CHAPTER 7

TRANSIENT SOLUTIONS OF A SOFTWARE MODEL WITH IMPERFECT DEBUGGING AND GENERATION OF ERRORS BY TWO SERVERS

7.1 INTRODUCTION

Software is a package which transforms a discrete set of inputs into a discrete set of outputs. As these software are developed by human and may be imperfect, since there is a difference between what the software package can do and what the user wants it to do. This deviation is called a software fault. There are several software reliability models available in the literature. Jelinski and Moranda (1972) were the first to introduce such a model. Later similar models were considered by Shooman (1977), Musa (1975) and Littlewood (1981). Further the problem of imperfect debugging for non-homogeneous Poisson process was given by Goel and Okumoto (1979). Several software reliability growth models have been proposed (modified exponential, S-shaped and inflection S-shaped) by Shanthikumar (1983), Yamada and Osaki (1983,1984). The main assumption in their models is that no new errors were introduced while removing the error. Subsequently Kuo and Kremer (1983) have relaxed this assumption by incorporating the concept of imperfect debugging and error generation. They have assumed that the generation of errors may lead the software to have infinite number of errors. Based on these results, birth-death process was easily described by Kendall (1948) followed by Chiang (1968). The purpose of this chapter, is to consider a software model where it is tested by two servers, the first M errors being debugged by the first server and the
remaining (M+1) to N errors by the second server. The transient solutions of the probabilities of the number of errors remaining in the software, mean number of errors and the like are obtained.

The layout of this chapter is as follows. In section 7.2, we describe the model under consideration while in section 7.3, the governing equations with their notation and solutions are given. The last section deals with a numerical example.

7.2 DESCRIPTION OF THE MODEL

In this chapter, we consider a software model with the following assumptions:

(i) The number of faults being finite (N).

(ii) The failure rate is proportional to the number of faults remaining in the software.

(iii) Debugging is imperfect and error generation will never lead the software to have infinite errors.

(iv) The software is tested by two servers with the first M errors being debugged by the first server and the remaining M+1 to N errors by the second server.

(v) The software has k faults initially and each time when a failure occurs, the fault causing that failure is instantaneously detected and further the failure rate of the software is equally affected by the errors remaining in the software.
When a failure occurs, instantaneously repair starts with the following probabilities:

(a) the fault content is reduced by one by the first (second) server with probability \( p_1 (p_2) \), \( p_1 \leq p_2 \); 

(b) the fault content remains unchanged with probability \( \Psi \) and 

(c) the fault content is increased by one by the first (second) server with probability \( \lambda_1 (\lambda_2) \), \( \lambda_1 \geq \lambda_2 \)

where

\[
\mu_1 + \Psi + \lambda_1 = 1, \quad \mu_2 + \Psi + \lambda_2 = 1, \quad \mu_1 \gg \Psi \gg \lambda_1 \text{ and } \mu_2 \gg \Psi \gg \lambda_2.
\]

Under these assumptions, we obtain the transient solutions of probabilities of the number of errors in the software, mean number of errors and the expected number of failures at any time. The model is also illustrated with a numerical example.

7.3 NOTATION, GOVERNING EQUATIONS AND THEIR SOLUTIONS

Let \( k \) be the initial fault content and \( N \) be the maximum fault content. Further let \( P_n(t) \) be the probability of \( n \) faults in the software at time \( t \) with \( \alpha \) being the failure rate per remaining software error or proportionality constant. Also let \( f(s) \) be the Laplace transforms of \( f(t) \).

Now using simple probabilistic arguments the differential-difference equations are given by
\[ P_0'(t) = \mu_1 \alpha P_1(t) \] (7.1)
\[ P_1'(t) = -(\alpha - \alpha' \Psi) P_1(t) + 2\alpha \mu_1 P_2(t) \] (7.2)
\[ P_n'(t) = -(n\alpha - n\alpha' \Psi) P_n(t) + (n-1) \alpha \lambda_1 P_{n-1}(t) + (n+1) \alpha \mu_1 P_{n+1}(t), \quad 2 \leq n \leq M-1 \] (7.3)
\[ P_M'(t) = -(M\alpha - M\alpha' \Psi) P_M(t) + (M-1) \alpha \lambda_1 P_{M-1}(t) + (M+1) \alpha \mu_2 P_{M+1}(t) \] (7.4)
\[ P_{n+1}'(t) = -(n\alpha - n\alpha' \Psi) P_n(t) + (n-1) \alpha \lambda_2 P_{n-1}(t) + (n+1) \alpha \mu_2 P_{n+1}(t), \quad M + 1 \leq n \leq N-1 \] (7.5)
\[ P_N'(t) = -(N\alpha - N\alpha' \Psi) P_N(t) + (N-1) \alpha \lambda_2 P_{N-1}(t) \] (7.6)

with their initial conditions given by
\[ P_k(0) = 1, \quad P_n(0) = 0 \text{ for } n \neq k, \quad (0 < n < N). \]

Now for studying the operating characteristics of the model, we need to calculate \( P_n(t), \quad 0 \leq n \leq N \). Hence we solve the above system of equations (7.1-7.6) by Laplace transforms which reduce to
\[ A(s) P(s) = I \]

where \( A(s) \) is a \((N+1 \times N+1)\) matrix of coefficients, \( P(s) \) is a \((N+1 \times 1)\) column vector in \( P_n(s) \) and \( I \) is a \((N+1 \times 1)\) column vector having unity in the \((k+1)\)th position and zero elsewhere.

Therefore one can determine \( P_n(s) \) as
\[ P_n(s) = \frac{|A_{n+1}(s)|}{|A(s)|}, \quad 0 \leq n \leq N \] (7.7)

where \(|A_{n+1}(s)|\) is obtained from \(|A(s)|\) by replacing \((n+1)\)th column by \( I_{k+1} \). Now \(|A(s)|\) can be expressed as \( s |D(s)| \) where \(|D(s)|\) is the determinant of a real, symmetric tri-diagonal matrix \( D(s) \) of order \( N \) with
negative off-diagonal elements. The diagonal elements are \((s + \alpha - \alpha \Psi)\), \((s + 2\alpha - 2\alpha \Psi)\), \(\ldots\), \((s + M\alpha - M\alpha \Psi)\), \(\ldots\), \((s + N\alpha - N\alpha \Psi)\) and their off-diagonal elements given by

\[
-\sqrt{2\alpha^2 \mu_1 \lambda_1}, -\sqrt{6\alpha^2 \mu_1 \lambda_1}, \ldots, -\sqrt{M(M-1) \lambda_1 \mu_1 \alpha^2},
\]

\[
-\sqrt{M(M+1) \alpha^2 \lambda_1 \mu_2}, -\sqrt{(M+1)(M+2) \alpha^2 \lambda_1 \mu_2}, \ldots, -\sqrt{N(N-1) \alpha^2 \lambda_2 \mu_2}.
\]

Hence the roots of the polynomial of \(|D(s)|\) is of degree \(N\) in \(s\) and are negative of the eigen values of matrix \(D\) obtained by replacing \(s\) with zero and is given by

\[
\begin{bmatrix}
1 & 2 & \ldots & M & \ldots & N \\
1 & \alpha - \alpha \Psi & -\sqrt{2\alpha^2 \lambda_1 \mu_1} & 0 & 0 \\
2 & -\sqrt{2\alpha^2 \lambda_1 \mu_1} & 2\alpha - 2\alpha \Psi & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
D = M & 0 & 0 & M\alpha - M\alpha \Psi & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
N & 0 & 0 & 0 & N\alpha - N\alpha \Psi
\end{bmatrix}
\]

It is observed that the eigen values of \(D\) are real and distinct and \(D\) is positive definite. Now let \(\delta_n (1 \leq n \leq N)\) denote the eigen values of \(D\) and let \(|A(s)|\) be written as

\[
|A(s)| = s^{N} \prod_{n=1}^{N} (s + \delta_n)
\]
so that
\[ P_m(s) = \frac{|A_{m+1}(s)|}{\sum_{n=1}^{N} \frac{a_{0n}}{s + \delta_n}}, \quad 0 \leq m \leq N. \] (7.8)

Hence we have
\[ P_0(s) = \frac{a_0}{s} + \sum_{n=1}^{N} \frac{a_{0n}}{s + \delta_n} \quad \text{and} \]
\[ P_m(s) = \sum_{n=1}^{N} \frac{a_{mn}}{s + \delta_n}, \quad 1 \leq m \leq N \] (7.9) (7.10)

where
\[ a_0 = \frac{|A_1(0)|}{\prod_{j=1}^{N} \delta_j} \quad \text{and} \]
\[ a_{mn} = \frac{|A_{m+1}(\delta_n)|}{\prod_{j=1}^{N} (\delta_j \cdot \delta_n)}, \quad 0 \leq m \leq N \] (7.11)
\[ \delta_n = \prod_{j=1}^{N} (\delta_j - \delta_n), \quad 1 \leq n \leq N. \]

Taking inverse Laplace transforms in (7.9) and (7.10) we obtain
\[ P_0(t) = \frac{|A_1(0)|}{\prod_{n=1}^{N} \delta_n} - \sum_{n=1}^{N} \frac{|A_1(\delta_n)| \exp(-\delta_n t)}{\prod_{j=1 \neq n}^{N} (\delta_j - \delta_n)} \] (7.12)
As $\delta_n (1 \leq n \leq N)$ are all positive, it may be observed that as $t \to \infty$, $P_m(t)$ vanishes for $1 \leq m \leq N$ which is also evident from (7.1-7.6). Using the expression for transient probabilities one may define the following characteristic measures.

(i) The mean number of faults remaining in the software at time $t$ given by

$$\sum_{n=0}^{N} n P_n(t)$$

(ii) The expected number of failures at time $t$ given by

$$\int_{0}^{t} \sum_{n=0}^{N} n P_n(x) \, dx = \begin{cases} k(1-e^{(\mu_1+\lambda_1)x})/(\mu_1-\lambda_1) & \text{(for the first server)} \\ k(1-e^{(\mu_2+\lambda_2)x})/(\mu_2-\lambda_2) & \text{(for the second server).} \end{cases}$$

Similarly other measures can also be defined.

### 7.4 NUMERICAL EXAMPLE

The above model is explained by taking $M=2, N=3, k=1, \alpha=1, \lambda_1=0.10, \lambda_2=0.05, \mu_1=0.60, \mu_2=0.65$ and $\Psi=0.30$. Using these values for the parameter involved in the model, the matrices $A(s)$ and $D(s)$ reduce to
\[
A(s) = \begin{bmatrix}
0 & 1 & 2 & 3 \\
0 & s - \mu_1 \alpha & 0 & 0 \\
1 & 0 & s + \alpha - \mu_1 \Psi & -2\alpha \mu_1 \\
2 & 0 & -\alpha \lambda_1 & s + 2\alpha - 2\alpha \Psi \\
3 & 0 & 0 & s + 2\alpha - 2\alpha \Psi
\end{bmatrix}
\]
\[
D(s) = \begin{bmatrix}
1 & 2 & 3 \\
1 & s + \alpha - \alpha \Psi & -\sqrt{2\alpha^2 \mu_1 \lambda_1} & 0 \\
2 & -\sqrt{2\alpha^2 \lambda_1 \mu_1} & s + 2\alpha - 2\alpha \Psi & -\sqrt{6\alpha^2 \mu_2 \lambda_2} \\
3 & 0 & -\sqrt{6\alpha^2 \mu_2 \lambda_2} & s + 3\alpha - 3\alpha \Psi
\end{bmatrix}
\]

with the corresponding eigen values given by

\[
\delta_1 = 0.537364, \, \delta_2 = 1.334432, \, \delta_3 = 2.328200
\]

obtained by putting \( s=0 \) in \( D(s) \). With these parameters, we have obtained the transient probabilities, the mean number of faults remaining in the software and the expected number of failures at time \( t \). These characteristics are shown in Tables 7.1, 7.2, and 7.3 respectively. From Table 7.1, we observe that the probability of no errors in the software increases as time \( t \) increases, as expected. Finally from Table 7.2, we observe that the mean number of faults remaining in the software decreases with time \( t \).
TABLE 7.1

Probability of faults remaining at time $t$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$P_0(t)$</th>
<th>$P_1(t)$</th>
<th>$P_2(t)$</th>
<th>$P_3(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1.0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.440409</td>
<td>0.521008</td>
<td>0.037265</td>
<td>0.001317</td>
</tr>
<tr>
<td>2</td>
<td>0.679360</td>
<td>0.288519</td>
<td>0.030550</td>
<td>0.001570</td>
</tr>
<tr>
<td>3</td>
<td>0.812101</td>
<td>0.164466</td>
<td>0.020060</td>
<td>0.001172</td>
</tr>
<tr>
<td>4</td>
<td>0.891931</td>
<td>0.095019</td>
<td>0.012293</td>
<td>0.000756</td>
</tr>
<tr>
<td>5</td>
<td>0.937430</td>
<td>0.055236</td>
<td>0.007332</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0.963475</td>
<td>0.032199</td>
<td>0.004323</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0.978665</td>
<td>0.018794</td>
<td>0.002537</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>0.987527</td>
<td>0.010976</td>
<td>0.001485</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>0.992708</td>
<td>0.006412</td>
<td>0.000868</td>
<td>0</td>
</tr>
</tbody>
</table>
TABLE 7.2
Mean number of faults remaining at time $t$

<table>
<thead>
<tr>
<th>$t$</th>
<th>Mean number of faults</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0</td>
</tr>
<tr>
<td>1</td>
<td>0.631482</td>
</tr>
<tr>
<td>2</td>
<td>0.393889</td>
</tr>
<tr>
<td>3</td>
<td>0.244226</td>
</tr>
<tr>
<td>4</td>
<td>0.150991</td>
</tr>
<tr>
<td>5</td>
<td>0.093218</td>
</tr>
<tr>
<td>6</td>
<td>0.057510</td>
</tr>
<tr>
<td>7</td>
<td>0.035469</td>
</tr>
<tr>
<td>8</td>
<td>0.021871</td>
</tr>
<tr>
<td>9</td>
<td>0.013486</td>
</tr>
<tr>
<td>t</td>
<td>First server</td>
</tr>
<tr>
<td>----</td>
<td>--------------</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.786939</td>
</tr>
<tr>
<td>2</td>
<td>1.264241</td>
</tr>
<tr>
<td>3</td>
<td>1.553740</td>
</tr>
<tr>
<td>4</td>
<td>1.729329</td>
</tr>
<tr>
<td>5</td>
<td>1.835830</td>
</tr>
<tr>
<td>6</td>
<td>1.900427</td>
</tr>
<tr>
<td>7</td>
<td>1.939605</td>
</tr>
<tr>
<td>8</td>
<td>1.963369</td>
</tr>
<tr>
<td>9</td>
<td>1.977782</td>
</tr>
</tbody>
</table>