CHAPTER 5

APPLICATION OF BOUNDARY ELEMENT METHODS FOR EVALUATION OF STRESS CONCENTRATIONS AROUND HOLES IN FRP PLATES UNDER IN-PLANE STATIC LOADS

5.1 INTRODUCTION

The estimation of stress concentrations around holes in composite plates and strips under in-plane loading is an important problem in aerospace and other fields of general engineering. The stress concentrations obviously depend upon the size and shape of the holes as well as the relative sizes of the hole and the plate. Classical continuum mechanics solutions for isotropic cases have been extended in literature to orthotropic and anisotropic materials. The application of Finite Element Methods to these problems of stress concentrations has also been established very well in literature. However recent studies in literature indicate that Boundary Element Methods (BEM) perhaps offer more effective methods of solutions for such problems. Recent literature refers to some investigations on BEM for isotropic materials. It is felt that not much of work has been done so far in application of BEM to composite plates. This study is a modest contribution in this direction.

Homogeneous elastic field problems may be alternately posed as infinite systems of boundary integral equations through suitable family of Kernel functions and integration by parts. As compared with Finite Element Method, Boundary Element Method is attractive in that freedoms occur only on the boundary of the problem, lending relative ease in problem definition and a possible reduction in overall freedoms as well as numerical calculations. The FEM technique is expensive and considerable effort is
generally needed to apply it to get stiffness degradation under cyclic reversed stresses in FRP.

The present work contains a review of the existing literature on Boundary Element Method applied to problems associated with elasto-plasticity of FRP structures. In case of FRP structures with cut-outs, stress concentrations occurring around holes would depend on the configuration of the hole, the type of load (static, dynamic or impact) and the geometric parameters.

In the present chapter, the stress concentration factors around holes in isotropic plates are determined using the fictitious stress method. Further an attempt is made to analyse an orthotropic plate. Experiments are conducted on FRP plates and the results are compared with the numerical values obtained.

5.2 BRIEF LITERATURE SURVEY

The stress concentration factors for holes, cutouts, notches etc., in plates subjected to various types of loadings have been obtained by R.E. Peterson [37]. He has solved two dimensional and three dimensional problems analytically as well as experimentally. The effect of circular holes on stress distribution has been discussed by S.Timoshenko and J.N.Goodier [38] analytically. R.M. Jones [39] has explained the various reasons why the stress concentration factor is more for composite materials. He has obtained tangential stress distribution around holes for anisotropic plates.

T.K. Hung [40] found a root modification scheme in order to avoid the limiting procedure which will occur when the characteristic roots are either equal or equal
to -1. R.E.Rowlands [41] analyzed glass epoxy composite plates both by experimental techniques and finite element numerical methods.

Many research engineers have worked on the basic formulation and theories behind the Boundary Element Method to bring it to its present state of efficiency. Alarcon and Brebbia [42] have given the fundamental algorithms of applying the method to basic equations of elasticity. T.A. Cruse et al [43] have reviewed the application of advanced Boundary Element Method. The above papers presented the algorithm without considering body forces. D.J.Danson [44] considered the problem with body forces and solved it by transferring from a domain to a boundary integral. An integral equation method applicable to boundary value problems of elastostatics has been given by F.J.Rizzo [45]. S.N.Vogel and F.J.Rizzo [46] have given formulation to three dimensional anisotropic elastostatic boundary value problems. Mastaka Tanaka et al [47] have contributed much towards the application of Boundary Element Method and have collected a number of References. A detailed analysis and solutions have been obtained by F.J.Rizzo and D.J.Shippy [48] for orthotropic and anisotropic plates. The Finite Element Method is being used widely in all design offices. C.A. Brebbia [49] has taken the advantage of using Boundary Element Method in combination with Finite Element Method to update the existing packages. C.A.Fellippa [50] has given several techniques for coupling Finite Element and Boundary Element solutions. C.Patterson and M.A.Sheikh [51] have shown clearly how the regular boundary integral equations could be derived. The same authors in another paper [52] have differentiated conventional and non-conforming boundary elements and have shown the advantages of non-conforming elements at boundary discontinuities. In popularizing the
C.A. Brebbia has contributed his part as the author of many books and by conducting six international conferences. The contribution of Brebbia is mainly towards direct Boundary Element Method, whereas P.K. Banerjee and R. Butterfield have explained clearly about the indirect Boundary Element Method. S.L. Crouch and A.M. Statfield have also given their contribution in developing both direct and indirect Boundary Element Methods in a very simple and straightforward manner using physical concept rather than mathematical treatment.

5.3 ISOTROPIC ANALYSIS

The application of BEM to two dimensional isotropic elastostatic problems are considered in the present work. An indirect BEM using constant element is demonstrated.

From the Kelvin's problem of a concentrated force acting at a point in an infinite elastic solid, it is possible to obtain analytical solutions to other problems by treating the Kelvin's solution as an influence function. In the present method, all the elements have been considered to be subjected to some fictitious stress components and their effects on the elements have been determined.

5.3.1 Kelvin's problem for plane strain

If $F_x$ and $F_y$ are the components of the force $F_1$, acting as a concentrated force on an infinite elastic solid, the solution can be expressed in terms of a function $g(x, y)$. 
The displacements are,

$$U_x = \frac{F_x}{2G} [(3-4v)g_{x} - x_{g,x}] + \frac{F_y}{2G} [-y_{g,x}]$$

$$U_y = \frac{F_x}{2G} [-x_{g,y}] + \frac{F_y}{2G} [(3-4v) - g_{y,y}]$$  \hspace{1cm} (3.1)$$

and the stresses are,

$$\sigma_{xx} = F_x [2(1-v)g_{x} - x_{g,xx}] + F_y [2y_{g,yy} - y_{g,xx}]$$

$$\sigma_{yy} = F_x [2y_{g,y} - y_{g,yy}] + F_y [2(1-v)g_{y,yy}]$$

$$\sigma_{xy} = F_x [(1-2v)g_{g,y} - x_{g,xy}] + F_y [(1-2v)g_{y,x} - y_{g,xy}]$$  \hspace{1cm} (3.2)$$

where $g(x,y)$ is defined as

$$g(x,y) = \frac{1}{\sqrt{4(1-v)(x^2 + y^2)^{1/2}}}$$  \hspace{1cm} (3.3)$$

The above displacements and stresses are due to force located at the origin of the coordinate axes. For a point located at $x = C_x$ and $y = C_y$, these solutions 3.1 and 3.2 will be appearing with a transformed coordinates $x - C_x$ and $y - C_y$. 
Now, the problem of constant tractions \( t_x = p_x \) and \( t_y = p_y \) applied to the line segment \( |x| \leq a, \ y=0 \) in an infinite elastic solid can be found by integrating the solutions to Kelvin's problem. The resultant force (Fig. 5.1) on an element centered at the point \( x=\xi, \ y=0 \) is then, \( F_1(\xi) = p_1 d\xi \).

Substituting the forces \( F_x(\xi) \) and \( F_y(\xi) \) into equations (3.1) and (3.2), duly replacing \( x \) by \( (x - \xi) \) and integrating the resultant expressions with respect to \( \xi \) between limits \(-a\) and \(+a\), the solutions can be obtained. The displacements and stresses are obtained in terms of a function \( f(x,y) \) where,

\[
\begin{align*}
\int_{-a}^{+a} g(x-\xi, y) d\xi 
\end{align*}
\]

Then the displacements and the stresses are given by,

\[
\begin{align*}
U_x &= \frac{p_x}{2G} [(3-4\upsilon)f, x - yf, y] + \frac{p_y}{2G} [-yf, x] \\
U_y &= \frac{p_x}{2G} [-yf, x] + \frac{p_y}{2G} [(3-4\upsilon)f - yf, y] \\
\sigma_{xx} &= p_x [(3-2\upsilon)f, x + yf, xy] + p_y [2\upsilon f, y + yf, yy]
\end{align*}
\]
Figure 5.1 Integral Solution to Kelvin's Problem

Figure 5.2 Stress boundary value problem
\[
\sigma_{yy} = p_x \left[ -(1-2v)f_{,x} + yf_{,xy} + p_y \left[ 2(1-v)f_{,y} - yf_{,yy} \right] \right]
\]

\[
\sigma_{xy} = p_x \left[ 2(1-v)f_{,y} + yf_{,yy} \right] + p_y \left[ (1-v)f_{,x} - yf_{,xy} \right]
\]

where \( f_{,x}, f_{,y}, f_{,xy}, f_{,xx} \) and \( f_{,yy} \) are the derivatives of the function \( f(x,y) \).

The equations 3.5 and 3.6 can be used to find the displacements and stresses in an infinite elastic solid, for the case that tractions \( t_1 = p_1 = (p_x, p_y) \) are applied over the line segment \(|(x)| \leq a, y=0\). The above equations form the basis of the boundary element method which is demonstrated in the subsequent sections.

### 5.3.2. Numerical procedure

The physical concept underlying this indirect method is explained below, with reference to a particular problem of holes in an infinite elastic plates. The same method can also be extended to finite bodies.

Figure 5.2 shows a stress boundary value problem of a hole in an infinite elastic body. The local coordinates \( n \) and \( s \) are perpendicular and tangent to curve \( C \), respectively which vary from point to point along the boundary. The inner boundary is subjected to a uniform normal stress \( \sigma_n = -p \), without shear stress \( \sigma_s \). The boundary is divided into \( N \) straight line segments. The length of a typical element 'i' is denoted by '2a'. Each element is subjected to a normal stress \( \sigma_n = -p \) along its entire length and is free from shear stress.
Figure 5.3 Numerical model

Figure 5.4 Line Segment Stress of Arbitrary Orientation
In the above system of $2N$ simultaneous linear algebraic equations, the stresses are fictitious. They are introduced to facilitate the numerical solution to a particular problem and have no physical significance with respect to the problem. But the linear combinations of the fictitious stresses specified by equation 3.8 have the physical meaning for the problem in question. The relations 3.9 are established using this criteria. Once these equations are established, the displacements and stresses at any point in the body as other linear combinations of the fictitious stresses $p_n(j)$ and $p_s(j)$ where $j=1$ to $N$ can be expressed.

5.3.3 Coordinate Transformations

The new coordinate system is chosen to have the axes $\overline{x}$ and $\overline{y}$ as shown in figure 5.4. Here the line segment is specified by the conditions $|\overline{x}| \leq a$, $y=0$. The normal stress $p_{\overline{y}}$ and a shear stress $p_{\overline{x}}$ are assumed to be acting on the element. The local $\overline{x},\overline{y}$ coordinate system is obtained by translation and rotation of the global $x,y$ coordinate system. The components of the translation are $C_x$ in the $x$ direction and $C_y$ in the $y$ direction. The rotational angle $\beta$ is taken as positive in the counter clockwise sense. This coordinate transformation is obtained by the relations:

\[ -p = \sum_{j=1}^{N} A_{ns}(ij) p_s(j) + \sum_{j=1}^{N} A_{nn}(ij) p_n(j) \]

\[ 0 = \sum_{j=1}^{N} A_{ss}(ij) p_s(j) + \sum_{j=1}^{N} A_{sn}(ij) p_n(j) \] (3.9)
\[
\bar{x} = (x-C_x) \cos \beta + (y-C_y) \sin \beta
\]
\[
\bar{y} = -(x-C_x) \sin \beta + (y-C_y) \cos \beta
\]

The displacements and stresses in the \(\bar{x}, \bar{y}\) coordinates system due to stresses \(p_x, p_y\) on the line segment \(|x| \leq a, \bar{y}=0\) can be obtained from equations 3.5 and 3.6.

The displacements are,

\[
U_x = \frac{p_x}{2G} [ (3-4\nu)f + yf, \bar{y}] + \frac{p_y}{2G} [ -\bar{y}f, \bar{x}] 
\]
\[
U_y = \frac{-p_x}{2G} [ -\bar{y}f, \bar{x}] + \frac{-p_y}{2G} [ (3-4\nu)f - yf, \bar{y}] 
\]

The stresses are

\[
\sigma_{xx} = p_x [ (3-2\nu)f, \bar{x} + yf, \bar{y}] + p_y [ 2yf, \bar{y} + \bar{y}f, \bar{y}] 
\]
\[
\sigma_{yy} = p_x [ -(1-2\nu)f, \bar{x} - \bar{y}f, \bar{y}] + p_y [ 2(1-\nu)f, \bar{y} - yf, \bar{y}] 
\]
\[
\sigma_{xy} = p_x [ (2(1-\nu)f, \bar{y} + yf, \bar{y}] + p_y [ (1-2\nu)f, \bar{x} - yf, \bar{y}] 
\]

in which the function \(f(x,y)\) is given by
\[
f(\bar{x}, \bar{y}) = \bar{F}_1 = \frac{-1}{4\pi(1-\nu)} \left[ \bar{y} \left( \arctan \frac{-\bar{y}}{\bar{x} - a} \right) \right]
\]

\[
- \arctan \frac{-\bar{y}}{\bar{x} + a} \right] - (\bar{x} - a) \ln \left[ (\bar{x} - a)^2 + \bar{y}^2 \right]^{1/2}
\]

\[
+ (\bar{x} + a) \ln \left[ (\bar{x} + a)^2 + \bar{y}^2 \right]^{1/2}
\]

and the derivatives by

\[
f_{,\bar{x}} = \bar{F}_2 = \frac{-1}{4\pi(1-\nu)} \left[ \ln (\bar{x} - a)^2 + \bar{y}^2 \right]^{1/2} - \ln \left( (\bar{x} + a)^2 + \bar{y}^2 \right)^{1/2}
\]

\[
f_{,\bar{y}} = \bar{F}_3 = \frac{-1}{4\pi(1-\nu)} \left[ \arctan \frac{-\bar{y}}{\bar{x} - a} - \arctan \frac{\bar{y}}{\bar{x} + a} \right]
\]

\[
f_{,\bar{x}\bar{y}} = \bar{F}_4 = \frac{1}{4\pi(1-\nu)} \left[ \frac{-\bar{y}}{(\bar{x} - a)^2 + \bar{y}^2} - \frac{\bar{y}}{(\bar{x} + a)^2 + \bar{y}^2} \right]
\]

\[
f_{,\bar{x}\bar{x}} = -f_{,\bar{y}\bar{y}} = \bar{F}_5 = \frac{1}{4\pi(1-\nu)} \left[ \frac{\bar{x} - a}{(\bar{x} - a)^2 + \bar{y}^2} - \frac{\bar{x} + a}{(\bar{x} + a)^2 + \bar{y}^2} \right]
\]

\[(3.13)\]
Using the equations 3.11 and 3.12 along with the derivatives specified in 3.13, the displacements and stresses are calculated by computing the $\bar{x}, \bar{y}$ coordinates of the point from the relations given by 3.10. These values refer to a local coordinate system and therefore are not in convenient form. Hence they are changed to global coordinate $(x, y)$ system by using the transformation

$$
U_x = U_{\bar{x}} \cos \beta - U_{\bar{y}} \sin \beta \\
U_y = U_{\bar{x}} \sin \beta + U_{\bar{y}} \cos \beta
$$

(3.14)

$$
\sigma_{xx} = \sigma_{\bar{xx}} \cos^2 \beta - 2 \sigma_{\bar{xy}} \sin \beta \cos \beta + \sigma_{\bar{yy}} \sin^2 \beta \\
\sigma_{yy} = \sigma_{\bar{xx}} \sin^2 \beta + 2 \sigma_{\bar{xy}} \sin \beta \cos \beta + \sigma_{\bar{yy}} \cos^2 \beta \\
\sigma_{xy} = \sigma_{yx} = (\sigma_{\bar{xx}} - \sigma_{\bar{yy}}) \sin \beta \cos \beta + \sigma_{\bar{xy}} (\cos^2 \beta - \sin^2 \beta)
$$

(3.15)

Now substituting the equations 3.11 into 3.14, the displacements are obtained.

$$
U_x = -\frac{p_{\bar{x}}}{2G} [(3-4v) \cos \beta \bar{F}_1 + \bar{y} (\sin \beta \bar{F}_2 + \cos \beta \bar{F}_3)] \\
+ \frac{p_{\bar{y}}}{2G} [-(3-4v) \sin \bar{F}_1 - \bar{y} (\cos \beta \bar{F}_2 - \sin \beta \bar{F}_3)]
$$
Substitution of 3.12 into 3.15 gives,

\[ u_y = \frac{p_x}{2G} [(3-4v)\sin \beta \ddot{F}_1 - \dot{y} (\cos \beta \ddot{F}_2 - \sin \beta \ddot{F}_3)] \]

\[ + \frac{p_y}{2G} [(3-4v)\cos \beta \ddot{F}_1 - \dot{y} (\sin \beta \ddot{F}_2 + \cos \beta \ddot{F}_3)] \]

Substitution of 3.12 into 3.15 gives,

\[ \sigma_{xx} = p_x [\ddot{F}_2 + 2(1-v) (\cos 2\beta \ddot{F}_2 - \sin 2\beta \ddot{F}_3) + \dot{y} (\cos 2\beta \ddot{F}_4 + \sin 2\beta \ddot{F}_5)] + p_y [\ddot{F}_3 - (1-2v) (\sin 2\beta \ddot{F}_2 + \cos 2\beta \ddot{F}_3) + \dot{y} (\sin 2\beta \ddot{F}_4 - \cos 2\beta \ddot{F}_5)] \]

\[ \sigma_{yy} = p_x [\ddot{F}_2 - 2(1-v) (\cos 2\beta \ddot{F}_2 - \sin 2\beta \ddot{F}_3) - \dot{y} (\cos 2\beta \ddot{F}_4 + \sin 2\beta \ddot{F}_5)] + p_y [\ddot{F}_3 + (1-2v) (\sin 2\beta \ddot{F}_2 + \cos 2\beta \ddot{F}_3) + \dot{y} (\sin 2\beta \ddot{F}_4 - \cos 2\beta \ddot{F}_5)] \]

\[ \sigma_{xy} = p_x [2(1-v) (\sin 2\beta \ddot{F}_2 + \cos 2\beta \ddot{F}_3) + \dot{y} (\sin 2\beta \ddot{F}_4 - \cos 2\beta \ddot{F}_3)] + p_y [-(1-v) (\cos 2\beta \ddot{F}_2 - \sin 2\beta \ddot{F}_3) - \dot{y} (\cos 2\beta \ddot{F}_4 + \sin 2\beta \ddot{F}_5)] \]

(3.17)
5.3.4. Boundary Influence Coefficients

The boundary element influence coefficients are obtained for an infinite body containing \( N \) boundary elements. Let the \( i^{th} \) and \( j^{th} \) elements have lengths \( 2a(i) \) and \( 2a(j) \) with orientations \((i)\) and \((j)\) and mid point coordinates \( x(i), y(i) \) and \( x(j), y(j) \) as shown in figure 5.5. The local co-ordinate axis \( X \) of any element is positive in the direction of traversal and angle \( \beta \) specifies the inclination of this axis with respect to the positive X-direction.

As already explained, let the fictitious stresses at the \( j^{th} \) element be \( p_X(j) \) and \( p_Y(j) \). Boundary influence coefficients are obtained by choosing global co-ordinate points \((x,y)\) to be the mid point of the \( i^{th} \) element and local co-ordinate points \((\bar{x},\bar{y})\) to be the mid point of the \( j^{th} \) element. Then equation (3.10) becomes,

\[
\begin{align*}
\bar{x} &= [x(i)-x(j)] \cos \beta(j) + [y(i)-y(j)] \sin \beta(j) \\
\bar{y} &= [(x(i)-x(j)] \sin \beta(j) + [y(i)-y(j)] \cos \beta(j)
\end{align*}
\] (3.18)

The components of displacement and stress at point \((i)\), relative to the \( x,y \) co-ordinate system at point \((j)\) are given directly by (3.11) and (3.12).

The local co-ordinate systems are related by the transformation.

\[
\begin{align*}
\bar{x}' &= x' \cos \psi + \bar{y} \sin \psi \\
\bar{y}' &= -x' \sin \psi + \bar{y} \cos \psi
\end{align*}
\] (3.19)

where \( \psi = \beta(i) - \beta(j) \)
The boundary influence coefficients for the displacements are obtained from (3.11) and (3.14) and after noting that \( U_x(i) \), \( U_s(i) \), and \( U_y(i) = U_n(i) \), also \( p_s(j) = p_x(j) \) and \( p_n(j) = p_y(j) \)

\[
U_s(i) = \frac{p_s(j)}{2G} 
- \frac{P_n(j)}{2G} \left[ (3-4v) \cos \psi_1 - y (\sin \psi F_2 - \cos \psi F_3) \right] 
+ \frac{P_n(j)}{2G} \left[ (3-4v) \sin \psi F_1 - y (\cos \psi F_2 + \sin \psi F_3) \right]
\]

\[
U_n(i) = \frac{p_n(j)}{2G} \left[ -(3-4v) \sin \psi F_1 - y (\cos \psi F_2 + \sin \psi F_3) \right] 
+ \frac{P_n(j)}{2G} \left[ (3-4v) \cos \psi F_1 + y (\sin \psi F_2 - \cos \psi F_3) \right]
\]

(3.20)

Similarly the boundary influence coefficients for the stresses \( \sigma_s(i) = \sigma_x(i) \), \( \sigma_y(i) = \sigma_y(i) \) are,

\[
\sigma_s(i) = p_s(j) \left[ -2(1-v) \sin 2\psi F_2 - \cos 2\psi F_3 \right] 
- y \left[ \sin 2\psi F_4 + \cos 2\psi F_5 \right]
+ P_n(j) \left[ (1-2v) \cos 2\psi F_2 + \sin 2\psi F_3 \right]
- y \left[ \cos 2\psi F_4 - \sin 2\psi F_5 \right]
\]
\[ \sigma_n(i) = p_s(j) \left[ F_2 - 2(1-\nu) \left( \cos 2\psi F_2 + \sin 2\psi F_3 \right) \right] \\
- \bar{y} \left( \cos 2\psi F_4 - \sin 2\psi F_5 \right) \right] \\
+p_n(j) \left[ F_3 - (1-2\nu) \left( \sin 2\psi F_2 - \cos 2\psi F_3 \right) \right] \\
+ \bar{y} \left( \sin 2\psi F_4 + \cos 2\psi F_5 \right) \right] \] (3.21)

The displacements and stresses at the \( i^{th} \) element are functions of the fictitious stress components \( p_s(j) \) and \( p_n(j) \) at all \( N \) elements. Therefore we can write from (3.20) and (3.21).

\[ U_s(i) = \sum_{j=1}^{N} B_{ss}(ij) p_s(j) + \sum_{j=1}^{N} B_{sn}(ij) p_n(j) \]  

\[ U_n(i) = \sum_{j=1}^{N} B_{ns}(ij) p_s(j) + \sum_{j=1}^{N} B_{nn}(ij) p_n(j) \] (3.22)

and

\[ \sigma_s(i) = \sum_{j=1}^{N} A_{ss}(ij) p_s(j) + \sum_{j=1}^{N} A_{sn}(ij) p_n(j) \] 

\[ \sigma_n(i) = \sum_{j=1}^{N} A_{ns}(ij) p_s(j) + \sum_{j=1}^{N} A_{nn}(ij) p_n(j) \] (3.23)

where the boundary influence coefficients \( B_{ss}(ij) \), etc., and \( A_{ss}(ij) \), etc., in these equations are specified by the expressions inside the brackets in (3.20) and (3.21).
Now in equations (3.20) and (3.21) the influences of the fictitious stresses $p_s(j)$ and $p_n(j)$ at the $j^{th}$ element on the displacements and stresses at the $j^{th}$ element itself are called 'element self-effects'. Element self-effects are obtained by equating (3.20) and (3.21) for the case that $\bar{x}, \bar{y}$ and $\gamma = \beta(i) - \beta(j)$ are all equal to zero.

Substituting these values, the required element self-effects are found as,

\[
U_s(i) = \frac{-(3-4v)}{4\pi G(1-v)} a(i) \mathcal{A}[a(i)] p_s(i) \\
U_n(i) = \frac{-(3-4v)}{4\pi G(1-v)} a(i) \mathcal{A}[a(i)] p_n(i) \tag{3.24}
\]

and

\[
\sigma_s(i) = \begin{cases} 
\frac{1}{2} p_s(i) & y=0+ \\
\frac{1}{2} p_s(i) & y=0- \\
\frac{1}{2} p_n(i) & y=0+ \\
\frac{1}{2} p_n(i) & y=0-
\end{cases}
\]

and the diagonal terms of the influence coefficients will be

\[
B_{ss}(jj) = B_{ns}(jj) = 0 \\
B_{ss}(jj) = B_{nn}(jj) = \frac{-(3-4v)}{4\pi G(1-v)} a(i) \mathcal{A}[a(i)] \tag{3.26}
\]
and

\begin{align}
\lambda_{sn}(jj) &= \lambda_{ns}(jj) = 0 \\
\lambda_{ss}(ii) &= \lambda_{nn}(jj) = +\frac{1}{2} \text{ for } y = 0^+ \quad (3.27)
\end{align}

Thus the coefficients \( \lambda_{ss}(jj) \) and \( \lambda_{nn}(jj) \) depend upon the way in which the curve \( C' \) in figure 5.5 is approached. It therefore appears that, the coefficient values will be different depending on the region which is being considered. Therefore in order to avoid different values for the coefficients \( \lambda_{ss} \) and \( \lambda_{nn} \), the boundary can be traversed in such a way that the outward normal points 'away' from the region of interest. It is possible by 'traversing the boundary of the finite body in the clockwise sense, and for that of an infinite body in the counterclockwise sense. By doing so, computer programming can be made simple as the coefficient values \( \lambda_{ss} \) and \( \lambda_{nn} \) are equal to \( +\frac{1}{2} \) for both types, i.e., interior and exterior problems.

5.3.5 Computation of Tangential Stress

Boundary coefficients \( \lambda_{ss} \)' etc., in (3.23) give shear and normal stresses at the \( i \)th boundary element. The tensor component \( \sigma_{x'x}(i) \) is called the 'tangential stress' and let it be denoted as \( \sigma_t(i) \). Determination of the tangential stress along the boundary is one of the main objectives in solving an elasticity problem. Substituting equation (3.12) in (3.15) we get,

\begin{align}
\sigma_t(i) &= p_s(j) [F_2 + 2(1-\nu)\cos 2\psi F_2 + \sin 2\psi F_3] \\
&\quad + \bar{y} (\cos 2\psi \bar{F}_4 - \sin 2\psi \bar{F}_5) + p_n(j) [F_3 + (1-2\nu)(\sin 2\psi \bar{F}_2 \\
&\quad - \cos 2\psi \bar{F}_3) - \bar{y} (\sin 2\psi \bar{F}_4 + \cos 2\psi \bar{F}_5)] \quad (3.28)
\end{align}
In the case of diagonal terms, as usual by setting \( \bar{x}, \bar{y} \) and equal to zero in equation (3.15) and \( A_{ts}, A_{tn} \) the influence coefficients for the tangential stresses are found out. They are,

\[
A_{ts}(jj) = A_{tn}(jj) = \frac{\nu}{(1-\nu)} \quad \text{for} \quad \bar{y} = 0 + (3.29)
\]

By following the convention already stated different values for the influence coefficient \( A_{ts} \) avoided.

5.3.6 Boundary Conditions

Equations (3.22) and (3.23) form the basis of a general numerical procedure for solving boundary value problems in plane elasticity. The particular type of boundary conditions prescribed for a problem govern the form of the system of algebraic equations that are to be solved. If the stresses \( \sigma_s(i) = b_s(i) \) and \( \sigma_n(i) = b_n(i) \) are prescribed at the \( i \)th boundary element, then \( b_s(i) \) and \( b_n(i) \) for \( \sigma_s(i) \) and \( \sigma_n(i) \) are to be substituted in (3.23). Similarly, if the displacements are \( u_s(i) = b_s(i) \) and \( u_n(i) = b_n(i) \), then the respective values will be substituted in (3.22).

Finally,

\[
b_s(i) = \sum_{j=1}^{N} C_{ss}(ij) p_s(j) + \sum_{j=1}^{N} C_{sn}(ij) p_n(j)
\]

\[
b_n(i) = \sum_{j=1}^{N} C_{ns}(ij) p_s(j) + \sum_{j=1}^{N} C_{nn}(ij) p_n(j) \quad (3.30)
\]
Where $i = 1$ to $N$

and $b_s(i)$ or $b_n(i) = \text{Known boundary values of stress or displacement},$

which ever is appropriate

and $C_{ss}, \ldots \text{etc.},$ are the corresponding influence coefficients. Equation (3.30) is applicable to both interior and exterior (cavity) problems.

5.3.7 - Symmetry conditions

Symmetry conditions frequently occur in practical problems. A symmetry exists for a certain problem when the elastic properties of the material, the geometric configuration of the boundaries and the loading conditions are all symmetrical with respect to the axis concerned.

A line of symmetry can be introduced with respect to a given boundary element by adding an image element at the appropriate location. This line of symmetry acts as a 'mirror', and the components of the fictitious stress at the image element are found by reflecting those of the actual element across this line. The result of this operation is that the shear component of the fictitious stress at the actual and image boundary elements are always equal in magnitude but opposite in sign. The normal components are always equal.

Incorporation of the symmetry conditions is explained by considering a case for which the line $y=y^*$ is a line of symmetry. (Figure 5.6). Only global coordinates are considered for illustration. Let the original boundary element be centered at the point $x = x(j)$ and with $y(j)$ where $y(j)$ $y^*$ and oriented at an angle $+\beta (j)$
Figure 5.5  Boundary Elements Orientation

Figure 5.6  Symmetry Conditions for the Line $Y = Y^*$
with respect to the $x$ direction. Let the fictitious stresses at this element be $\sigma_s(j)$ and $\sigma_n(j)$ as shown in figure 5.6.

Now, the mid point of the image element is located at the point $x = x'(j), y = y'(j)$. This element is oriented at angle $\beta'(j) = \pi - \beta(j)$ from the $X$-direction and has fictitious stresses, $\sigma_s'(j) = -\sigma_s(j)$ and $\sigma_n' = \sigma_n(j)$. X co-ordinates of both original and image element are equal i.e. $x(j) = x'(j)$. The $y$ co-ordinate of the image element is found by noting that the actual and image elements are equidistant from the line of symmetry, $y = y^*$ from this figure. Then,

$$x'(j) = x(j)$$
$$y'(j) = y^* - [\tilde{y}(j) - y^*] = 2y^* - \tilde{y}(j) \quad (3.31)$$

The boundary displacement $u_s(i)$ and $u_n(i)$ and stresses $\sigma_s(i)$, $\sigma_n(i)$, $\sigma_i(i)$ at the point $x = x(i)$ and $y = y(i)$ due to fictitious stresses $\sigma_s'(j)$ and $\sigma_n'(j)$ at the image element can now be computed in terms of local co-ordinates

$$\bar{x}' = [x(i) - x'(j)] \cos \beta'(j) + [y(i) - y'(j)] \sin \beta'(j)$$
$$= [x(i) - x'(j)] \cos (\pi - \beta(j)) +$$
$$[y(i) - 2y^* + y(j)] \sin (\pi - \beta(j))$$

$$\bar{y}' = -[x(i) - x'(j)] \sin \beta'(j) + [y(i) - y'(j)] \cos \beta'(j)$$
$$= -[x(i) - x'(j)] \sin (\pi - \beta(j)) +$$
$$[y(i) - 2y^* + y(j)] \cos (\pi - \beta(j))$$
Accordingly the functions $F_k(x', y')$ from equation (3.13) can be modified. While calculating the influence coefficients, hence displacements and stresses, the sign change for shear component of fictitious stress at the image element must be observed. Influence coefficients $B_{ss}(ij)$, etc., in (3.22), $A_{ss}(ij)$, etc., in (3.23) and $A_{ts}(ij)$, etc., automatically incorporate the effects of image element and they can be computed accordingly. A similar procedure is followed for the symmetry condition $x=x^*$ and $x=x^*$, $y=y^*$ (three image boundary elements).

5.3.8 Program Structure

Based on the physical concepts explained above, a Fortran program for isotropic linearly elastic problems is written. The program uses constant elements as already explained. The problem of a circular hole is an infinite width plate under a uniaxial stress at infinity is solved. For this problem, the field stresses are taken as $\sigma_{xx} = p$ and $\sigma_{yy} = \sigma_{xy} = 0$ at infinity. As the problem is symmetrical about $x$ and $y$ axes only one quarter of the circular boundary is considered. This boundary is divided into 12 elements.

The analytical solution for the tangential stress $\sigma_t = \sigma_{\theta \theta}$ along the boundary of the hole is available which calculates,

$$\sigma_{\theta \theta} = p \left(1 - 2 \cos 2\theta\right) \text{ at } r = R$$

$\theta$ is measured with respect to $X$-axis. These values are plotted in figures 9 and 10.
Computations of the program are carried out basically in 6 steps. These are explained in the flow chart shown in figure 5.7.

Basically, the program consists of one main program and three sub-routines.

Main program reads all the input required by the program, defines the boundary element location, sets the system of algebraic equation, calculates the required boundary displacements and stresses, computes the displacements and stresses at other specified points and finally prints the results.

At first it calculates the centre point of each element, its length, its angle with respect to the global y co-ordinate.

Then it adjusts the stress boundary values duly accounting for initial stresses. After that it calculates the influence coefficients calling the subroutine INITL and subroutine INFLO. Then it sets up the system of algebraic equations.

Calling the subroutine GAUSS it solves the system of algebraic equations (3.22), (3.23), (3.14) and (3.15). Using the similar procedure the displacements and stresses at field points are calculated. The imaging process is done by making successive calls to INFL, after defining co-ordinates and inclinations of the image elements.

Subroutine INITL : It is used to initialize the influence coefficients.
Figure 5.7 Macroflow Chart of Isotropic Program
Subroutine INFL : Using equations (3.16) and (3.17) the influence coefficients are calculated. The results are used in the main program.

Subroutine GAUSS : This subroutine solves the system of algebraic equations defined in the main program by the use of Gauss elimination method.

The variables used in the program together with their meaning are given below:

- **PR** and **E** : Poisson's ratio and Young's modulus respectively.
- **NBES** : Number of straight line boundary elements.
- **NOS** : Number of other line segments where the displacements and stresses are required to be computed.
- **KSYM** : Index of symmetry
  - If there is no symmetry, KSYM=1
  - If x=XSYM is a line of symmetry, KSYM=2
  - If y=YSYM is a line of symmetry, KSYM=3
  - If x=XSYM and y=YSYM, both are lines of symmetry, KSYM=4
- **XSYM** : Location of line of symmetry parallel to y axis
- **YSYM** : Location of line of symmetry parallel to x axis
- **PXX, PYY, PXY** : Initial stresses in the region concerned.
- **NUM** : Number of equally spaced boundary elements along straight line segments.
- **XB, YB** : X and Y co-ordinates of beginning of the line segment.
XE,YE : x and y co-ordinates of end of the line segment

KODE : An index used to define the boundary values
= 1 if $s$ and $n$ are prescribed
= 2 if $U_s$ and $U_n$ are prescribed
= 3 if $U_s$ and $n$ are prescribed
= 4 if $s$ and $U_n$ are prescribed

BVS : Resultant shear stress ($\sigma_s$) or shear displacement ($u_s$).

BVN : Resultant normal stress ($\sigma_n$) or normal displacement ($U_n$).

5.3.9 Results

Using the program explained above the problem of a circular hole in an infinite width plate is analysed. The plate dimensions (Figure 5.8) are 304 mm x 97 mm. Circular hole is of 19 mm diameter. Young's modulus of the material is taken as $0.170 \times 10^6$ kg/cm² and Poisson's ratio as 0.2.

The boundary of the circular hole is divided into 12 elements. 11 field points are chosen along the X-direction and 7 field points are chosen along the Y-direction. A uniform tensile stress of 1000 kg/cm² is applied in the X-direction. The elementary discretization of the boundary is in the counterclockwise direction. Using the program explained, the tangential stress values are obtained. These values are compared with analytical values as already described. The results are agreeing quite well as could be seen from the Tables 5.1 and 5.2. The maximum error is of 10% only which may be reduced by taking more number of elements along the boundary of hole.
<table>
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<th>Element</th>
<th>X</th>
<th>Y</th>
<th>$\sigma_{xx}$ kg/cm$^2$</th>
<th>Isotropic Analysis</th>
<th>Orthotropic Analysis</th>
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TABLE 5.2 STRESS AT SPECIFIED POINTS IN THE BODY

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<th>Points</th>
<th>X</th>
<th>Y</th>
<th>$\sigma_{xx}$ (kg/cm$^2$)</th>
<th>Isotropic Analysis</th>
<th>Orthotropic Analysis</th>
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<td>988.1</td>
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<td>990.0</td>
<td>106.1</td>
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<td>4.8500</td>
<td>1022.0</td>
<td>201.4</td>
<td></td>
</tr>
</tbody>
</table>
Figure 5.8 Plate Discretization

All dimensions in mm
Figure 5.9 Stress Concentration Around The Boundary of Hole
Figure 5.10 Tangential Stress Distribution Around The Hole
5.4 ORTHOTROPIC ANALYSIS

In the theory of linear elasticity, according to Hooke's law, for an anisotropic material, the stresses $\{\sigma\}$ are related to the strains $\{\varepsilon\}$ by

$$\{\sigma\} = [C] \{\varepsilon\} \tag{4.1}$$

where $[C]$ contains 36 elastic constants. Using the strain energy principle, the $[C]$ matrix becomes symmetric and the constants reduce to 21. For an orthotropic material, there are 9 independent elastic constants and five for transversely isotropic materials.

In the present work, the solutions of Kelvin's problem for an orthotropic (transversely isotropic) solid are used as the basis of the fictitious stress method and a program has been developed.

5.4.1 Stress-strain relations

Stress-strain relations for the orthotropic elastic material in a state of plane stress can be written as (by setting $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$)

$$\varepsilon_{xx} = \frac{-1}{E_x} \sigma_{xx} - \frac{\nu}{E_y} \sigma_{yy}$$

$$\varepsilon_{yy} = \frac{-\nu}{E_x} \sigma_{xx} + \frac{\sigma_{yy}}{E_y}$$

$$\varepsilon_{xy} = \frac{\sigma_{xy}}{G_{xy}} \tag{4.1}$$
The equations (4.1) are often written as,

\[ \varepsilon_{xx} = S_{11} \sigma_{xx} + S_{12} \sigma_{yy} \]
\[ \varepsilon_{yy} = S_{12} \sigma_{xx} + S_{22} \sigma_{yy} \]
\[ \varepsilon_{xy} = \frac{1}{2} S_{66} \sigma_{xy} \]  

(4.2)

The equations (4.2) can be inverted to give

\[ \sigma_{xx} = (S_{22} \varepsilon_{xx} - S_{12} \varepsilon_{yy})/(S_{11} S_{22} - S_{12}^2) \]
\[ \sigma_{yy} = (-S_{12} \varepsilon_{xx} + S_{11} \varepsilon_{yy})/(S_{11} S_{22} - S_{12}^2) \]
\[ \sigma_{xy} = 2 \varepsilon_{xy}/S_{66} \]  

(4.3)

A state of plane strain for the x-y plane is defined by the conditions \( \varepsilon_{zz} = \varepsilon_{xz} = 0 \) and these relations can be written as,

\[ \sigma_{xx} = C_{11} \varepsilon_{xx} + C_{12} \varepsilon_{yy} \]
\[ \sigma_{yy} = C_{12} \varepsilon_{xx} + C_{22} \varepsilon_{yy} \]
\[ \sigma_{xy} = 2 C_{66} \varepsilon_{xy} \]  

(4.4)

where

\[ C_{11} = (S_{22} - S_{23}^2/S_{33}) S_o^2 \]
\[ C_{12} = -(S_{12} - S_{13} S_{23}/S_{33})/S_o^2 \]
\[ C_{22} = (S_{11} - S_{13}^2/S_{33})/S_o^2 \]
The constants $C_{11}, C_{12}, C_{22}$ and $C_{66}$ for a problem in plane stress can be obtained by setting $S_{13} = S_{23} = 0$ from equations (4.4) and (4.5).

These are

\[ C_{11} = \frac{S_{12}}{D} \]

\[ C_{12} = \frac{-S_{12}}{D} \]

\[ C_{22} = \frac{S_{11}}{D} \]

\[ C_{66} = \frac{1}{S_{66}} \]

where

\[ D = S_{11} S_{22} - S_{12}^2 \]
5.4.2. Kelvin's Solution for Plane Strain

Green and also Rizzo and Shippy have given the solution to the problem of a line of concentrated force in an orthotropic elastic solid in the following form.

\[ U_x = \frac{-q_1q_2}{2\pi C_{66}(q_1-q_2)} \left[ \frac{r_1}{q_1} \ln \left( \frac{x^2+y_1^2}{r_1^2} \right) - \frac{r_2}{q_2} \ln \left( \frac{x^2+y_2^2}{r_2^2} \right) \right] \]

\[ U_y = \frac{-q_1q_2}{2\pi C_{66}(q_1-q_2)} \left[ \frac{r_1}{q_1} \ln \left( \frac{x^2+y_1^2}{r_1^2} \right) - \frac{r_2}{q_2} \ln \left( \frac{x^2+y_2^2}{r_2^2} \right) \right] \]

\[ \sigma_{xx} = \frac{q_1q_2}{2\pi(q_1-q_2)} \left[ \frac{1}{r_1} \ln \left( \frac{x^2+y_1^2}{r_1^2} \right) - \frac{1}{r_2} \ln \left( \frac{x^2+y_2^2}{r_2^2} \right) \right] \]

\[ \sigma_{xy} = \frac{-q_1q_2}{2\pi(q_1-q_2)} \left[ \frac{r_1(1+q_1)}{q_1} \ln \left( \frac{x^2+y_1^2}{r_1^2} \right) - \frac{r_2(1+q_2)}{q_2} \ln \left( \frac{x^2+y_2^2}{r_2^2} \right) \right] \]
\[
\begin{align*}
\sigma_{x y} &= \frac{+q_1 q_2}{2\pi(q_1-q_2)} \left[ \frac{(q_1 + q_1^2)}{q_1 (x^2 + y_1^2)} - \frac{(1 + q_2)}{q_2 (x^2 + y_2^2)} \right] \\
F_x &= \frac{1}{2\pi(q_1-q_2)} \left[ \frac{(1+q_1) y_1}{q_1 (x^2 + y_1^2)} - \frac{(1+q_2) y_2}{q_2 (x^2 + y_2^2)} \right] \\
F_y &= \frac{1}{2\pi(q_1-q_2)} \left[ \frac{(1+q_1) x}{r_1 (x^2 + y_2^2)} - \frac{(1+q_2) x}{r_2 (x^2 + y_2^2)} \right]
\end{align*}
\]

(4.8)

\(F_x\) and \(F_y\) are the components of \(F\) acting as a concentrated force and \(y_1, y_2\) are two \(Y\)-co-ordinates defined as,

\[y_1 = \frac{y}{r_1^2} \quad y_2 = \frac{y}{r_2^2}\]

where \(r_1^2\) and \(r_2^2\) are the two roots of the quadratic equation in \(r^2\).

\[C_{11} C_{66} r^4 + (C_{12} + 2C_{66}) r^2 + C_{22} C_{66} = 0\]

(4.9)

The constants \(q_1\) and \(q_2\) are related to \(r_1^2\) and \(r_2^2\) by
\[ q_1 = \frac{(C_{11} r_1^2 - C_{66})}{(C_{12} + C_{66})} \]
\[ q_2 = \frac{(C_{11} r_2^2 - C_{66})}{(C_{12} + C_{66})} \]  

(4.10)

It can be shown that the constants \( r_1^2 \) and \( r_2^2 \) in (4.9) may be either both real and positive or are complex conjugates with positive real parts. However (4.7) and (4.8) are written under the assumption that \( r_1^2 \) and \( r_2^2 \) are both real and positive.

5.4.3 Actual Formulation - Based on the fictitious stress Method

In the isotropic analysis of the fictitious stress method, the solution was found for a problem of constant stresses applied to an arbitrarily oriented, finite line segment in an infinite solid. The solution is obtained from integration of the results of Kelvin's solution. The geometry used in carrying out the integration is already shown in figure 5.4.

The two co-ordinate systems are related by the transformation equations.

\[ x = \bar{x} \cos \beta - \bar{y} \sin \beta \]
\[ y = \bar{x} \sin \beta + \bar{y} \cos \beta \]  

(4.11)

The stresses applied at first are specified with reference to the \( x, y \) system and afterwards it will be transformed to \( \bar{x}, \bar{y} \) system by using the relations.
\[ P_x = p_x \cos \beta - p_y \sin \beta \]
\[ P_y = p_x \sin \beta + p_y \cos \beta \]
\[ (4.12) \]

The displacements and stresses due to constant stresses \( p_x \) and \( p_y \) applied to line segment \( |x| < a, y = 0 \) can be determined as already explained in section 5.3.1. Following the similar procedure explained in section 5.3.1 finally the displacements are obtained as

\[
U_x = \frac{-\{q_2 r_1 I_1(r_1) - q_1 r_2 I_1(r_2)\} \cos \beta}{2\pi c_{66}(q_1 - q_2)}
\]

\[
- \{I_2(r_1) - I_2(r_2)\} \sin \beta
\]

\[
+ \frac{-\{q_2 r_1 I_1(r_1) - q_1 r_2 I_1(r_2)\} \sin \beta}{2\pi c_{66}(q_1 - q_2)}
\]

\[
- \{I_2(r_1) - I_2(r_2)\} \cos \beta
\]

\[
U_y = \frac{-\{I_2(r_1) - I_2(r_2)q_1 q_2\} \cos \beta}{2\pi c_{66}(q_1 - q_2)}
\]

\[
\frac{q_1}{r_1} - \frac{q_2}{r_2} \{I_1(r_1) - I_1(r_2)\} \sin \beta
\]
Also, the stresses can be finally written as,

\[
\sigma_{xx} = \frac{\p_y}{2\pi c_{66}(q_1-q_2)} \left[ \left\{ \frac{q_2}{r_1} \right\} + \frac{q_1}{r_2} \right] (1+q_1) I_3(r_1) \sin \beta + \frac{q_2}{r_2} \right] (1+q_1) I_3(r_2) \sin \beta + \frac{(1+q_1)}{r_1} I_4(r_1) \left\{ \frac{q_1}{r_2} \right\} \cos \beta + \frac{(1+q_2)}{r_2} I_4(r_2) \left\{ \frac{q_2}{r_1} \right\} \cos \beta \]

(4.13)
\[
\sigma_{yy} = \frac{P_x}{2\pi(q_1-q_2)} \left\{ \begin{array}{c}
- q_1 r_2 (1+q_2) I_3(r_2) \cos \beta - (1+q_1) I_4(r_1) \\
- (1+q_2) I_4(r_2) \sin \beta \\
+ \frac{P_y}{2\pi(q_1-q_2)} \left\{ \begin{array}{c}
q_2 r_1 (1+q_1) I_3(r_1) \\
q_2 r_1 (1+q_1) I_3(r_1) - q_1 (1+q_2) I_4(r_2) \cos \beta \\
- (1+q_2) I_4(r_2) \sin \beta \\
\end{array} \right. \\
\end{array} \right. 
\]

\[
\sigma_{xy} = \frac{P_x}{2\pi(q_1-q_2)} \left\{ \begin{array}{c}
(1+q_1) \left\{ \begin{array}{c}
I_3(r_1) - I_3(r_2) \sin \beta \\
\end{array} \right. \\
- I_4(r_1) - q_1 (1+q_2) I_4(r_2) \cos \beta \\
\end{array} \right. 
\]

\[
\sigma_{yx} = \frac{P_y}{2\pi(q_1-q_2)} \left\{ \begin{array}{c}
(1+q_1) \left\{ \begin{array}{c}
I_3(r_1) - I_3(r_2) \cos \beta \\
\end{array} \right. \\
- I_4(r_1) - q_1 (1+q_2) I_4(r_2) \sin \beta \\
\end{array} \right. 
\]

(4.14)
Various functions used in the above equations are defined as

\[ I_1 (r_i) = \frac{\bar{y}}{r_i \lambda_i} \left[ \theta_1 (r_i) - \theta_2 (r_i) \right] \]

\[ - (\bar{x} - a + \frac{B_i \bar{y}}{2 \lambda_i}) \ln R_1 (r_i) \]

\[ + (\bar{x} + a + \frac{B_i \bar{y}}{2 \lambda_i}) \ln R_2 (r_i) \]

\[ I_2 (r_i) = (\bar{x} - a + \frac{B_i \bar{y}}{2 \lambda_i}) \frac{\theta_1 (r_i)}{\lambda_i} - (\bar{x} + a + \frac{B_i \bar{y}}{2 \lambda_i}) \frac{\theta_2 (r_i)}{\lambda_i} \]

\[ - \frac{\bar{y}}{r_i \lambda_i} \ln \left[ \frac{R_1 (r_i)}{R_2 (r_i)} \right] \]

\[ I_3 (r_i) = - \frac{\cos \beta}{\lambda_i} \ln \left( \frac{R_1 (r_i)}{R_2 (r_i)} \right) \]

\[ - \frac{\sin \beta}{r_i \lambda_i} \left[ \theta_1 (r_i) - \theta_2 (r_i) \right] \]

\[ I_4 (r_i) = - \frac{\sin \beta}{r_i \lambda_i} \ln \left( \frac{R_1 (r_i)}{R_2 (r_i)} \right) \]

\[ + \frac{\cos \beta}{\lambda_i} \left[ \theta_1 (r_i) - \theta_2 (r_i) \right] \]

(4.15)
and

\[ R_1(r_i) = \left[ A_i(\bar{x}-a)^2 + B_i(\bar{x}-a)\bar{y} + C_i\bar{y}^2 \right]^{\frac{1}{2}} \]

\[ R_2(r_i) = \left[ A_i(\bar{x}+a)^2 + B_i(\bar{x}+a)\bar{y} + C_i\bar{y}^2 \right]^{\frac{1}{2}} \quad (4.16) \]

\[ \theta_1(r_i) = \arctan \frac{\bar{y}}{r_i A_i} \]

\[ (\bar{x}-a) + \frac{1}{2} B_i \bar{y}/A_i \]

\[ \theta_2(r_i) = \arctan \frac{\bar{y}}{r_i A_i} \]

\[ (\bar{x}+a) + \frac{1}{2} B_i \bar{y}/A_i \quad (4.17) \]

in which

\[ A_i = \left( r_i^2 \cos^2 \beta + \sin^2 \beta \right)/r_i^2 \]

\[ B_i = \left( 1 - r_i^2 \right) \sin 2\beta / r_i^2 \]

\[ C_i = \left( r_i^2 \sin^2 \beta + \cos^2 \beta \right) / r_i^2 \quad (4.18) \]

for \( i = 1 \) or 2

The equations (4.13) and (4.14) are now comparable to equations (3.16) and (3.17) of section 5.3.3 of the isotropic analysis.
These equations can be used to compute general influence coefficients for orthotropic bodies. The other procedure follows in the same way as already discussed in section 5.3.4 of the isotropic case.

5.4.4 Program Structure for Orthotropic Bodies

A program has been described in FORTRAN IV for the analysis of plane orthotropic bodies using the fictitious stress method detailed above with constant elements. The computations of the program are carried out in 6 steps in a very similar way as explained in the case of isotropic case.

The macroflow chart is shown in the figure 5.11. Details of the main program and other subroutines (three) are the same as that of isotropic case. The main program calculates all the constants and other values that are required in the calculation of general influence coefficients. But subroutine COEFF differs entirely from the isotropic case which make use of equations (4.13) and (4.14) for displacements and stress influence coefficients with other useful formulae from equations (4.15) to (4.18).

5.4.5 Results and Discussions

Using the program demonstrated above, the stress analysis of a circular hole in an infinite orthotropic elastic plate under uniaxial tension is made. The elastic constants are obtained from the experimental results and are as follows:

\[
\begin{align*}
C_{11} & = 4.09847 \times 10^5 \text{ kg/cm}^2 \\
C_{12} & = 0.31763 \times 10^5 \text{ kg/cm}^2
\end{align*}
\]
Figure 5.11 Macroflow Chart of Orthotropic Program
\[ C_{22} = 1.02462 \times 10^5 \text{ kg/cm}^2 \]
\[ C_{66} = 0.800 \times 10^5 \text{ kg/cm}^2 \]

Substituting the above values in the equation (4.9), the values \( r_i \) are obtained as

\[ r_1 = 0.83059 \]
\[ r_2 = 0.55546 \]

The numerical results have been obtained using a 12 element approximation to one-quarter of the circular boundary. The maximum error is around 20%.

5.5 EXPERIMENTAL WORK

Experimental analysis has been done on glass/epoxy composite material. Strain gauge technique has been employed to find the strains and stresses around the boundary of the hole to determine the stress concentration factors.

5.5.1 Specimen Preparation

All the specimens have been prepared from glass fibres in the form of rovings. Araldite LY 556 and Hardener HT 972 in the ratio of 100:27 by weight have been used as matrix material. Laminates are prepared by using hand-lay-up technique. Curing has been done at elevated temperature under constant pressure. All the specimens are cut from the laminates.
5.5.2. Determination of elastic constants

Three specimens are cut from the laminate. For one specimen, the fibre direction is parallel to the axis of the specimen and for the other it is perpendicular to the axis and for the third it is inclined at 45° to the axis. Strain gauges are fixed on to the specimens (Figure 5.12) and they are connected to a strain indicator. Each specimen is tensile tested by a Universal Testing Machine. The strain values are recorded. The stresses, Young's Modulus, Poisson's ratios and the Rigidity modulus are calculated and tabulated (Table 5.3, 5.4 and 5.5).

5.5.3 Testing of Specimen

A specimen is prepared and a circular hole is drilled at the centre of the specimen. Strain gauges are fixed at selected points to measure the stresses so that they could be compared with the numerical values (Figure 5.13). The gauges are connected to a multichannel strain indicator. Tensile load is applied to the specimen and the strain values are recorded. The stress concentration factors are calculated and tabulated (Table 5.6). The variation of SCF is shown in Figure 5.14.

5.5.4 Results and Discussions

From the experiment, it is found that the maximum measured stress concentration factor is 1.75 at a distance of 125 mm from the centre of the hole whose diameter is 19mm. Preliminary calculations using proper extrapolation techniques would indicate that the value of the stress concentration factor at the hole would be around 3.85.
Figure 5.12 Specimens to Determine Elastic Constants

\[ \sigma_1 = \frac{P}{A} \]
\[ \epsilon_1 = \frac{\sigma_1}{E_1} \]
\[ \nu = -\frac{\epsilon_2}{12} \frac{E_1}{E_1} \]

\[ \sigma_2 = \frac{P}{A} \]
\[ \epsilon_2 = \frac{\sigma_2}{E_2} \frac{E_1}{E_1} \]
\[ \nu_{21} = -\frac{E_1}{E_2} \]

\[ E_x = \frac{P}{A} \frac{E_x}{E_x} \]

\[ G_{12} = \frac{1}{\frac{4}{E_w} E_1 E_2 - \frac{2^{1/2} E_1 E_2}{E_1}} \]

-P Axial Load
### Table 5.3 Elastic Constants for 0° Fibre Direction

<table>
<thead>
<tr>
<th>Load (kg)</th>
<th>$\sigma_{xx}$ (kg/cm$^2$)</th>
<th>$\varepsilon_1$ ($10^{-6}$)</th>
<th>$\varepsilon_2$ ($10^{-6}$)</th>
<th>$E_{11}$ (kg/cm$^2$)</th>
<th>$\nu_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>262</td>
<td>650</td>
<td>200</td>
<td>0.40</td>
<td>0.31</td>
</tr>
<tr>
<td>400</td>
<td>525</td>
<td>1200</td>
<td>400</td>
<td>0.43</td>
<td>0.32</td>
</tr>
<tr>
<td>600</td>
<td>787</td>
<td>2000</td>
<td>600</td>
<td>0.39</td>
<td>0.30</td>
</tr>
<tr>
<td>800</td>
<td>1050</td>
<td>2800</td>
<td>870</td>
<td>0.37</td>
<td>0.31</td>
</tr>
</tbody>
</table>

### Table 5.4 Elastic Constants for 90° Fibre Direction

<table>
<thead>
<tr>
<th>Load (kg)</th>
<th>$\sigma_{xx}$ (kg/cm$^2$)</th>
<th>$\varepsilon_1$ ($10^{-6}$)</th>
<th>$\varepsilon_2$ ($10^{-6}$)</th>
<th>$E_{22}$ (kg/cm$^2$)</th>
<th>$\nu_{21}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>262</td>
<td>2460</td>
<td>190</td>
<td>0.106</td>
<td>0.077</td>
</tr>
<tr>
<td>400</td>
<td>525</td>
<td>5280</td>
<td>412</td>
<td>0.099</td>
<td>0.078</td>
</tr>
<tr>
<td>600</td>
<td>787</td>
<td>8000</td>
<td>640</td>
<td>0.098</td>
<td>0.080</td>
</tr>
<tr>
<td>800</td>
<td>1050</td>
<td>9500</td>
<td>665</td>
<td>0.110</td>
<td>0.070</td>
</tr>
</tbody>
</table>
### TABLE 5.5 ELASTIC CONSTANTS FOR 45° FIBRE DIRECTION

<table>
<thead>
<tr>
<th>Load (kg)</th>
<th>$^0\sigma_{xx}$</th>
<th>$^0\epsilon_x$</th>
<th>$^0E_x$</th>
<th>$^0G_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>262</td>
<td>1540</td>
<td>0.106</td>
<td>0.079</td>
</tr>
<tr>
<td>400</td>
<td>525</td>
<td>3045</td>
<td>0.172</td>
<td>0.081</td>
</tr>
<tr>
<td>600</td>
<td>787</td>
<td>8000</td>
<td>0.175</td>
<td>0.084</td>
</tr>
<tr>
<td>800</td>
<td>1050</td>
<td>6150</td>
<td>0.170</td>
<td>0.075</td>
</tr>
</tbody>
</table>

### TABLE 5.6 TEST SPECIMEN READINGS

<table>
<thead>
<tr>
<th>Gauge Location from centre of hole (mm)</th>
<th>Strain ($\times 10^{-6}$)</th>
<th>Stress ($\times 10^6$ kg/cm$^2$)</th>
<th>SCF</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.5</td>
<td>878</td>
<td>351</td>
<td>1.75</td>
</tr>
<tr>
<td>16.5</td>
<td>647</td>
<td>258</td>
<td>1.29</td>
</tr>
<tr>
<td>21.5</td>
<td>582</td>
<td>232</td>
<td>1.16</td>
</tr>
<tr>
<td>25.5</td>
<td>557</td>
<td>222</td>
<td>1.11</td>
</tr>
<tr>
<td>30.0</td>
<td>550</td>
<td>220</td>
<td>1.09</td>
</tr>
<tr>
<td>34.0</td>
<td>537</td>
<td>214</td>
<td>1.07</td>
</tr>
<tr>
<td>38.5</td>
<td>522</td>
<td>208</td>
<td>1.04</td>
</tr>
<tr>
<td>45.0</td>
<td>512</td>
<td>204</td>
<td>1.02</td>
</tr>
</tbody>
</table>
Figure 5.13 Test Specimen
Figure 5.14 Variation of SCF along the line $y/a$.
5.6 CONCLUDING REMARKS

Computer program has been developed for isotropic analysis using Boundary Element Method. The tangential stress distribution around the boundary of the hole has been obtained which agrees with the analytical results already available.

The above program is extended to orthotropic analysis. The results obtained are not very satisfactory. This requires further investigations. It is certainly possible to improve the results by using more number of elements. The experimental results vary approximately by 20% with the numerical results. Experimental results could also be improved by using miniature gauges and making strain measurements closer to the boundary of the hole.