CHAPTER 1

PRELIMINARIES

1.0. Introduction

A brief account of fuzzy set, intuitionistic fuzzy set, decision theory, optimization theory and related topics which are used in the subsequent chapters is given in this opening chapter.

In classical Mathematics, a set is characterized by its characteristic function. If $X$ denotes the universe of discourse, then a subset $A$ of $X$ is expressed as

$$A = \{(x, \chi_A(x)) / x \in X\}$$

where the characteristic function of $A$, $\chi_A : X \rightarrow \{0, 1\}$ is defined as

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Here $\{0, 1\}$ is called the valuation set or membership set. Lotfi A. Zadeh generalized the characteristic function such that, the values
assigned to the elements of the universal set fall within a specified range \([0,1]\), and this value indicate the membership grades of these elements in the given set.

Now we give the formal definition of fuzzy set as follows.

### 1.1. Fuzzy sets

#### 1.1.1. Definition.**

Let \(X\) be any non empty set and \(I = [0, 1]\). A fuzzy (sub) set \(A\) of \(X\) is given by,

\[
A = \{ (x, \mu_A(x)) / x \in X \}
\]

where \(\mu_A : X \to I\) is called the membership function of \(A\) and \([0, 1]\) is called the membership set. \(\mu_A(x)\) is called the degree of membership of the element \(x\) in \(A\) and \(0 \leq \mu_A(x) \leq 1\) for each \(x \in X\).

In other words, any fuzzy subset of \(X\) with membership set \(I = [0,1]\) is a member of \(I^X\), where \(I^X\) denotes the set of all functions from \(X\) to \(I\). Also a fuzzy set is characterized by its membership function just as a classical set is characterized by its characteristic function.

#### 1.1.2. Note.**

We can use either \(A(x)\) or \(\mu_A(x)\) for membership value of \(x\). In this work \(\mu_A(x)\) denotes membership value or grade of \(x\) in \(A\), otherwise stated.
1.1.3. **Definition.** For a fuzzy set $A$ defined on any given set $X$, and for any number $\alpha \in [0,1]$, the $\alpha$ cut of $A$, denoted by $^{\alpha}A$, is defined as

$$^{\alpha}A = \{ x : x \in X \text{ such that } \mu_A(x) \geq \alpha \}$$

1.1.4. **Definition.** For a fuzzy set $A$ in $X$, and for any number $\alpha \in [0,1]$, the strong $\alpha$ cut of $A$, denoted by $^{\alpha+}A$, is defined as

$$^{\alpha+}A = \{ x : x \in X \text{ such that } \mu_A(x) > \alpha \}$$

1.1.5. **Note.** The total ordering of values of $\alpha \in [0,1]$, is inversely preserved by set inclusion of the corresponding $\alpha$ cuts as well as strong $\alpha$ cuts.

That is, for any set $A$, and for any pair of distinct values $\alpha_1, \alpha_2 \in [0,1]$ such that $\alpha_1 < \alpha_2$, we have

$$^{\alpha_1}A \supseteq {^{\alpha_2}A}$$

and,

$$^{\alpha_1+}A \supseteq {^{\alpha_2+}A}$$

1.1.6. **Note.** An obvious consequence of this property is that all $\alpha$ cuts and strong $\alpha$ cuts of any fuzzy set form two distinct families of nested crisp sets.
1.1.7. **Definition.** The *support* of a fuzzy set $A$, denoted by $\text{supp}(A)$, within a universal set $X$ is the crisp set that contains all the elements of $X$ that have nonzero membership grades in $A$.

Clearly, $\text{supp}(A)$ of $A$ is the strong $\alpha$ cut of $A$ for $\alpha = 0$. The *height*, denoted by $h(A)$, of a fuzzy set $A$ is the largest membership grade obtained by any element in that set. Formally,

$$h(A) = \sup_{x \in X} \mu_A(x)$$

1.1.8. **Definition.** A fuzzy set $A$ is said to be *normal*, if $h(A) = 1$. That is, if

$$\sup_{x \in X} \mu_A(x) = 1$$

1.1.9. **Definition.** Let $A$ be any fuzzy set in $X$, for the crisp sets $\alpha A$ and $\alpha^+ A$ we define the fuzzy sets denoted by $\alpha A$ and $\alpha^+ A$ as

$$\alpha A(x) = \alpha \cdot \alpha A(x)$$

and, $\alpha^+ A(x) = \alpha \cdot \alpha^+ A(x)$

1.1.10. **Definition.** Let $A$ be any fuzzy set in $X$. Then the level set of $A$ denoted by $\land(A)$, is defined as

$$\land(A) = \{\alpha/ \mu_A(x) = \alpha \text{ for some } x \in X\}$$

19
1.1.11. **Definition.** A fuzzy set $A$ on $\mathbb{R}$, the set of real numbers, is said to be a fuzzy number if $A$ satisfies the following properties

(i) $A$ must be a normal fuzzy subset of $\mathbb{R}$;

(ii) each $\alpha$ cut of $A$ must be closed interval for every $\alpha \in (0,1]$;

(iii) the support of $A$, $0^+A$, must be bounded.

We now state the three decomposition theorems as put forward by Lotfi A. Zadeh

1.1.12. **First Decomposition Theorem of fuzzy sets.**

For any fuzzy set $A$,

$$A = \bigcup_{\alpha \in [0,1]} \alpha A$$

1.1.13. **Second Decomposition Theorem of fuzzy sets.**

For any fuzzy set $A$,

$$A = \bigcup_{\alpha \in [0,1]} \alpha + A$$

1.1.14. **Third Decomposition Theorem of fuzzy sets.**

For any fuzzy set $A$,

$$A = \bigcup_{\alpha \in \Lambda(A)} \alpha A ;$$

where $\Lambda(A)$ is the level set of $A$. 

20
1.1.15. Drastic union.

Let \( x \in X \), the universal set, for two fuzzy sets \( A \) and \( B \), such that \( \mu_A(x) = a, \mu_B(x) = b \) then, the drastic union of \( A \) and \( B \) denoted by \( A \cup_D B \) is defined as

\[
(A \cup_D B) (x) = a \quad \text{when } b = 0 \\
= b \quad \text{when } a = 0 \text{ and} \\
= 1 \quad \text{otherwise.}
\]

1.1.16. Drastic intersection.

Let \( x \in X \), the universal set, for two fuzzy sets \( A \) and \( B \), such that \( \mu_A(x) = a, \mu_B(x) = b \) then, the drastic intersection of \( A \) and \( B \) denoted by \( A \cap_D B \) is defined as

\[
(A \cap_D B) (x) = a \quad \text{when } b = 1 \\
= b \quad \text{when } a = 1 \text{ and} \\
= 0 \quad \text{otherwise.}
\]

Now we will outline the concept of Intuitionistic fuzzy set in the next section.
1.2. Intuitionistic fuzzy sets

The theory of fuzzy sets proposed by L. A. Zadeh was extended further by K. T. Atanassov and he introduced intuitionistic fuzzy sets. Formally an intuitionistic fuzzy set is defined as follows.

1.2.1. Definition. Let \( X \) be a given set. An Intuitionistic fuzzy set \( A \) in \( X \) is given by,

\[
A = \{ (x, \mu_A(x), \nu_A(x)) / x \in X \}
\]

where \( \mu_A, \nu_A : X \rightarrow [0, 1] \), and \( 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \). Also \( \mu_A(x) \) is the degree of membership of the element \( x \) in \( A \) and \( \nu_A(x) \) is the degree of non membership of \( x \) in \( A \).

For each \( x \in X \), \( \pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \) is called the degree of hesitation.

Now we give the algebra of IFS

1.2.2. Algebra of Intuitionistic Fuzzy Sets. If \( A \) and \( B \) are IFSs in \( X \), then

(1) \( A \subset B \) iff \( \forall x \in X, \mu_A(x) \leq \mu_B(x) \) and \( \nu_A(x) \geq \nu_B(x) \)

(Inclusion)

(2) \( A = B \) iff \( \forall x \in X, \mu_A(x) = \mu_B(x) \) and \( \nu_A(x) = \nu_B(x) \)

(Equality)
(3) $\overline{A} = \{(x, \nu_A(x), \mu_A(x)), x \in X\}$ (Complement of A)

(4) $A \cap B = \{(x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)))/x \in X\}$ (Intersection)

(5) $A \cup B = \{(x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x)))/x \in X\}$ (Union)

(6) $A \cdot B = \{(x, \mu_A(x) \cdot \mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x) \cdot \nu_B(x)) / x \in X\}$ (Product)

(7) $nA = \{(x, 1 - (1 - \mu_A(x))^n, (\nu_A(x))^n)/x \in X\}$ (Scalar Multiplication)

(8) $A^n = \{(x, (\mu_A(x))^n, 1 - (1 - \nu_A(x))^n)/x \in X\}$ (Power)

(9) $A + B =$

\[
\{(x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), \nu_A(x) \cdot \nu_B(x)) / x \in X\} \text{ (Sum)}
\]

Now we recall the definition of $(\alpha, \beta)$ cut of an IFS $A$ and its properties [53].

1.2.3. Definition. Let $A$ be any IFS then, its $(\alpha, \beta)$ cut is defined by

$$C_{\alpha, \beta} = \{x : x \in X \text{ such that } \mu_A(x) \geq \alpha, \ \nu_A(x) \leq \beta\}$$

Let us denote it by $(\alpha, \beta)A$ for short.

That is, $(\alpha, \beta)A = \{x : x \in X \text{ such that } \mu_A(x) \geq \alpha, \ \nu_A(x) \leq \beta\}$
1.2.4. Properties of \((\alpha, \beta)\) cut of an IFS.

If A and B are two IFS, then the following properties hold

(i) \((\alpha, \beta)A \subseteq (\delta, \theta)A\) if \(\alpha \geq \delta\) and \(\beta \leq \theta\)

(ii) \((1-\beta, \beta)A \subseteq (\alpha, \beta)A \subseteq (\alpha, 1-\alpha)A\)

(iii) \(A \subseteq B \Rightarrow (\alpha, \beta)A \subseteq (\alpha, \beta)B\)

(iv) \((\alpha, \beta)(A \cap B) = (\alpha, \beta)A \cap (\alpha, \beta)B\)

(v) \((\alpha, \beta)(A \cup B) \supseteq (\alpha, \beta)A \cup (\alpha, \beta)B\) , equality hold if, \(\alpha + \beta = 1\)

(vi) \((\alpha, \beta)(\cap A_i) = \cap (\alpha, \beta)A_i\)

(vii) \((0, 1)A = X\) where \(\alpha, \beta \in [0, 1]\) with \(\alpha + \beta \leq 1\)

The proofs of the above are obvious.

Now we define an intuitionistic fuzzy number (IFN for short) as follows.

1.2.5. Definition. An intuitionistic fuzzy set (IFS) \(A = (\mu_A, \nu_A)\) of \(\mathbb{R}\) is said to be an intuitionistic fuzzy number if \(\mu_A\) and \(\nu_A\) are fuzzy numbers with \(\mu_A, \nu_A \in [0, 1]\) and \(\mu_A + \nu_A \leq 1\)

Intuitionistic fuzzy numbers, each of which is characterized by the degree of membership and the degree of non-membership of an element, are a very useful means to depict the decision information in the process of decision making.
1.2.6. **Remark** The definition of interval valued intuitionistic fuzzy set (IVIFS) will be given later.

1.2.7. **Definition.** Let $X$ be any given set. A set of $(\alpha, \beta)$ level, generated by an IFS $A$ in $X$ is defined as

$$N_{\alpha,\beta}(A) = \{(x, \mu_A(x), \nu_A(x))/x \in X, \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta\}$$

where $\alpha, \beta \in [0, 1]$ are fixed numbers such that $\alpha + \beta \leq 1$.

1.2.8. **Definition.** Let $X$ be any given set. If $A$ and $B$ are two IVIFSs in $X$, then the union and intersection of $A$ and $B$ can be defined as

$$A \cup B = \{(x, [\max(\mu_{AL}(x), \mu_{BL}(x)), \max(\mu_{AU}(x), \mu_{BU}(x))], [\min(\nu_{AL}(x), \nu_{BL}(x)), \min(\nu_{AU}(x), \nu_{BU}(x))])\}$$

and

$$A \cap B = \{(x, [\min(\mu_{AL}(x), \mu_{BL}(x)), \min(\mu_{AU}(x), \mu_{BU}(x))], [\max(\nu_{AL}(x), \nu_{BL}(x)), \max(\nu_{AU}(x), \nu_{BU}(x))])\}$$

1.2.9. **Definition.** For every IFS $A$ over the universe $X$, the Closure $C(A)$ and Interior $I(A)$ are defined as

$$C(A) = \{(x, \max_{y \in X} \mu_A(y), \min_{y \in X} \nu_A(y))/x \in X\}$$

and

$$I(A) = \{(x, \min_{y \in X} \mu_A(y), \max_{y \in X} \nu_A(y))/x \in X\}$$
1.2.10. Definition. Let A be an IFS over the universal set X, the operators boxplus (⊞(A)) and boxtimes (⊠(A)) are defined by

\[ \forall x \in X, \ (\forall x, \ \frac{\mu_A(x)}{2}, \frac{\nu_A(x)+1}{2})/x \in X \]

and

\[ \forall x \in X, \ (\forall x, \ \frac{\mu_A(x)+1}{2}, \frac{\nu_A(x)}{2})/x \in X \].

We discuss some of the aggregation operators in the next section.

1.3. Intuitionistic Fuzzy Aggregation Operators

The following definitions are from [66],[64],[68]

1.3.1. Definition. Let \( A_j \) (j = 1, 2, ..., n) be IFSs. The arithmetic average operator denoted by \( F(A_1, A_2, ..., A_n) \) is defined by

\[ F(A_1, A_2, ..., A_n) = (1 - \prod (1 - \mu_{A_j}(x)), \prod (\nu_{A_j}(x))). \]

1.3.2. Definition. Let \( A_j \) (j = 1, 2, ..., n) be IFSs. The geometric average operator denoted by \( G(A_1, A_2, ..., A_n) \) is defined by

\[ G(A_1, A_2, ..., A_n) = (\prod \mu_{A_j}(x), 1 - \prod (1 - \nu_{A_j}(x))). \]

1.3.3. Definition. Let \( A_j \) (j = 1, 2, ..., n) be IFSs. The weighted arithmetic average operator denoted by \( F_w(A_1, A_2, ..., A_n) \) is defined by
\[ F_w(A_1, A_2, \ldots A_n) = (1 - \prod (1 - \mu_{A_j}(x))^{w_j}, \prod (\nu_{A_j}(x))^{w_j}). \]

where \( w_j \) is the weight of \( A_j (j = 1, 2, \ldots n) \), \( w_j \in [0, 1] \)
and \( \sum_{j=1}^{n} w_j = 1 \)

1.3.4. Definition. Let \( A_j (j = 1, 2, \ldots n) \) be IFSs. The weighted geometric average operator denoted by \( G_w(A_1, A_2, \ldots A_n) \) is defined by
\[ G_w(A_1, A_2, \ldots A_n) = (\prod (\mu_{A_j}(x))^{w_j}, 1 - \prod (1 - \nu_{A_j}(x))^{w_j}). \]
Where \( w_j \) is the weight of \( A_j (j = 1, 2, \ldots n) \), \( w_j \in [0, 1] \)
and \( \sum_{j=1}^{n} w_j = 1 \).

1.3.5. Definition. Let \( A_j (j = 1, 2, \ldots n) \) be IFSs. The intuitionistic fuzzy weighted averaging (IFWA) operator denoted by \( IFWA_w(A_1, A_2, \ldots A_n) \) is defined by
\[ IFWA_w(A_1, A_2, \ldots A_n) = (1 - \prod_{j=1}^{n} (1 - \mu_{A_j}(x))^{w_j}, 1 - \prod_{j=1}^{n} (\nu_{A_j}(x))^{w_j}). \]

where \( w_j \) is the weight of \( A_j (j = 1, 2, \ldots n) \), \( w_j \in [0, 1] \)
and \( \sum_{j=1}^{n} w_j = 1 \).

1.3.6. Definition. Assume \( a_i = (\mu_{a_i}, \nu_{a_i}) \) are IFNs, and \( b_i = (\mu_{b_i}, \nu_{b_i}) \)
are ordered IFNs of \( a_i = (\mu_{a_i}, \nu_{a_i}) \) from large to small for \( i = 1, \ldots, n \)
(i) If \( \omega = (\omega_1, \ldots, \omega_n) \) is the weight vector of position, then the ag-
Aggregation operator of intuitionistic fuzzy ordered weighted average is defined by

\[ IFOWA_{\omega}(a_1, ..., a_n) = \omega_1 b_1 + ... + \omega_n b_n \]

(ii) If \( \omega = (\omega_1, ..., \omega_n) \) is the exponential weight vector of position, then the aggregation operator of intuitionistic fuzzy ordered weighted geometric is defined by

\[ IFOWG_{\omega}(a_1, ..., a_n) = b_1^{\omega_1} ... b_n^{\omega_n} \]

1.3.7. Note. Even after aggregation, the aggregation operators \( F, G, F_w, G_w, IFWA_w, IFOWA_{\omega} \) and \( IFOWG_{\omega} \) are still IF numbers.

Several authors like Xu and Lakshmana Gomathi Nayagam studied comparison between two intuitionistic fuzzy numbers, to rank them. For that they define score function and accuracy function. Bigger the score indicates, larger the intuitionistic fuzzy number. Thus, score function can be used as a useful tool to compare intuitionistic fuzzy numbers. However, in certain situations, i.e., if the score values of two intuitionistic fuzzy numbers are equal, Xu introduced accuracy function for comparing these two intuitionistic fuzzy numbers.

Now we introduce some score function and accuracy function.
1.4. Score function and Accuracy function

The following definitions are from [39], [40] and [75]

1.4.1. Definition. Z. S. Xu, defined score functions and accuracy functions from the set of alternatives to the set of real numbers as follows.

For an alternative A, represented by an IFN A, with degree of membership(a) and degree of non membership(b) denoted by A = (a,b), the score function value of A is given by

\[ S(A) = a - b \]

The next is the definition of an accuracy function given by Xu.

1.4.2. Definition. Let A = (a,b) be an IFN representing an alternative A. The accuracy function value of A, denoted by H(A) is given by

\[ H(A) = a + b \]

Here score function and accuracy function are functions from the set of IFNs considered, (say) set of alternatives, to the set of real numbers. Let \( A_1 = (a_1, b_1) \) and \( A_2 = (a_2, b_2) \) be two IF numbers. Then, \( A_1 \subset A_2 \) if \( S(A_1) < S(A_2) \) and vice versa. If \( S(A_1) = S(A_2) \),
then find the value of accuracy function to make a decision. In such case $A_1 \subset A_2$ if $S(A_1) = S(A_2)$ and $H(A_1) < H(A_2)$.

1.4.3. **Definition.** Let $A = ([a, b], [c, d])$ be an interval-valued intuitionistic fuzzy number. Based on the hesitancy degree, an accuracy function $H$ defined on the set of IVIFNs (set of alternatives) to the set of real numbers is expressed as

$$H(A) = \frac{a + b + c + d}{2}$$

1.4.4. **Definition.** Let $A = ([a, b], [c, d])$ be an interval-valued intuitionistic fuzzy number. Based on the hesitancy degree, an accuracy function $M$ defined on the set of IVIFNs considered (set of alternatives) to the set of real numbers is given by

$$M(A) = a + b - 1 + \frac{c + d}{2}$$

1.4.5. **Definition.** Let $A = ([a, b], [c, d])$ be an interval-valued intuitionistic fuzzy number. Based on the hesitancy degree, an accuracy function $L$, defined as above is given by

$$L(A) = \frac{a + b - d(1-b) - c(1-a)}{2}$$

Let us now recall score function and accuracy function given by Zhi Pei and Li Zheng for ranking of alternatives.
1.4.6. **Definition.** Consider a decision making model with ‘n’ alternatives \{A_1, ..., A_n\}, with respect to the ‘m’ attributes \{T_1, ..., T_m\}. If \(\mu_{ij}\) and \(\nu_{ij}\) denote the degree to which \(i^{th}\) alternative satisfies \(j^{th}\) attribute and the degree to which \(i^{th}\) alternative does not satisfies \(j^{th}\) attribute respectively. Also if the IFN \((\mu_i, \nu_i)\) for the \(i^{th}\) alternative \(A_i\) is

\[
\mu_i = \sum_{j=1}^{m} w_j \mu_{ij} \quad \text{and} \quad \nu_i = \sum_{j=1}^{m} w_j \nu_{ij}
\]

where \(w_j\) is the normalized weight of the \(j^{th}\) attribute, which satisfying \(w_j > 0\) and \(\sum_{j=1}^{m} w_j = 1\). Then the score function value and accuracy function value for the \(i^{th}\) alternative \(A_i\) are respectively defined by

\[
S_i = \alpha \mu_i - \beta \nu_i
\]

and

\[
H_i = \alpha \mu_i - \gamma \pi_i
\]

where \(\alpha, \beta\) and \(\gamma\) stands respectively for the relative importance of the degree of membership, non membership and hesitation. Also \(\alpha, \beta, \gamma \in [0, 1]\) with \(\alpha \geq \beta \geq \gamma\) and \(\alpha + \beta + \gamma = 1\). We can find \(\alpha, \beta\) and \(\gamma\) by solving the following linear programming model, which could be solved by simplex method as in [81].
\[
\max \sum_{i=1}^{n} S_i = \sum_{i=1}^{n} (\alpha \mu_i - \beta \nu_i)
\]
such that,
\[
\alpha' \leq \alpha \leq \alpha'', \quad \beta' \leq \beta \leq \beta'', \quad \gamma' \leq \gamma \leq \gamma'', \quad \alpha \geq \beta \geq \gamma, \\
\alpha + \beta + \gamma = 1.
\]

where the objective function is the sum of the score function values for all the alternatives. By substituting the values of \(\alpha\) and \(\beta\) in score function, Zhi Pei and Li Zheng [81] ranked the alternatives \(A_i\).

If \(S_i = S_j\) for \(i \neq j\), they used accuracy function together with the score function to rank the alternatives.

Score function and accuracy function help in ranking of alternatives, which are important factors in decision making.

Now we discuss the basic concepts of decision making.

### 1.5. Decision Making.

A decision problem consists of two stages, the *structural stage* and the *decisional stage*. Although much of the decision models are
developed using classical mathematical tools, several decision models are being developed with the help of fuzzy and intuitionistic fuzzy tools. Decision problem which involve a single decision maker is generally referred as *individual decision making problems* and those involve several decision makers is called *multiperson decision making problem*.

We studied individual as well as multiperson decision making in this work. Selection of the best alternative from various alternatives is the problem involved here. Each alternative has to satisfy various criteria, sometimes these criteria are such that, they have different level of importance. We call them weighted criteria.

**1.6. Optimization model: Assignment and Transportation model.** Most of the definitions in this section are taken from [2], [3] and [4]

**1.6.1. Definition.** In classical sense, optimization model consists of objective function(s) and a set of constraints. The value of the variable which satisfies the constraints and optimizes (maximize or minimize) the objective function(s) is called the optimal solution of the model. In 1994 Plamen P. Angelov studied optimization prob-
lem in the fuzzy context [3], and later in 1995 he extended it to intuitionistic fuzzy case [4]. According to Plamen, in intuitionistic fuzzy optimization theory, we have to maximize the degree of acceptance of the IF objective(s) and constraints and to minimize the degree of rejection of IF objective(s) and constraints.

That is,

\[
\begin{align*}
\max_{x \in \mathbb{R}^n} \left( \mu_i(x) \right) \quad & \text{where} \quad i = 1, \ldots, p + q \\
\min_{x \in \mathbb{R}^n} \left( \nu_i(x) \right) \quad & \text{where} \quad i = 1, \ldots, p + q
\end{align*}
\]

subject to

\[
\begin{align*}
\nu_i(x) & \geq 0 \quad i = 1, \ldots, p + q \\
\mu_i(x) & \geq \nu_i(x) \quad i = 1, \ldots, p + q \\
\mu_i(x) + \nu_i(x) & \leq 1 \quad i = 1, \ldots, p + q
\end{align*}
\]

where,

- \( x \) denotes unknowns,
- \( \mu_i(x) \) denotes degree of membership of \( x \) to the \( i^{th} \) IFS,
- \( \nu_i(x) \) denotes degree of rejection of \( x \) to the \( i^{th} \) IFS,
- \( p \) denotes the number of objectives,
- \( q \) denotes the number of constraints.

Among various optimization models in literature, we have studied, Assignment model and Transportation model.
Mathematical model of a classical assignment problem is defined as follows.

1.6.2. Definition\[83].

Assume that there are $n$ jobs and $n$ persons. Each job must be assigned to one and only one person and each person has to perform one and only one job. Let $c_{ij}$ be the cost of assigning $i^{th}$ person to $j^{th}$ job. The problem is to find an assignment (which job should be assigned to which person) so that the total cost for performing all jobs is minimum. Assume that, $i^{th}$ person is assigned to $j^{th}$ job and is denoted by $x_{ij}$. Also and let

$$x_{ij} = 1 \text{ if } i^{th} \text{ person is assigned to } j^{th} \text{ job}$$

and $x_{ij} = 0 \text{ if } i^{th} \text{ person is not assigned to } j^{th} \text{ job}$

Then, the mathematical model of assignment problem in crisp environment is,

$$\text{Min } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}x_{ij}$$

subject to,

$$\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, ... n$$

$$\sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, ... n$$
\[ x_{ij} \in \{0, 1\} \text{ for } i,j = 1,2,\ldots,n \]

Our IF assignment model is an extension of crisp assignment model, we adopt the Hungarian method to IF context.

1.6.3. Computational Procedure. The Hungarian method which is actually rooted in the simplex method consists of the following steps, as in [29].

Step-1

For the original cost matrix, identify minimum cost in each row, and subtract it from all the entries of the row.

Step-2

For the matrix resulting from step 1, identify minimum cost in each column, and subtract it from all the entries of the column.

Step-3

Identify the optimal assignment as the one associated with the zero elements of the matrix obtained in step 2.

Step-2a

If no feasible assignment (with all zero entries) is obtained from steps 1 and 2,

(i) Draw the minimum number of horizontal and vertical lines in the reduced matrix obtained from step 2, that will cover all the zero
entries.

(ii) Select the smallest uncovered element, and subtract it from every uncovered element; then add it to every element at the intersection of two lines.

(iii) If no feasible assignment can be found among the resulting zero entries, repeat step 2a. Otherwise, go to step 3 to determine the optimal assignment.

The Hungarian method is applied in Intuitionistic fuzzy Assignment model in section 7.2.

In this classical assignment model, the variables, assignment cost coefficients, are crisp values. However, in real life situations, the parameters of assignment problem are imprecise numbers, as time/cost for doing a job by a person might vary due to different reasons. Therefore assignment cost coefficients are usually considered as uncertain and vague. If cost coefficients are fuzzy numbers, then total assignment cost becomes fuzzy also. Now the problem is how to achieve a minimum total cost under fuzzy cost \( \tilde{c}_{ij} \). Then the above classical assignment problem turns into the following fuzzy assignment problem.
Min $\tilde{Z} = \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{c}_{ij}x_{ij}$

subject to

$\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, ... n$

$\sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, ... n$

$x_{ij} \in \{0, 1\}$ for $i,j = 1,2,\ldots,n$

1.7. Fundamental Theorems of Fuzzy Assignment Problem [58].

The solution to fuzzy assignment problem is fundamentally based on the following two theorems.

1.7.1. Theorem. Fuzzy Reduction Theorem on Assignment Problem.

The fuzzy assignment minimizes the total fuzzy cost for the new fuzzy cost matrix. It also minimizes the total fuzzy cost for the original fuzzy cost matrix.

1.7.2. Theorem. If $x_{ij}, i= 1,2,\ldots,n; j= 1,2,\ldots,n$ is an optimal solution for an assignment problem with cost $\tilde{c}_{ij}$, then it is also optimal for the problem with cost $\tilde{c}'_{ij}$, where
\[ \tilde{c}_{ij} = \tilde{c}_{ij} \] for \( i, j = 1, 2, ... n; j \neq k \) and

\[ \tilde{c}'_{ij} = \tilde{c}_{ij} - \tilde{A}, \] where \( \tilde{A} \) is a fuzzy constant.

In chapter 7 we discuss, how we can solve the same problem in intuitionistic fuzzy context.

### 1.8. Mathematical formulation of Intuitionistic fuzzy transportation problem.

Consider an intuitionistic fuzzy transportation problem (IFTP) with \( m \) origins and \( n \) destinations. Let \( \tilde{c}_{ij} \) be the cost of transporting one unit of the product from \( i^{th} \) origin to \( j^{th} \) destination. Let \( a_i \) be the total availability of the product at the \( i^{th} \) origin and \( b_j \) be the total demand of the product at \( j^{th} \) destination. Let \( x_{ij} \) be the quantity transported from \( i^{th} \) origin to \( j^{th} \) destination so as to minimize the total intuitionistic fuzzy transportation cost. Therefore the IFTP in which the decision maker is uncertain about the precise values of transportation cost from \( i^{th} \) origin to \( j^{th} \) destination but sure about the supply and demand of the product can be defined as

\[
\text{Minimize } \sum_{i=1}^{m} \sum_{j=1}^{n} \tilde{c}_{ij} x_{ij}
\]

subject to

\[
\sum_{j=1}^{n} x_{ij} \leq a_i \quad i = 1, 2, ... m
\]
\[ \sum_{i=1}^{m} x_{ij} \geq b_j \quad j = 1, 2, \ldots, n \]
\[ x_{ij} \geq 0 \quad \forall \ i, j \]

If \( \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \), then IFTP is said to be balanced, otherwise it is said to be unbalanced IFTP.

The next chapter deals with \((\alpha, \beta)\) cut and strong \((\alpha, \beta)\) cut of intuitionistic fuzzy sets and their properties. Using these cuts, we define three decomposition theorems for an intuitionistic fuzzy set.