}b_{11} = \gamma_3 \]
\[ b_{00}b_{11} = \gamma_4 \] \hspace{1cm} (30)

Consider the following nonlacunary version $B'(Z)$ of the polynomial $B(Z)$ in (2.29),

\[ B'(Z) = b'_{00} + b'_{01}Z + xZ^2 + b'_{10}Z^3 + b'_{11}Z^4 \] \hspace{1cm} (2.31)

The stability test given in Section 2.2 can now be applied on the polynomial in (2.31).

Since $B'(Z)$ is non lacunary, a non-zero value for $x$ has to be found such that $B'(Z)$ has the same autocorrelation coefficient as $B(Z)$ and is also stable.

The auto correlation constraint equations are

\[ b'_{00}^2 + b'_{01}^2 + x^2 + b'_{10}^2 + b'_{11}^2 = \gamma_0 \]
\[ b'_{00}b'_{01} + b'_{01}x + xb'_{10} + b'_{10}b'_{11} = \gamma_1 \]
\[ b'_{00}x + b'_{01}b'_{10} + xb'_{11} = \gamma_2 \]
\[ b'_{00}b'_{10} + b'_{01}b'_{11} = \gamma_3 \]
\[ b'_{00}b'_{11} = \gamma_4 \] \hspace{1cm} (2.32)
Let $x = 0$ be set in (2.32) the autocorrelation coefficient equations of the type given in (2.30) can be got, which as mentioned in Section 2.4 are exactly the same as those of the polynomial $B(Z^2, Z)$ except one extra equation which does not have $b'_{00}$ in it. The Lagrange method of optimization will yield exactly the same values for $b'_B$s.

That is,

\[
\begin{align*}
b'_0 &= b_0 \\
b'_{01} &= b_{01} \\
b'_{10} &= b_{01} \\
b'_{11} &= b_{11}
\end{align*}
\]

and the maximum value of $b'_{00}$ is equal to $b_{00}$. But the polynomial $B(Z^3, Z)$ as given in (2.31) may not be stable since it is lacunary.

If $x$ is not set to 0, but if equations (2.32) are solved for $b'_{00}$, they effectively result in four equations containing four unknowns. These four equations are solved for the four unknowns namely $b'_{01}$, $x$, $b'_{10}$ and $b'_{11}$. It is reasonable to assume that a real solution exists with $x$ being infinitesimally small. Then maximum value of $b'_{00}$ can be got from the equation $\frac{\partial L(b'_{00}, \lambda_j)}{\partial b'_{00}} = 0$, by a suitable choice for $\lambda_j$.

The nonlacunary polynomial $B'(Z)$ thus obtained will be stable or marginally stable $B(Z)$ of (2.23) will not be a stable polynomial since in the family of $B'(Z)$ only one polynomial can be stable. If in the optimization process $x$ automatically takes a value of zero, then $B'(Z)$ and $B(Z)$ will be the same, and
so $B(Z)$ will be stable, or marginally stable. Similar arguments can be extended to any polynomial $B(Z^L, Z)$ $L \geq 3$ in which there may be more missing terms. The corresponding non-lacunary versions will have more $x_j$ which may turn out to be infinitesimally small or may vanish in the optimization process. So it can be concluded that any $B(Z^L, Z)$, $L \geq 3$ may be unstable or stable if $B(Z^2, Z)$ is stable. This is true with respect to any $N^{th}$ degree 2-D polynomial.

**Example 4:** Consider the 2-D polynomial $B(Z_1, Z_2) = Z_1Z_2 + 2(Z_1 + Z_2) + 2$ which has zeros on the unit bicircle. It is found that $B(Z^2, Z)$ is a stable polynomial where as $B(Z^3, Z)$ is an unstable polynomial.

**Example 5:** Consider the 2-D polynomial

$$B(Z_1, Z_2) = Z_1^2Z_2^2 + \sqrt{(3-1)}Z_1Z_2(Z_1 + Z_2) + \sqrt{(3-2)}(Z_1^2 + Z_2^2)$$

$$+ (100 + \sqrt{3}) Z_1Z_2 + (\sqrt{3-1})(Z_1 + Z_2) + 101$$

this has zeros on the unit bicircle. In this case also $B(Z^3, Z)$ is a stable polynomial where as $B(Z^2, Z)$ is a marginally stable polynomial since it has zeros on the unit circle.

**Example 6:** The polynomial $B(Z_1, Z_2) = Z_1Z_2 + 1.5Z_1 + 1.5Z_2 + 2$

$B(Z_1, Z_2)$ has zeros on the unit bicircle and has its form preserving polynomial $B(Z^2, Z)$ stable. It can be seen that in this case $B(Z^3, Z)$ is marginally stable.

### 2.6 A SIMPLE STABILITY TEST FOR 2-D POLYNOMIALS

In this section a very important Theorem will be proved to simplify greatly the stability testing of 2-D first quadrant polynomials. The same
arguments used to test the stability of 1-D polynomial in section (2.2), namely that the 1-D polynomial of a 'family' is stable or marginally stable if and only if its constant coefficient is the highest in magnitude. Even though a given 2-D polynomial is not factorizable, the same concept that it has a 'family' of polynomials of its own having the same autocorrelation coefficients as the given one is made use of. The 2-D polynomial of this 'family' having the highest magnitude for its constant coefficient can be assumed to be obtainable by maximizing the constant coefficient, subject to the 2-D autocorrelation coefficient constraints using computer-aided optimization though not algebraically. In this case if one does not take into account the autocorrelation constraint equations not containing the constant coefficient, one has the same number of unknown coefficients and so the optimum does exist. It is justifiable in not taking into account the constraint equations not containing the constant coefficient because if one views the computer aided maximization of the constant coefficient as equivalent to Lagrange multiplier method, these constraint equations do not contribute to any additional Lagrange multipliers.

So the following Theorem is stated and proved.

**Theorem 4**: Let $B(Z_1,Z_2)$ be a first quadrant 2-D polynomial written as

$$B(Z_1,Z_2) = \sum_{i=0}^{N} \sum_{j=0}^{N} b_{ij} Z_1^i Z_2^j.$$  

$B(Z_1,Z_2)$ is stable if and only if $B(Z^{2N+1},Z)$ and/or $B(Z, Z^{2N+1})$ is stable.

**Proof**: Let $B(Z_1,Z_2)$ be a non-lacunary 2-D polynomial of degree $N$ in both the variables. Then the 1-D form preserving polynomial $B(Z^{N+1},Z)$ will also be non-
lacunary. In the case of a non-lacunary 1-D polynomial, it is a stable polynomial provided its constant coefficient is the highest in magnitude among the family of 1-D polynomials having the same autocorrelation coefficients (Robinson, E.A, 1967). If $B(Z_1, Z_2)$ is a 2-D polynomial devoid of zeros on the unit bidisc $T^2$, then according to Mersereau’s Theorem cited in section 2.3, all 1-D polynomials $B(Z^L, Z)$, $L \geq N+1$ should be stable. According to observation (i) given in section (2.4), if the 1-D polynomial $B(Z^L, Z)$, $L = N+1$ is unstable, then all the polynomials $B(Z^L, Z)$, $L > 2N+1$ will also be unstable. If $B(Z^L, Z)$, $L = N+1$ is stable among the other polynomials $B(Z^L, Z)$, $L > N+1$ at least one polynomial has to be unstable if $B(Z_1, Z_2)$ has zeros on the unit bicircle. This one 1-D polynomial is $B(Z^{2N+1}, Z)$, since this polynomial is a representative of the 2-D polynomial $B(Z_1, Z_2)$ having the same 1-D autocorrelation coefficients as the 2-D autocorrelation coefficients of $B(Z_1, Z_2)$. This is because, in view of the discussion presented at the beginning of this section, if the coefficient $b_{00}$ of $B(Z_1, Z_2)$ is not the highest, the 1-D polynomial $B(Z^{2N+1}, Z)$ having the same $b_{00}$ as its constant coefficient will be unstable and if $b_{00}$ is the highest $B(Z^{2N+1}, Z)$ will be stable as discussed in section (2.2). The same discussion holds good for $B(Z, Z^{2N+1})$ and so if $B(Z_1, Z_2)$ has zeros on the unit bicircle $B(Z^{2N+1}, Z)$ and/or $B(Z, Z^{2N+1})$ will have to be unstable. In fact the stability condition (ii) of Theorem 3 is superfluous when once the condition (i) of the Theorem 3 is tested for.

Hence the proof of Theorem-4.

Example 7: Consider the 2-D polynomial $B(Z_1, Z_2)=1 - 1.2Z_2 + 0.5Z_2^2 - 1.5Z_1 + 1.8Z_1Z_2 - 0.75Z_1Z_2^2 + 0.6Z_1^2 - 0.72Z_1^2Z_2 + 0.2718Z_1^2Z_2^2$ (Huang, 1981, pp-129.b3)
This is a very critical polynomial since this 2-D polynomial is found to be barely unstable. But 1-D tests of Theorem-4 showed in a simple way that $B(Z^{2N+1}, Z)$ and $B(Z, Z^{2N+1})$ are clearly unstable. Jury-Marden algorithm was used by replacing $Z_i$ by $1/Z_i$ in the above 1-D polynomial.

For this above example, the Jury’s Mapping algorithm required checking the root distribution of fifty 1-D polynomials before it was established that it is an unstable polynomial.

2.7 STABILITY TESTING OF NON SYMMETRIC HALF PLANE (NSHP) FILTERS

The NSHP filters can be tested for stability by transforming them into quarter plane filters using the method suggested by Connor, Huang (1978) and applying Theorem 4.

2.8 CONCLUSIONS

In this chapter a new test for stability of a 1-D polynomial based on Lagrange method of optimization is dealt only to get insight into the method. Mersereau's method of testing 2-D polynomials for stability is dealt by using what are known as 1-D form preserving polynomials. Some new results were arrived for simplifying the 2-D polynomial stability testing. A Theorem (Theorem -4) was proved which will be very useful in considerably reducing the complexity of testing for stability of 2-D polynomials.
In this chapter an attempt has been made to present a method which will reduce the complexity of testing of 2-D polynomials for stability. The algebraic method available now in the literature Huang (1981) is not efficient particularly when N>4. This is because the complexity of computations involved and also the inaccuracies that creep in a big way due to the finite precision involved in the computers. So whenever the degree of the polynomial N is large numerical methods like row and column algorithm or row and column concatenation algorithm are more practical and are recommended. Even these methods do not assure (in some cases) that the polynomial has no zeros on the unit bicircle $T^2$. So these methods to a large extent, test only the necessary condition for the non existence of zeros on $T^2$. They are not exact enough to test for sufficiency. This is so because in the case of row and column concatenation test one has to test infinite number of 1-D polynomials for root distribution for it to be sufficient also.

In this chapter the row and column concatenation method is simplified and it reduces the number of 1-D polynomials to be tested for root distribution to one in order to check if the given first quadrant 2-D polynomial is devoid of zeros on the unit bicircle $T^2$. With the result, the method of testing presented in this chapter tests for both the necessary and sufficient conditions. The 1-D polynomial root distribution can be tested by Jury's table method (Huang, 1981).