CHAPTER 4

APPLICATION OF SCHWARZ-CHRISTOFFEL TRANSFORMATION FOR ESTIMATING THE EFFECT OF SEPTUM THICKNESS ON IMPEDANCE OF A SYMMETRIC TEM CELL

4.1 INTRODUCTION

This chapter presents a method of finding the effect of septum thickness on characteristic impedance of a symmetric TEM cell using the Schwarz-Christoffel transformation. In this formulation, the cross-section of the TEM cell is transformed into a parallel plate configuration through three successive transformations. The characteristic impedance is determined from the capacitance of this parallel plate configuration. As the thickness of the septum increases, the spacing between the two parallel plates decreases from its value corresponding to zero septum thickness. The amount of reduction in spacing is chosen so as to give the correct thickness for the septum for impedance calculation.

Equating the real and imaginary parts of the incomplete elliptic integrals and associated complex quantities appearing in the transformation, equipotential contours and electric flux lines around the septum are obtained. The thickness and width of the septum are found from the co-ordinates on the equipotential line which is coincident with the septum boundaries. Numerical results of the variation in characteristic
impedance with septum thickness are presented and are found in good agreement with those obtained using the methods described in Chapters 2 and 3.

4.2 CO-ORDINATES ON THE EQUIPOTENTIAL AND ELECTRIC FLUX LINES

A cross-sectional view of uniform region of a rectangular symmetric TEM cell is shown in Fig.4.1. In the first instance, it is assumed that the inner conductor is of negligible thickness and the medium is air. Because of the symmetry with respect to x and y axes, the capacitance between the upper conductor AFED and the inner conductor BC will be calculated. The total capacitance is then twice this value. The characteristic impedance is expressed in terms of the line capacitance. The region ABCDEF is mapped into the upper half of a complex T-plane using the Schwarz-Christoffel transformation (Tippet and Chang 1976, Byrd and Friedman 1971):

\[ Z = X + jY = B \Phi F(\phi, k) + C z \]  

\[ \Phi = \sin^{-1} T \] \[ T = T_r + jT_i \]

Use of Equation (4.3) and comparison of the corresponding points on Z and T-planes lead to the following relations
Fig. 4.1 Cross-section of a TEM cell

\[ \frac{1}{k} \quad -1 \quad -\alpha \quad 0 \quad \alpha \quad 1 \quad \frac{1}{k} \]

\[ F \quad V=0 \quad A \quad B \quad V=V_0 \quad C \quad D \quad V=0 \quad E \]

Fig. 4.2 Complex T-plane for upper half of TEM cell

\[ -\frac{1}{\alpha} \quad -1 \quad 0 \quad 1 \quad \frac{1}{\alpha} \]

\[ V=0 \quad A \quad B \quad V=V_0 \quad C \quad D \quad V=0 \]

Fig. 4.3 Complex S-plane

Fig. 4.4 Complex W-plane of parallel plate configuration
\[ C = 0 \] \hspace{1cm} \text{(4.4)}

\[ B = \frac{a}{K(k)} \] \hspace{1cm} \text{(4.5)}

\[ F(\sin^{-1} \alpha, k) = K(k) \cdot \frac{w}{a} \] \hspace{1cm} \text{(4.6)}

\[ \frac{K(k')}{K(k)} = \frac{b}{a} \] \hspace{1cm} \text{(4.7)}

where \( K(k), K(k') \) = complete elliptic integrals of first kind with moduli \( k \) and \( k' \), respectively, \( \alpha \) is a constant \((0 < \alpha < 1)\) and

\[ k' = (1-k^2)^{1/2} \] \hspace{1cm} \text{(4.8)}

From Equations (4.1), (4.4) and (4.5)

\[ Z = a \frac{F(\phi, k)}{K(k)} \] \hspace{1cm} \text{(4.9)}

For convenience, Fig.4.2 is further transformed from the complex \( T \)-plane to a complex \( S \)-plane defined by

\[ S = T/\alpha = S_r + jS_i \] \hspace{1cm} \text{(4.10)}

as shown in Fig.4.3. Finally, the upper half of the \( S \)-plane is mapped into a parallel plate configuration (Fig 4.4) in \( W \)-plane using the transformation

\[ W = U + jV = A_w F(\phi, \alpha) + B_w \] \hspace{1cm} \text{(4.11)}

where

\[ \phi = \sin^{-1} S \] \hspace{1cm} \text{(4.12)}

and \( A_w, B_w \) are arbitrary constants to be determined. Use of Equation (4.11) and comparison of the corresponding points on \( S \)-plane and \( W \)-plane lead to the following Equations
\[ A_w = -1/k(\alpha) \quad \ldots (4.13) \]
\[ B_w = jV'_o \quad \ldots (4.14) \]
\[ V'_o = \frac{K(\alpha')}{K(\alpha)} \quad \ldots (4.15) \]

where \( V'_o \) is the plate spacing in \( W \)-plane and

\[ \alpha' = \sqrt{1 - \alpha^2} \quad \ldots (4.16) \]

Using Equations (4.13)-(4.15) in (4.11) the expression for the complex potential function is obtained as

\[ W = U' + jV' = - \frac{1}{K(\alpha)} F(\phi, \alpha) + j \frac{K(\alpha')}{K(\alpha)} \quad \ldots (4.17) \]

Writing complex arguments \( \phi \) and \( \theta \) in terms of real and imaginary parts:

\[ \phi = \phi_r + j\phi_i \quad \ldots (4.18) \]
\[ \theta = \theta_r + j\theta_i \quad \ldots (4.19) \]

the incomplete elliptic integrals can be separated in terms of real and imaginary parts (Byrd and Friedman 1971):

\[ F(\phi, k) = F(\phi_r + j\phi_i, k) = F(\xi, k) + jF(\eta, k') \quad \ldots (4.20) \]
\[ F(\theta, k) = F(\theta_r + j\theta_i, k) = F(\phi, k) + jF(\psi, \alpha') \quad \ldots (4.21) \]

with

\[ T = T_r + jT_i = \sin \phi = \sin \phi_r \cosh \phi_i + j \cos \phi_r \sinh \phi_i \quad \ldots (4.22) \]
\[ S = S_r + jS_i = \sin \phi_r \cosh \phi_i + j \cos \phi_r \sinh \phi_i \]

....(4.23)

Separating real and imaginary parts from the above Equations and carrying out simplification, it can be shown that

\[ T_r = \sin \phi_r \cosh \phi_i = \frac{\sin \xi (1 - k^2 \sin^2 \eta)^{1/2}}{\cos^2 \eta + k^2 \sin^2 \xi \sin^2 \eta} \]

....(4.24)

\[ T_i = \cos \phi_r \sinh \phi_i = \frac{\cos \xi \sin \eta \cos \eta (1 - k^2 \sin^2 \xi)^{1/2}}{\cos^2 \eta + k^2 \sin^2 \xi \sin^2 \eta} \]

....(4.25)

\[ S_r = \sin \phi_r \cosh \phi_i = \frac{\sin \delta (1 - \alpha^2 \sin^2 \nu)^{1/2}}{\cos^2 \nu + \alpha^2 \sin^2 \delta \sin^2 \nu} \]

....(4.26)

\[ S_i = \cos \phi_r \sinh \phi_i = \frac{\cos \delta \sin \nu \cos \nu (1 - \alpha^2 \sin^2 \delta)^{1/2}}{\cos^2 \nu + \alpha^2 \sin^2 \delta \sin^2 \nu} \]

....(4.27)

From Equations (4.17) and (4.21) the Equation for the complex potential function is obtained as

\[ W = U' + jV' = -\frac{1}{K(\alpha)} F(\delta, k) + j \frac{F(\nu, \alpha')}{K(\alpha)} \]

....(4.28)

in the transformed parallel plate configuration. Separating the real and imaginary parts and using Equation (4.15), it is found that

\[ \frac{V'}{V_o} = 1 - \frac{F(\nu, \alpha')}{K(\alpha')} \]

....(4.29)
and
\[ U' = -\frac{F(\delta, \alpha)}{K(\alpha)} \] ....(4.30)

Contours \( V'/V_o' = \) constant in Equation (4.29) represent the equipotential lines and those of \( U' = \) constant in Equation (4.30) represent the electric flux lines in \( W \)-plane. Here \( V'/V_o' = 0 \) represents the outer conductor and \( V'/V_o' = 1 \) represents the inner conductor of zero thickness at potential \( V_o' \), so that \( 0 \leq V'/V_o' \leq 1 \), \( \pi/2 \geq \nu \geq 0 \). For the electric flux lines, the range of \( U' \) is \(-1 \leq U' \leq 1\) for \(-\pi/2 \leq \delta \leq \pi/2\). Therefore, equipotential contours are obtained for a given value of \( \nu \) and varying \( \delta \) in the range \(-\pi/2 \leq \delta \leq \pi/2\). Similarly, the electric flux lines are obtained for a given value of \( \delta \) and varying \( \nu \) in the range \( \pi/2 \geq \nu \geq 0 \). In order to construct the equipotential contours and the flux lines in \( XY \) plane, following procedure is used. The Equations (4.24) and (4.25) lead to a set of simultaneous non-linear Equations

\[ F_1(\xi, \eta) = T_r \left[ \cos^2 \xi + k^2 \sin^2 \xi \sin^2 \eta \right] - \sin \xi \left( 1-k^2 \sin^2 \eta \right)^{1/2} = 0 \] ....(4.31)

\[ F_2(\xi, \eta) = T_1 \left[ \cos^2 \xi + k^2 \sin^2 \xi \sin^2 \eta \right] - \cos \xi \sin \eta \cos \eta \left( 1-k^2 \sin^2 \xi \right)^{1/2} = 0 \] ....(4.32)

The solution of the simultaneous non-linear Equations (4.31) and (4.32) gives the unknown \( \xi, \eta \) of the \( Z \)-plane when \( T_r \) and \( T_1 \) are known. This solution is obtained from the minimisation of the following objective function

\[ F(\xi, \eta) = \frac{1}{2} \left[ F_1^2(\xi, \eta) + F_2^2(\xi, \eta) \right] \] ....(4.33)
The coordinates of the point of minimum of the function $P(\xi, \eta)$ should satisfy the Equations (4.31) and (4.32). In the minimisation procedure a starting point is selected and appropriate direction of search is evaluated by computing the first derivative. The first derivative of $P$ is zero at the point of minimum in the range $0 \leq \xi \leq \pi/2$ and $\pi/2 \geq \eta \geq 0$. This corresponds to the first quadrant of the $Z$-plane. The one-dimensional search is continued until a minimum is located and a convergence is achieved.

For given ratios $b/a$ and $w/a$ of the TEM cell, values of $k, k', K(k), \alpha, \alpha'$ and $K(\alpha')$ can be determined from Equations (4.6) - (4.8) and (4.16). For a given equipotential contour $V'/V'_0$, the value of $\nu$ is obtained from Equation (4.29). By varying $\delta$ between $\pi/2$ to $-\pi/2$ the real and imaginary parts of $S$ are determined from Equations (4.26) and (4.27) with a knowledge of $\alpha$ and $\alpha'$. Subsequently, the real and imaginary parts of $T$ are found through the Equation (4.10). The $x,y$ coordinates on an equipotential contour are determined by solving Equation (4.33) following the above procedure and then using Equations (4.20) and (4.9). Similarly, the electric flux lines are obtained by fixing $\delta$ and varying $\nu$ from $\pi/2$ to $0$. When the ratio $V'/V'_0$ increases, the corresponding equipotential contour approaches the septum surface, when $V'/V'_0$ approaches unity.

4.3 EXPRESSION FOR CHARACTERISTIC IMPEDANCE

From Fig.4.4 it is evident that the capacitance per unit length of the TEM cell is equal to twice that of upper half of the TEM cell for zero thickness of the inner conductor and is given by the ordinary parallel plate capacitor formula, that is,
Using Equations (4.15) and (4.34), the characteristic impedance of the symmetric TEM cell with zero thickness of the inner conductor is expressed by

\[ Z_0 = \frac{1}{\mu_0 \varepsilon_0} \frac{C}{V'} \]

\[ = 30\pi V' \]

\[ = 30\pi \frac{K(\alpha')}{K(\alpha)} \]

\[ \ldots (4.35) \]

For taking the effect of thickness of the inner conductor of a TEM cell into account, the infinitely thin inner conductor is replaced by a conductor which coincides with one of the equipotential contours corresponding to \( V'/V'_o \to 1 \). An equipotential contour \( V' = V_1 \) is now taken to be the cross-section of the actual inner conductor in the \( W \)-plane (Fig. 4.4). The separation \( \Delta V \) of the two contours \( V' = V'_o \) and \( V' = V_1 \) in the \( W \)-plane has to be chosen so as to give the correct thickness \((2t)\) of the inner conductor cross-section. Since the equipotential representing the inner conductor cross-section passes through the \( x \)-axis at a distance \( w' > w \) (Fig. 4.5), the width of the inner conductor has been modified from its zero thickness width \( 2w \) to a new width \( 2w' \) (Collin 1960). If \( \Delta V << V'_o \), the capacitance per unit length in the \( W \)-plane is changed by a very small amount (Collin 1960):

\[ \Delta C = \frac{4\varepsilon_0 \Delta V}{V'_o (V'_o - \Delta V)} \]

\[ \ldots (4.36) \]
The total capacitance per unit length of the TEM cell having finitely thick inner conductor corresponding to $\Delta V$ is

$$C = \frac{4\varepsilon_0}{V_o - \Delta V} \quad \text{....(4.37)}$$

Using Equations (4.35) and (4.37) the expression for the characteristic impedance of the symmetric TEM cell becomes:

$$Z_o = 30n(V_o - \Delta V) = 30n \frac{K(\alpha')}{K(\alpha)} \left[ \frac{V}{V_o} \right] \quad \text{....(4.38)}$$

The thickness $2t$ and width $2w'$ of the inner conductor can be determined from the y and x intercepts, respectively, of the contour of equipotential in XY-plane. The characteristic impedance for a given thickness $2t$ and modified width $2w'$ can be calculated from Equation (4.38) for corresponding values of $\Delta V = V_o - V_1$, $K(\alpha)$ and $K(\alpha')$.

4.4 RESULTS AND DISCUSSION

A symmetrical rectangular TEM cell with the ratio $b/a=0.5$ and $w/a=0.6$ is chosen. The equipotential lines are shown in Fig.4.5 for $V'/V_o'=0.1$, $0.5$, $0.9$ and $0.95$. Because of symmetry only equipotentials in the one quadrant are drawn in the cross-section of the cell. It is seen that in the region close to the septum the equipotential contour becomes almost flat near the centre of the septum and curve towards the edges. Thus in the vicinity of the septum the equipotential lines are taking approximately the shape of the periphery of the septum cross-section. The thickness $2t$ of the septum is determined from the vertical axis intercept of a
Fig. 4.5 Equipotential lines in the one quarter cross-section of a symmetric TEM cell ($b/a = 0.5, w/a = 0.6$).
equipotential contour (Fig. 4.6a), which represents the boundaries of the septum. The corresponding modified width $2w'$ of the septum is determined from the horizontal axis intercept of the same equipotential contour (Fig. 4.6b). The equipotentials corresponding to $V'/V_0 = 0.950, 0.970, 0.980, 0.990$ and $0.995$ are chosen for the septum boundaries to select correspondingly different values of $t/w'$.

A plot of characteristic impedance $Z_0$ vs $t/w'$ computed by this method is shown in Fig. 4.7. Results for $Z_0$ obtained by the methods of Chapters 2 and 3 (viz. energy concept and method of moment, respectively) are also compared with those of present chapter. In all the cases $Z_0$ decreases with increase of thickness of the septum. The results obtained from all the three approaches closely agree each other for zero and small thickness of the septum. The values of the impedance determined in chapter 2 by using energy concept is found some what higher than those of other two methods as the thickness of the septum increases. A typical increase in impedance value in chapter 2 is $0.5$ ohm as compared with the MOM result for a value of $t/w'=0.02$. This is due to the fact that in chapter 2 the charges on the side surfaces of the septum were neglected and hence the total charge in the impedance computation was less. Fig. 4.8 shows that the electric flux lines are parallel in the central region of the cross-section. This indicates that the fields are uniform in this region as it was shown in chapter 3. The field distribution becomes non-uniform towards the edges of the septum.

Since a major portion of the equipotential line representing the contour of the septum is flat and is rounded near the edges, this analysis gives fairly
accurate results for the characteristic impedance of the rectangular symmetric TEM cell with septum having finite thickness with rounded corners. The rounded corners of the equipotential contour are often a better approximation of the cross-section of the inner conductor than an ideal rectangular cross-section as explained by Collin (1960). The analysis has the advantage that the same mapping function used to map the infinitely thin centre conductor, also maps the desired equipotential contour. The final mapping of the conductor cross-section is normally a simple contour in the final mapped representation of the TEM cell cross-section. The boundary conditions are automatically satisfied. For the design of TEM cell or rectangular co-axial line, the effect of finite thickness of the septum on the characteristic impedance can be estimated by this method. The method is computationally very efficient.
Fig. 4.6a Equipotential lines in the vicinity of inner conductor (one quarter domain, Expanded y-scale, b/a=0.5 and w/a=0.6)
Fig. 4.6b Equipotential lines in the vicinity of inner conductor (one quarter domain, Expanded x and y - scales, b/a=0.5 and w/a=0.6)
Fig. 4.7 Variation of characteristic impedance with septum thickness for a symmetric TEM cell (b/a=0.5 and w/a=0.6)
Fig. 4.8 Electric flux lines in one quarter cross-section of a symmetric TEM cell (b/a=0.5, w/a=0.6)