Chapter 5
Theoretical Analysis for Crack Assessment Using Longitudinal Vibration of Beam

5.1 INTRODUCTION

Some time the longitudinal vibration measurement based crack assessment of structure is simple than transverse vibration measurement.

This chapter describes the analytical method for assessment of crack location and crack size using longitudinal vibration of beam.

5.2 LONGITUDINAL VIBRATION OF BEAM

To derive the differential equation of motion for the longitudinal vibration of bar, consider a thin and uniform slender beam as shown in Fig. 5.1.

The beam can execute axial or longitudinal vibrations when the equilibrium condition is disturbed axially due to the axial forces.

Figure 5.1 Displacement of element of the beam
Fig. 5.2 shows the free-body diagram of a differential element of this beam of length ‘dx’. The equilibrium position of the element is denoted by ‘x’ and the deformed position is ‘u’. That is, if ‘u’ is the displacement at ‘x’, then the displacement at \( x + dx \) is \( u + \frac{\partial u}{\partial x} \, dx \). In other words, the deformed length of the element is \( \frac{\partial u}{\partial x} \, dx \) greater than the original length.

![Free body diagram of beam](image)

**Figure 5.2 Free body diagram of beam**

If ‘u’ is displacement at a distance ‘x’ from left and it becomes \([u+(\partial u/\partial x)dx]\) at a distance \([x + dx]\). It is clear that the element ‘dx’ has changed its position by an amount \((\partial u/\partial x)dx\). So strain of the element is given by,

\[
\varepsilon = \frac{\partial u}{\partial x} \]

Let,

\[ A = \text{cross sectional area of the bar} \]

\[ \rho = \text{density of the material} \]

\[ E = \text{modulus of elasticity of the material} \]

\[ F = \text{force acting axially on the bar} \]
Net force acting on the element is,

\[ [F + \frac{\partial F}{\partial x}] - F = (\text{Mass}) \times (\text{Acceleration of the Element}) \]

\[ \frac{\partial F}{\partial x} \, dx = \frac{dm}{dx} \times \frac{\partial^2 u}{\partial t^2} \]

\[ \frac{\partial F}{\partial x} \, dx = (\rho \, dx \, A) \times \frac{\partial^2 u}{\partial t^2} \]

...(5.2)

We know that \( F/A = \sigma \), where \( \sigma \) is the stress so,

\[ F = \sigma \, A \]

\[ \frac{\partial F}{\partial x} = \frac{\partial \sigma}{\partial x} \, A \]

\[ \frac{\partial F}{\partial x} \, dx = \frac{\partial \sigma}{\partial x} \, dxA \]

...(5.3)

Equation (5.2) can be written with the help of above equation as,

\[ \frac{\partial \sigma}{\partial x} \, dxA = (\rho dxA) \frac{\partial^2 u}{\partial t^2} \]

...(5.4)

According to Hook’s law,

\[ \frac{\text{stress}}{\text{strain}} = E \]
\[ \frac{\sigma}{\varepsilon} = E \]

\[ \sigma = \varepsilon E \]

\[ \frac{\partial \sigma}{\partial x} = \frac{\partial \varepsilon}{\partial x} E \]

\[ (\frac{\partial \sigma}{\partial x})dxA = (\frac{\partial \varepsilon}{\partial x})dxE \]

\[ \cdots (5.5) \]

With the help of equation (5.4) and (5.5) we get,

\[ (\frac{\partial \varepsilon}{\partial x})dxE = \rho dxA \frac{\partial^2 u}{\partial t^2} \]

\[ \frac{E}{\rho} \frac{\partial \varepsilon}{\partial x} = \frac{\partial^2 u}{\partial t^2} \]

\[ \therefore \frac{E}{\rho} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial^2 u}{\partial t^2} \]

\[ \therefore \frac{E}{\rho} \left( \frac{\partial^2 u}{\partial x^2} \right) = \frac{\partial^2 u}{\partial t^2} \]

\[ \therefore \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \]
Where,

\[ c^2 = \frac{E}{\rho} \]

\[ \therefore c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \]

\[ \cdots (5.6) \]

Solution of equation (5.6) can be obtained by assuming \( u \) to be function of \( U(x) \) and \( T(t) \).

\[ \therefore u(x, t) = U(x).T(t) \]

This is an assumption. To prove it right,

\[ u = UT \]

Therefore from above equation we get,

\[ \frac{\partial^2 u}{\partial x^2} = T \frac{\partial^2 U}{\partial x^2} \quad \text{and} \]

\[ \frac{\partial^2 u}{\partial t^2} = U \frac{\partial^2 T}{\partial t^2} \]

Substituting these in equation (5.6) we get,

\[ c^2 . T \frac{\partial^2 U}{\partial x^2} = U \frac{\partial^2 T}{\partial t^2} \]

\[ \therefore \frac{c^2}{U} \frac{\partial^2 U}{\partial x^2} = \frac{1}{T} \frac{\partial^2 T}{\partial t^2} \]

\[ \cdots (5.7) \]
Left hand side is a function of ‘x’ alone and right hand side is a function of ‘t’ alone. These two can only be equal if each expression is a constant. This constant may be positive, zero or negative. If we take it positive constant or zero, we do not get vibratory motion, which is contrary to our observations with practical system.

Therefore this constant has to be negative constant, \(-\omega^2\).

\[
\mathbf{c}^2 \frac{\partial^2 U}{\partial x^2} \frac{1}{U} \frac{\partial^2 T}{T} \partial t^2 = -\omega^2
\]

\[
\therefore \frac{c^2}{U} \frac{\partial^2 U}{\partial x^2} = -\omega^2
\]

\[
\therefore \frac{c^2}{U} \frac{\partial^2 U}{\partial x^2} + \omega^2 = 0
\]

\[
\therefore \frac{\partial^2 U}{\partial x^2} + \frac{\omega^2}{c^2} U = 0
\]

Solving above equation we get,

\[
\therefore U = a_1 \sin \left( \frac{\omega}{c} x \right) + a_2 \cos \left( \frac{\omega}{c} x \right)
\]

\[\ldots (5.8)\]
5.3 IDENTIFICATION OF CRACK LOCATION IN CANTILEVER BEAM BY MEASUREMENT OF LONGITUDINAL VIBRATION

The physical model, which will be considered, is a cantilever uniform Euler-Bernoulli beam as shown in Fig. 5.3. The length of the beam is ‘L’ and it has a crack at a distance ‘L₁’ from the left support. The beam has constant cross-section area ‘A’ and geometric moment of inertia ‘I’. Its material properties i.e. Young’s modulus ‘E’ and mass density ‘ρ’ are also constant.

![Diagram of cracked cantilever beam]

**Figure 5.3 Model of cracked cantilever beam**

The equation (5.8) of the motion of Euler-Bernoulli is does not hold near the crack, due to abrupt change of the cross-section. The beam can be treated as two uniform beams, connected by a linear spring at the crack location. The equation (5.8) is then valid for each segment of the beam separately, with the appropriate boundary condition. The left segment of the beam will be designated by subscript 1, the right one by 2. The end points are designated by A and C, and the crack section by B. This kind of modeling for the cracked beam has the advantage of using the exact solution throughout the beam, except for a narrow region near the crack, where the true stress-strain field is approximated by spring.
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From equation (5.8) for two beam segments we get,

\[
\therefore U_1(x) = a_1 \sin \left( \frac{\omega}{c} x \right) + a_2 \cos \left( \frac{\omega}{c} x \right) \quad \text{(5.9a)}
\]

\[
\therefore U_2(x) = a_3 \sin \left( \frac{\omega}{c} x \right) + a_4 \cos \left( \frac{\omega}{c} x \right) \quad \text{(5.9b)}
\]

Where ‘x’ is the origin for both segments from the fixed end A. The coefficients \(a_i\) can be found by substituting this solution in boundary conditions. The boundary conditions cantilever beam are as follows. The boundary conditions consistent with model of Fig. 5.3, i.e. no displacement at fixed end A and no axial force at free end C are,

\[U_{1A} = U_{2C} = 0\]

Where the subscripts 1 and 2 denote the two beam segments. The prevailing conditions at the crack cross-section are axial force continuity and at the extension-force relation for the equivalent spring,

\[U_{1B} = U_{2B} \quad \text{and} \quad U_{2B} - U_{1B} = \Theta_1 U'_{1B}\]

Substituting above boundary conditions in equations (5.9a) and (5.9b) and equating the system determinant to zero and obtains algebraic equations for the natural frequencies of the cracked beam,

\[
\cos \gamma + \Theta_1 \gamma \cos(\gamma L_1/L)[\sin(\gamma L_1/L)\cos \gamma - \cos(\gamma L_1/L)\sin \gamma] = 0 \quad \text{(5.10)}
\]

Where,

\[\gamma = (\omega L/c)\]

\(\Theta\) is non-dimensional flexibility = \((EI/K_sL)\)

\(e\) is non dimensional crack location = \((L_1 - L)/L\)
By using non-dimensional crack distance from mid-beam, ‘e’ as the frequency, equation can be simplified to,

\[ 2\cos\gamma - \Theta_A (\sin\gamma - \sin\gamma) = 0 \]  
\[ \text{...(5.11)} \]

Following the same procedure for transverse vibrations, starting by differential equation with respect to \( \Theta_A \) and assuming that the original beam was uncracked, one obtains a differential equation,

\[ 2\sin(\Delta\gamma_1/\gamma_1) = -(\sin\gamma_1 - \sin\gamma_1) \Delta \Theta_A \]  
\[ \text{...(5.12)} \]

For the first axial mode \( \gamma_1 = \pi / 2 \), and since \( \sin\gamma \) is linearly proportional to the axial frequency \( f_A \), Therefore equation (5.12) yields

\[ 2(\Delta f_{A1}/f_{A1}) = -[1 + \sin(e \pi / 2)] \Delta \Theta_A \]  
\[ \text{...(5.13)} \]

And for second axial mode \( \gamma_2 = 3 \pi /2 \), Therefore equation (5.12) yields

\[ 2(\Delta f_{A2}/f_{A2}) = -[1 + \sin(3e \pi /2)] \Delta \Theta_A \]  
\[ \text{...(5.14)} \]

It can be seen that at clamped end (\( e = -1 \)) and at the middle (\( e = 0 \)) the two relative frequency variations are equal \( (\Delta f_{A1}/f_{A1}) = (\Delta f_{A2}/f_{A2}) \)

They both are zero at the free end (\( e =1 \))

Now dividing equation (5.14) by (5.13),

\[ R_A = \frac{\Delta f_{A2}/f_{A2}}{\Delta f_{A1}/f_{A1}} = \frac{1 + \sin\left(\frac{3e \pi}{2}\right)}{1 - \sin\left(\frac{e \pi}{2}\right)} \]  
\[ \text{...(5.15)} \]
With the non-dimensional crack section flexibility denoted by $\Theta$, the angular displacement between the two beam segments can be related to the moment at this section by,

$$R_A = \frac{1 + \sin \left(\frac{3e\pi}{2}\right)}{1 - \sin \left(\frac{e\pi}{2}\right)}$$

Let,

$$\theta = e\pi /2,$$

$$(5.16)$$

$$\therefore R_A = \frac{1 + \sin 3\theta}{1 - \sin \theta}$$

$$\therefore R_A = (1 + \sin 3\theta)(1 + \sin \theta + \sin^2 \theta + \sin^3 \theta + \ldots.)$$

$$(\therefore \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots.)$$

$$(1 + 3 \sin \theta - 4 \sin^3 \theta)(1 + \sin \theta + \sin^2 \theta + \sin^3 \theta + \ldots.)$$

$$= (1 + \sin \theta + \sin^2 \theta + \sin^3 \theta + \ldots.) + (3 \sin \theta + 3 \sin^2 \theta + 3 \sin^3 \theta + \ldots.) + (-4 \sin^3 \theta - 4 \sin^4 \theta + \ldots.)$$

$$= (1 + 4 \sin \theta + 4 \sin^2 \theta + \ldots.)$$

$$\therefore R_A = (1 + 2 \sin \theta)^2 + \ldots.$$(higher degree terms

$$\pm \sqrt{R_A} = (1 + 2 \sin \theta)$$

$$\sin \theta = \frac{\pm \sqrt{R_A} - 1}{2}$$
\[ \theta = \sin^{-1}\left[ \frac{\pm \sqrt{R_A} - 1}{2} \right] \]  

\( \ldots(5.17) \)

Like the conclusion in the transverse analysis this relation proves that the ratio at relative variations at two axial modes depends only on the crack location and is absolutely independent of crack geometry or any beam property. It is interesting to note that the value of this ratio satisfies \(0 < R_A < 1\) for \(-1 < e < 0\) and \(1 < R_A\) for \(0 < e < 1\), so that \(R_A\) may provide the additional data needed in the bending mode analysis in order to specify which half of the beam is damaged.

From equation (5.16) and (5.17) we can write for crack location,

\[ e = \frac{2}{\pi} \sin^{-1}\left[ \frac{\pm \sqrt{R_A} - 1}{2} \right] \]

\[ e = \frac{2}{\pi} \sin^{-1}\left[ \frac{\pm (\Delta f_{A2}/f_{A2})/(\Delta f_{A1}/f_{A1}) - 1}{2} \right] \quad \ldots(5.18) \]

Above equation is called as equation of crack location ‘e’ of the cantilever beam in longitudinal vibration. Where \(\Delta f_{An} = f_{An} - f_{An}^\sim\), \(f_{An}\) and \(f_{An}^\sim\) are the axial natural frequencies of uncracked and cracked beam.

This relation suggest that the ratio of the relative vibrations of two modes depends solely on the location of the crack and is independent on crack geometry or beam properties, no information is required even about the configuration of crack i.e. whether it is one sided, two sided or starts on the side faces of the beam.
5.4 IDENTIFICATION OF CRACK SIZE IN CANTILEVER BEAM BY MEASUREMENT OF LONGITUDINAL VIBRATION

Consider a typical beam structure that has been damaged by a discrete crack. Based on the consideration of the characteristic equations of the physical model shown in Fig. 5.3, the relationship between the change in eigenfrequency ratio $\Delta f_{An}/f_{An}$ and the crack location ‘x’ and stiffness of crack ‘K’ is given by Jialou and Liang (1993).

$$\frac{\Delta f_{An}}{f_{An}} = 2g_n(x) \frac{1}{K} \quad (5.19)$$

Where,

- $x$ is non-dimensional crack location $= L_1/L$ or $\frac{(e+1)}{2}$

- $K = \frac{(K_x L)}{EA} \quad (5.20)$

$\Delta f_{An}$ is difference between uncracked and cracked beam $= f_{An} - \tilde{f}_{An}$

$f_{An}$ and $\tilde{f}_{An}$ are the natural frequencies of uncracked and cracked beam.

The $g_n(x)$ function in equation (5.19) can be expressed in terms of mode shape $\varphi(x)$ for the intact beam which is given by Jialou and Liang et al. (1993).

$$g_n(x) = \frac{1}{4} \int_0^1 \frac{[\varphi_n^*(x)]^2}{\int_0^1 [\varphi_n^*(x)]^2 dx} dx \quad (5.21)$$
From elementary beam theory, the mode shapes of beams with typical homogeneous boundary conditions can be easily calculated. For cantilever beam the mode shape is \( \phi_n = \sin\left(\frac{n \pi x}{2}\right) \).

Now, put \( \phi_n(x) = \sin\left(\frac{n \pi x}{2}\right) \) in equation (5.21)

\[
\therefore g_n(x) = \frac{1}{4} \int_0^1 \left[ \frac{d^2}{dx^2} (\sin n\pi x) \right]^2 dx
\]

Consider first,

\[
I_1 = \left[ \frac{d^2}{dx^2} (\sin \frac{n \pi x}{2}) \right]^2
\]

\[
\therefore I_1 = \left[ \frac{d}{dx} (\cos \pi x) \right]^2
\]

\[
\therefore I_1 = [n\pi (-\sin \pi x)^2]^2
\]

\[
\therefore I_1 = n^4 \pi^4 (\sin^2 n\pi x)
\]
Now consider,

\[
I_2 = \int_0^1 \left( \frac{d^2}{dx^2} \left( \sin n\pi x \right) \right)^2 \, dx
\]

\[
\therefore I_2 = \int_0^1 n^4 \pi^4 \left( \sin^2 n\pi x \right) \, dx
\]

\[
\therefore I_2 = n^4 \pi^4 \int_0^1 \left( \sin^2 n\pi x \right) \, dx
\]

\[
\therefore I_2 = n^4 \pi^4 \int_0^1 \frac{1 - \cos 2n\pi x}{2} \, dx
\]

\[
\therefore I_2 = \frac{n^4 \pi^4}{2} \int_0^1 (1 - \cos 2n\pi x) \, dx
\]

\[
\therefore I_2 = \frac{n^4 \pi^4}{2} \left[ \int_0^1 dx - \int_0^1 (\cos 2n\pi x) \, dx \right]
\]

\[
\therefore I_2 = \frac{n^4 \pi^4}{2} \left\{ 1 - 0 \right\}
\]

\[
\therefore I_2 = \frac{n^4 \pi^4}{2}
\]

Now,

\[
\therefore g_n(x) = \frac{1}{4} \left[ \frac{I_1}{I_2} \right]
\]
\[ g_n(x) = \frac{\sin^2(n/2\pi x)}{2} \]

Put this value of \( g_n(x) \) in equation 5.19 we get,

\[ \frac{\Delta f_{An}}{f_{An}} = \sin^2 \left( \frac{n}{2} \pi x \right) \frac{1}{K} \]

Put \( K = (K_x L)/EA \) we get,

\[ \frac{\Delta f_{An}}{f_{An}} = \sin^2 \left( \frac{n}{2} \pi x \right) \frac{EA}{K_x L} \quad \text{...(5.23)} \]

The equation (5.23) which relates the change in eigenfrequency, crack location and stiffness of the crack.

Therefore for a cantilever beam, equation (5.23) gives the relationship between the changes in eigenfrequencies and the crack location and stiffness of crack.

Rizos et.al. (1990) given the spring stiffness \( K_x \) in the vicinity of the cracked section of a beam having width ‘b’, height ‘h’ and crack depth ‘a’. From the crack strain energy function,

\[ K_x = \frac{EA}{(5.346h)f(a/h)} \]

Put this value in equation (5.23) we get,

\[ \frac{\Delta f_{An}}{f_{An}} = \sin^2 \left( \frac{n}{2} \pi x \right) \frac{5.346h.f(a/h)}{L} \]
\[
\frac{\Delta f_{An}}{f_{An}} = 5.346 \sin^2 \left( \frac{n \pi (e+1)}{2} \right) \frac{h f(a/h)}{L} \quad \text{...(5.24)}
\]

Where, \( f(a/h) = 1.8624(a/h)^2 - 3.95(a/h)^3 + 16.375(a/h)^4 - 37.226(a/h)^5 + 78.81(a/h)^6 - 126.9(a/h)^7 + 172(a/h)^8 - 143.97(a/h)^9 + 66.56(a/h)^{10} \)

Put this value in equation (5.24) and neglecting higher order values, the equation becomes,

\[
(a/h)^2 = \frac{\Delta f_{An}/f_{An}}{9.9563 \sin^2 \left( \frac{n \pi (e+1)}{2} \right) \frac{h}{L}} \quad \text{...(5.25)}
\]

Using equation (5.25) we can easily find out the crack depth ratio \((a/h)\) if we know the natural frequencies of uncracked and cracked beam and crack location for cantilever beam.

### 5.5 Identification of Crack Location in Simply Supported or Free-Free Beam by Measurement of Longitudinal Vibration

In case of axial vibration of beam, the boundary condition for simply supported and free-free beam is same; therefore analysis of cracked beam in axial vibration for simply supported and free-free beam is same.

The physical model, which will be considered, is a simply supported or free-free uniform Euler-Bernoulli beam as shown in Fig.5.4.
The equation (5.8) of the motion of Euler-Bernoulli is does not hold near the crack, due to abrupt change of the cross-section. The beam can be treated as two uniform beams, connected by a linear spring at the crack location. The equation (5.8) is then valid for each segment of the beam separately, with the appropriate boundary condition. The left segment of the beam will be designated by subscript 1, the right one by 2. The end points are designated by A and C, and the crack section by B. This kind of modeling for the cracked beam has the advantage of using the exact solution throughout the beam, except for a narrow region near the crack, where the true stress-strain field is approximated by spring.

From equation (5.9a and 5.9b) for two beam segments we get,

\[ U_1(x) = a_1 \sin \left( \frac{\omega}{c} x \right) + a_2 \cos \left( \frac{\omega}{c} x \right) \]

\[ U_2(x) = a_3 \sin \left( \frac{\omega}{c} x \right) + a_4 \cos \left( \frac{\omega}{c} x \right) \]
Where ‘x’ is the origin for both segments from the support A. The coefficients $a_i$ can be found by substituting this solution in boundary conditions. The boundary conditions for simply supported beam or free-free beam for axial vibrations are as follows.

For the free vibrations of the beam, there is no external excitation and consequently no axial force at the supports,

$$\dot{U}_{1A} = \dot{U}_{2C} = 0$$

The continuity conditions at the crack position the displacement, moments and shear forces are,

$$\dot{U}_{1B} = \dot{U}_{2B}$$

With the non-dimensional crack section flexibility denoted by $\Theta$, the angular displacement between the two beam segments can be related to the force at this section by,

$$U_{2B} - U_{1B} = \Theta A L \dot{U}_{2B}$$

Substituting above boundary conditions in equations (5.9a) and (5.9b) and equating the system determinant to zero and obtains algebraic equations for the natural frequencies of the cracked beam,

$$4\sin\beta \sinh\beta + \beta \Theta_A [\sinh\beta (\cos\beta - \cose\beta) + \sin\beta (\cosh\beta - \coshe\beta)] = 0$$

Where,

$\Theta_A$ is non-dimensional axial flexibility = $(EA/K_xL)$

$\beta$ is non dimensional frequency parameter = $\lambda L$

$e$ is non dimensional crack location = $(L_1 - L/2)/(L/2)$
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For constant crack location, a partial differentiation with respect to $\Theta_A$ yields,

$$4(\cos \beta \sinh \beta + \sin \beta \cosh \beta) \left( \frac{\partial \beta}{\partial \Theta_A} \right) + \beta \left[ \sinh \beta (\cos \beta - \cose \beta) + \sin \beta (\cosh \beta - \coshe \beta) \right] + \Theta_A \left( \frac{\partial}{\partial \Theta_A} \right) \left( \frac{\partial \beta}{\partial \Theta_A} \right) = 0$$

If it is assumed that the original beam was uncracked, with negligible equivalent flexibility, the nominal values of $\Theta_A$ in above equation become zero. Then put $\Theta_A = 0$ and $\beta = n\pi$ we get,

$$4 \cos (\beta / \Theta_A) + \beta [\cos \beta - \cose \beta] = 0$$

This can be written as a difference equation,

$$4 \cos \beta \left( \Delta \beta / \beta \right) = [\cose \beta - \cos \beta] \Delta \Theta_A$$

By definition of the non-dimensional frequency parameter it is given that $2\Delta \beta / \beta = \Delta f_A / f_A$.

Putting this value in above equation and can be rewritten for the $i^{th}$ mode as,

$$2 \cos \beta_i \left( \Delta f_A / f_A \right) = [\cose \beta_i - \cos \beta_i] \Delta \Theta_A$$

For the first natural mode $\beta_1 = 1\pi$, above equation yields,

$$-2(\Delta f_A / f_A1) = (\cos 2e\pi + 1) \Delta \Theta_A$$

$$2(\Delta f_A / f_A1) = - (1 + \cos 2e\pi) \Delta \Theta_A$$

...(5.26)

And for the second mode $\beta_2 = 2\pi$, above equation yields,

$$2(\Delta f_A / f_A2) = (\cos 2e\pi - 1) \Delta \Theta_A$$

$$2(\Delta f_A / f_A2) = - (1 - \cos 2e\pi) \Delta \Theta_A$$

...(5.27)
Dividing equation (5.27) by equation (5.26) we get,

\[ \frac{\Delta f_{A2}/f_{A2}}{\Delta f_{A1}/f_{A1}} = \frac{1 - \cos 2 \epsilon \pi}{1 + \cos \epsilon \pi} \]

\[ \frac{\Delta f_{A2}/f_{A2}}{\Delta f_{A1}/f_{A1}} = 1 - \cos \epsilon \pi \]

\[ \cos \epsilon \pi = \left[ 1 - \frac{\Delta f_{A2}/f_{A2}}{\Delta f_{A1}/f_{A1}} \right] \frac{1}{2} \]

\[ \epsilon = \frac{1}{\pi} \cos^{-1} \left[ 1 - \frac{\Delta f_{A2}/f_{A2}}{\Delta f_{A1}/f_{A1}} \right] \]

... (5.28)

Above equation is called as equation of crack location ‘\( \epsilon \)’ of the simply supported and free-free beam. Where \( \Delta f_{An} = f_{An} - f_{An}^- \), \( f_{An} \) and \( f_{An}^- \) are the axial natural frequencies of uncracked and cracked beam.

This relation suggest that the ratio of the relative vibrations of two modes depends solely on the location of the crack and is independent on crack geometry or beam properties, no information is required even about the configuration of crack i.e. whether it is one sided, two sided or starts on the side faces of the beam.
5.6 IDENTIFICATION OF CRACK SIZE IN SIMPLY SUPPORTED OR FREE-FREE BEAM BY MEASUREMENT OF LONGITUDINAL VIBRATION

Consider a typical beam structure that has been damaged by a discrete crack. Based on the consideration of the characteristic equations of the physical model shown in Fig. 5.4, the relationship between the change in eigenfrequency ratio $\Delta f_{An}/f_{An}$ and the crack location ‘x’ and stiffness of crack ‘K’ is given by Jialou and Liang (1993).

$$\frac{\Delta f_{An}}{f_{An}} = 2g_n(x) \frac{1}{K} \quad \text{...(5.29)}$$

Where,

- $g_n(x)$ is the function in equation (5.29) can be expressed in terms of mode shape $\varphi(x)$ for the intact beam which is given by Jialou and Liang et al. (1993).

$$K = (K_xL)/EA \quad \text{...(5.30)}$$

$\Delta f_n$ is difference between uncracked and cracked beam = $f_{An} - f_{An}^-$

$f_{An}$ and $f_{An}^-$ are the natural frequencies of uncracked and cracked beam.

The $g_n(x)$ function in equation (5.29) can be expressed in terms of mode shape $\varphi(x)$ for the intact beam which is given by Jialou and Liang et al. (1993).

$$g_n(x) = \frac{1}{4} \int_0^1 \left[ \varphi_n^*(x) \right]^2 \, dx \quad \text{...(5.31)}$$
From elementary beam theory, the mode shapes of beams with typical homogeneous boundary conditions can be easily calculated. For simply supported beam the mode shape is \( \varphi_n = \sin(n \pi x) \).

Now,

Put \( \varphi_n(x) = \sin(n \pi x) \) in equation (5.31)

\[
\varphi_n(x) = \sin(n \pi x)
\]

\[
\therefore g_n(x) = \frac{1}{4} \left[ \frac{d^2}{dx^2} (\sin n\pi x) \right]^2
\]

\[
\therefore I_1 = \left[ \frac{d^2}{dx^2} (\sin n\pi x) \right]^2
\]

\[
\therefore I_1 = \left[ \frac{d}{dx} (\cos n\pi x) \right]^2
\]

\[
\therefore I_1 = \left[ n\pi (-\sin n\pi x) \right]^2
\]

\[
\therefore I_1 = n^4 \pi^4 (\sin^2 n\pi x)
\]

Now consider,

\[
\therefore I_2 = \int_0^1 \left[ \frac{d^2}{dx^2} (\sin n\pi x) \right]^2 dx
\]

\[
\therefore I_2 = \int_0^1 n^4 \pi^4 (\sin^2 n\pi x) dx
\]

\[
\therefore I_2 = n^4 \pi^4 \int_0^1 (\sin^2 n\pi x) dx
\]
\[ I_2 = n^4 \pi^4 \int_0^1 \frac{1 - \cos 2n\pi x}{2} dx \]

\[ \therefore I_2 = \frac{n^4 \pi^4}{2} \int_0^1 (1 - \cos 2n\pi x) dx \]

\[ I_2 = \frac{n^4 \pi^4}{2} \left[ \int_0^1 dx - \int_0^1 (\cos 2n\pi x) dx \right] \]

\[ \therefore I_2 = \frac{n^4 \pi^4}{2} \left[ \left[ x \right]_0^1 - \left[ \frac{\sin 2n\pi x}{2n\pi} \right]_0^1 \right] \]

\[ \therefore I_2 = \frac{n^4 \pi^4}{2} \left[ 1 - 0 \right] \]

\[ \therefore I_2 = \frac{n^4 \pi^4}{2} \]

Now,

\[ g_n(x) = \frac{1}{4} \left[ \frac{I_1}{I_2} \right] \]

\[ \therefore g_n(x) = \frac{\sin^2 (n\pi x)}{2} \]

Put this value of \( g_n(x) \) in equation (5.29) we get,

\[ \frac{\Delta f_{An}}{f_{An}} = \sin^2 (n\pi x) \frac{1}{K} \]

Put \( K = \frac{K_\alpha L}{EA} \) we get,

\[ \frac{\Delta f_{An}}{f_{An}} = \sin^2 (n\pi x) \frac{EA}{K_\alpha L} \]  

\( \ldots (5.33) \)
The equation (5.33) which relates the change in eigen frequency, crack location and stiffness of the crack.

Therefore for a simply supported or free-free beam, equation (5.33) gives the relationship between the changes in eigenfrequencies and the crack location and stiffness of crack.

Rizos et al. (1990) given the spring stiffness $K_x$ in the vicinity of the cracked section of a beam having width ‘b’, height ‘h’ and crack depth ‘a’. From the crack strain energy function,

$$K_x = \frac{EA}{(5.346h)f(a/h)}$$

Put this value in equation (5.33) we get,

$$\frac{\Delta f_{An}}{f_{An}} = \sin^2(n \pi x) \frac{5.346h.f(a/h)}{L}$$

$$\frac{\Delta f_{An}}{f_{An}} = 5.346 \sin^2 [n \pi (e+1)/2] \frac{h}{L} . f(a/h) \quad \ldots(5.34)$$

Where, $f(a/h) = 1.8624(a/h)^2 - 3.95(a/h)^3 + 16.375(a/h)^4 - 37.226(a/h)^5 + 78.81(a/h)^6 - 126.9(a/h)^7 + 172(a/h)^8 - 143.97(a/h)^9 + 66.56(a/h)^{10}$

Put this value in equation (5.34) and neglecting higher order values, the equation becomes,

$$(a/h)^2 = \frac{\Delta f_{An}/f_{An}}{9.9563 \sin^2 [n \pi (e+1)/2] \frac{h}{L}} \quad \ldots(5.35)$$

Using equation (5.35) we can easily find out the crack depth ratio (a/h) if we know the natural frequencies of uncracked and cracked beam and crack location for simply supported or free-free beam.
5.7 SUMMARY

This chapter gives the theoretical formulation of the method based on longitudinal vibration of beam, which gives the relation between natural frequencies, crack location and crack size for cantilever beam and simply supported or free-free beam which are as follows.

For Cantilever Beam:

Crack location from equation (5.18),

\[
e = \frac{2}{\pi} \sin^{-1}\left( \pm \frac{\sqrt{(\Delta f_{A2}/f_{A2})/(\Delta f_{A1}/f_{A1}) - 1}}{2} \right)
\]

Crack size from equation (5.25),

\[
(a/h)^2 = \frac{\Delta f_{An}/f_{An}}{9.9563 \sin^2 \left[ \frac{n (e + 1) h}{2} \right] \frac{L}{L}}
\]

For Simply Supported or Free-Free Beam:

Crack location from equation (5.28),

\[
e = \frac{1}{\pi} \cos^{-1}\left[ 1 - \frac{(\Delta f_{A2}/f_{A2})/(\Delta f_{A1}/f_{A1})}{2} \right]
\]

Crack size from equation (5.35),

\[
(a/h)^2 = \frac{\Delta f_{An}/f_{An}}{9.9563 \sin^2 \left[ n \pi (e + 1)/2 \right] \frac{h}{L}}
\]