Chapter 4
Theoretical Analysis for Crack Assessment Using Transverse Vibration of Beam

4.1 INTRODUCTION

The literature survey implies the frequency measurement based crack assessment of large structure is faster, feasible and reliable. Modal frequencies are used for monitoring the crack because modal frequencies are properties of the whole component. The natural frequency of a component decreases as a result of crack. The reduction in modal frequencies depends on crack location, crack size and number of cracks. If modal frequencies of the cracked components are known it is possible to detect the crack in components. This chapter describes the analytical method for assessment of crack location and crack size using transverse vibration of beam.

4.2 TRANSVERSE VIBRATION OF BEAM

To derive the differential equation of motion for the bending or transverse vibration of beam, consider an element of beam of length ‘dx’ with force and moments acting as shown in Fig. 4.1.
Assuming the displacement of any point on the beam will be small and motion occurs only in a direction normal to the axis of the beam. Also neglect the inertia effects of rotation of any section of the beam.

The lateral displacement at any point of the beam is represented as,

\[ y = y(x, t) \]

From strength of materials, the beam curvature and the moment \( M \) are related by,

\[ M = EI \frac{\partial^2 y}{\partial x^2} \]  \quad \ldots (4.1) \]

Where,

\( M \) is the bending moment at any transverse section.

\( E \) is the modulus of elasticity of the beam material.
I is the moment of inertia of the cross-sectional area of the beam about the axis of bending.

EI is the flexural stiffness of the beam.

Fig. 4.1 (b) shows an element of beam section with bending moments M, shear forces V, and external load per unit length w(x, t).

Taking the sum of the moments about the left end of the section in Fig. 4.2 (b) we get,

$$\frac{\partial M}{\partial x} \, dx - V \, dx - \frac{\partial V}{\partial x} (dx)^2 - \frac{w \, (dx)^2}{2} = 0 \quad \ldots (4.2)$$

Neglecting the higher-order terms containing \((dx)^2\) we obtain,

$$V = \frac{\partial M}{\partial x} \quad \ldots (4.3)$$

If the mass of the beam per unit length i.e. density is denoted as \(\rho\), then the equation of motion in the vertical direction as given by Newton’s second law is,

$$\frac{\partial V}{\partial x} \, dx + w \, dx = \rho \, dx \frac{\partial^2 y}{\partial t^2} \quad \ldots (4.4)$$

Using Equations (4.1) and (4.3), Equation (4.4) becomes,

$$\frac{\partial^2}{\partial x^2} \left( -EI \frac{\partial^2 y}{\partial x^2} \right) + w = \rho \frac{\partial^2 y}{\partial t^2} \quad \ldots (4.5)$$

or

$$\frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 y}{\partial x^2} \right) + \rho \frac{\partial^2 y}{\partial t^2} = w \quad \ldots (4.6)$$

Assuming the properties of the beam are constant along its length, Equation (4.6) becomes,
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\[
\frac{EI \, \partial^4 y}{\partial x^4} + \rho \, \frac{\partial^2 y}{\partial t^2} = w \quad \ldots (4.7)
\]

The case of free vibration is obtained by setting \( w(x, t) = 0 \). Hence

\[
\frac{EI \, \partial^4 y}{\partial x^4} + \rho \, \frac{\partial^2 y}{\partial t^2} = 0 \quad \ldots (4.8)
\]

The equation (4.8) is called governing equation of motion of Euler-Bernoulli’s. The general solution of the equation (4.8) is obtained by method of separation of variables i.e. by putting,

\[Y(x, t) = Y(x), T(t)\]

For simplicity put \( y = Y \cdot T \)

\[\therefore \frac{\partial^2 y}{\partial x^2} = T \cdot \frac{\partial^2 Y}{\partial x^2} \quad \text{and} \quad \frac{\partial^2 y}{\partial t^2} = Y \cdot \frac{\partial^2 T}{\partial t^2}\]

\[\therefore \frac{\partial^4 y}{\partial x^4} = T \cdot \frac{\partial^4 Y}{\partial x^4}\]

Put in equation (4.8) we get,

\[\frac{EI \, T \, \frac{\partial^4 Y}{\partial x^4}}{\rho A \, Y \, \frac{\partial^2 T}{\partial t^2}} = 0\]

\[\therefore \frac{1 \, \frac{\partial^2 T}{T \, \partial t^2}}{\frac{\partial^4 Y}{\rho A \, Y \, \partial x^4}} = -\frac{EI}{\rho A} \cdot \frac{1 \, \frac{\partial^4 Y}{\partial x^4}}{\partial t^2}\]
The left hand side of the above equation is function if ‘T’ alone and the right hand side is function of ‘Y’ alone. It can be shown that this is possible only if each side of this equation is equal to negative constant say $-\omega^2$ where ‘$\omega$’ is a real number.

\[
-\frac{EI}{\rho A} \frac{\partial^4 Y}{\partial x^4} = \frac{1}{T} \frac{\partial^2 T}{\partial t^2} = -\omega^2
\]

\[
\Rightarrow \frac{EI}{\rho A} \frac{\partial^4 Y}{\partial x^4} = \omega^2
\]

\[
\Rightarrow \frac{\partial^4 Y}{\partial x^4} - \lambda^4 Y = 0
\]

\[
\Rightarrow \frac{\partial^4 Y}{\partial x^4} - \lambda^4 Y = 0 \tag{4.9}
\]

Where,

\[
\lambda^4 = \frac{\rho A\omega^2}{EI}
\]

Now,

\[
\therefore (D^4 - \lambda^4) Y = 0
\]

\[
\therefore D^4 - \lambda^4 = 0
\]

\[
\therefore (D^2 - \lambda^2) (D^2 + \lambda^2) = 0
\]

\[
\therefore (D - \lambda) (D + \lambda) (D^2 + \lambda^2) = 0
\]

\[
\therefore D = \lambda, \ D = -\lambda, \ D = 0 \pm i \lambda
\]

\[
\therefore Y(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x} + e^{0x} (c_3 \cos \lambda x + c_4 \sin \lambda x)
\]

But,

\[
e^{\pm \lambda x} = (\cosh \lambda x \pm \sinh \lambda x)
\]

\[
\therefore Y(x) = c_1 (\cosh \lambda x + \sinh \lambda x) + c_2 (\cosh \lambda x - \sinh \lambda x) + c_3 \cos \lambda x + c_4 \sin \lambda x
\]

\[
\therefore Y(x) = c_1 \cosh \lambda x + c_1 \sinh \lambda x + c_2 \cosh \lambda x - c_2 \sinh \lambda x + c_3 \cos \lambda x + c_4 \sin \lambda x
\]

\[
\therefore Y(x) = (c_1 + c_2) \cosh \lambda x + (c_1 - c_2) \sinh \lambda x + c_3 \cos \lambda x + c_4 \sin \lambda x
\]
\[ Y(x) = (c_1 + c_2) \cosh \lambda x + (c_1 - c_2) \sinh \lambda x + c_3 \cos \lambda x + c_4 \sin \lambda x \]

\[ Y(x) = a_1 \sin \lambda x + a_2 \cos \lambda x + a_3 \sinh \lambda x + a_4 \cosh \lambda x \]  

... (4.10)

The equation (4.10) is called as equation of harmonic motion of the beam.

Where, \( a_1 = c_4 \), \( a_2 = c_3 \), \( a_3 = (c_1 - c_2) \), \( a_4 = (c_1 + c_2) \) are constants and can be found by substituting this solution in the boundary condition and hence we get different values of \( Y \) for the range of \( x = 0 \) to \( l \) for each modes, hence the mode shapes are found out.

### 4.3 IDENTIFICATION OF CRACK LOCATION IN SIMPLY SUPPORTED BEAM BY MEASUREMENT OF TRANSVERSE VIBRATION

The physical model, which will be considered, is a simply supported uniform Euler-Bernoulli beam as shown in Fig.4.2. The length of the beam is ‘\( L \)’ and it has a crack at a distance ‘\( L_1 \)’ from the left support. The beam has constant cross-section area ‘\( A \)’ and geometric moment of inertia ‘\( I \)’. Its material properties i.e. Young’s modulus ‘\( E \)’ and mass density ‘\( \rho \)’ are also constant.

![Figure 4.2 Model of cracked simply supported beam](image)
The equation (4.10) of the motion of Euler-Bernoulli is does not hold near the crack, due to abrupt change of the cross-section. The beam can be treated as two uniform beams, connected by a torsional spring at the crack location. The equation (4.10) is then valid for each segment of the beam separately, with the appropriate boundary condition. The left segment of the beam will be designated by subscript 1, the right one by 2. The end points are designated by A and C, and the crack section by B. This kind of modeling for the cracked beam has the advantage of using the exact solution throughout the beam, except for a narrow region near the crack, where the true stress-strain field is approximated by spring.

From equation (4.10) for two beam segments we get,

\[ Y_1(x) = a_1 \sin \lambda x + a_2 \cos \lambda x + a_3 \sinh \lambda x + a_4 \cosh \lambda x \quad \ldots \text{(4.11a)} \]

\[ Y_2(x) = a_5 \sin \lambda x + a_6 \cos \lambda x + a_7 \sinh \lambda x + a_8 \cosh \lambda x \quad \ldots \text{(4.11b)} \]

Where ‘x’ is the origin for both segments from the support A, and \( \lambda^4 = \rho A \omega^2/EI \). The coefficients \( a_i \) can be found by substituting this solution in boundary conditions. The boundary conditions for simply supported beam are as follows.

For the free vibrations of the beam, there is no external excitation and consequently no displacement and no moments at the supports.

\[ Y_{1A} = Y_{2C} = 0, \quad \text{and} \]

\[ Y_{1A}'' = Y_{2C}'' = 0 \]

The continuity conditions at the crack position the displacement, moments and shear forces are,

\[ Y_{1B} = Y_{2B}, \quad Y_{1B}''' = Y_{2B}''', \quad Y_{1B}'''' = Y_{2B}'''' \]
With the non-dimensional crack section flexibility denoted by $\Theta$, the angular displacement between the two beam segments can be related to the moment at this section by,

$$Y"_{2B} + \Theta L Y"_{2B} = Y'_{1B}$$

Substituting above boundary conditions in the equation (4.11a and 4.11b) and obtain a set of eight homogeneous linear algebraic equations for the eight unknown coefficients. The natural frequencies are then calculated by equating the set determinant to zero.

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
1 & 0 & -1 & 0 \\
cosh(e\beta) & \sinh(e\beta) & \cos(e\beta) & \sin(e\beta) \\
\sinh(e\beta) & \cosh(e\beta) & \sin(e\beta) & -\cos(e\beta) \\
cosh(e\beta) & \sinh(e\beta) & -\cos(e\beta) & -\sin(e\beta) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
\frac{\sin(e\beta)}{\Theta \beta} + \cosh(e\beta) \\
\frac{\cos(e\beta)}{\Theta \beta} - \sinh(e\beta) \\
\frac{\sinh(e\beta)}{\Theta \beta} + \cosh(e\beta) \\
\frac{-\cos(e\beta)}{\Theta \beta} - \sin(e\beta) \\
\end{bmatrix} = 0
\]
Where,

\[ \Theta \] is non-dimensional flexibility = \( \frac{EI}{K_r L} \)

\[ \beta \] is non dimensional frequency parameter = \( \lambda L \)

\[ e \] is non dimensional crack location = \( \frac{L_1 - L/2}{L/2} \)

The linear set of equation reduces to a single trigonometric equation,

\[ 4\sin\beta \sinh\beta + \beta \Theta [\sinh\beta (\cos\beta - \cosec\beta) + \sin\beta (\cosh\beta - \coshe\beta)] = 0 \]

The above equation is finding out by solving the above matrix by using a computational package MATHCAD-7.

For constant crack location, a partial differentiation with respect to \( \Theta \) yields,

\[ 4(\cos\beta \sinh\beta + \sin\beta \cosh\beta) (\frac{\partial \beta}{\partial \Theta}) + \beta [\sinh\beta (\cos\beta - \cosec\beta) + \sin\beta (\cosh\beta - \coshe\beta)] + \Theta \]

\[ (\frac{\partial}{\partial \beta}) \{ \beta \sinh\beta (\cos\beta - \cosec\beta) + \sin\beta (\cosh\beta - \coshe\beta) \} (\frac{\partial \beta}{\partial \Theta}) = 0 \]

If it is assumed that the original beam was uncracked, with negligible equivalent flexibility, the nominal values of \( \Theta \) in above equation become zero. Then put \( \Theta = 0 \) and \( \beta = n\pi \) we get,

\[ 4 \cos (\frac{\partial \beta}{\partial \Theta}) + \beta [\cos\beta - \cosec\beta] = 0 \]

This can be written as a difference equation

\[ 4 \cos\beta (\Delta \beta/\beta) = [\cosec\beta - \cos\beta] \Delta \Theta \]
By definition of the non-dimensional frequency parameter it is given that 

$$2\Delta \beta / \beta = \Delta f / f.$$  

Putting this value in above equation and can be rewritten for the $i^{th}$ mode as,

$$2\cos \beta_i (\Delta f / f_i) = [\cose \beta_i - \cos \beta_i] \Delta \Theta$$

For the first natural mode $\beta_1 = 1\pi$, above equation yields,

$$-2(\Delta f / f_1) = (\cos 2\pi + 1) \Delta \Theta$$

$$2(\Delta f / f_1) = - (1 + \cos 2\pi) \Delta \Theta \quad \text{... (4.12)}$$

And for the second mode $\beta_2 = 2\pi$, above equation yields,

$$2(\Delta f / f_2) = (\cos 2\pi - 1) \Delta \Theta$$

$$2(\Delta f / f_2) = - (1 - \cos 2\pi) \Delta \Theta \quad \text{... (4.13)}$$

Dividing equation (4.13) by equation (4.12) we get,

$$\frac{(\Delta f_2 / f_2)}{(\Delta f_1 / f_1)} = \frac{(1 - \cos 2\pi)}{(1 + \cos 2\pi)}$$

Now,

$$\therefore \frac{(\Delta f_2 / f_2)}{(\Delta f_1 / f_1)} = \frac{2\sin^2 \pi}{1 + \cos \pi}$$

$$\therefore \frac{(\Delta f_2 / f_2)}{(\Delta f_1 / f_1)} = \frac{2(1 - \cos^2 \pi)}{1 + \cos \pi}$$
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\[ \frac{(\Delta f_2/f_2)}{(\Delta f_1/f_1)} = \frac{2(1-\cos\pi)(1+\cos\pi)}{(1+\cos\pi)} \]

\[ \frac{(\Delta f_2/f_2)\cdot (\Delta f_1/f_1)}{2} = 2(1-\cos\pi) \]

\[ \frac{(\Delta f_2/f_2)/(\Delta f_1/f_1)}{2} = 1-\cos\pi \]

\[ \cos\pi = \left[ \frac{1 - (\Delta f_2/f_2)/(\Delta f_1/f_1)}{2} \right] \]

\[ e = \frac{1}{\pi} \cos^{-1}\left[ 1 - \frac{(\Delta f_2/f_2)/(\Delta f_1/f_1)}{2} \right] \]  ...(4.14)

Above equation is called as equation of crack location ‘e’ of the simply supported beam. Where \( \Delta f_n = f_n - \tilde{f}_n \), \( f_n \) and \( \tilde{f}_n \) are the natural frequencies of uncracked and cracked beam.

This relation suggest that the ratio of the relative vibrations of two modes depends solely on the location of the crack and is independent on crack geometry or beam properties, no information is required even about the configuration of crack i.e. whether it is one sided, two sided or starts on the side faces of the beam.

4.4 IDENTIFICATION OF CRACK SIZE IN SIMPLY SUPPORTED BEAM BY MEASUREMENT OF TRANSVERSE VIBRATION

Consider a typical beam structure that has been damaged by a discrete crack. Based on the consideration of the characteristic equations of the physical model
shown in Fig. 4.2, the relationship between the change in eigenfrequency ratio $\Delta f_n/f_n$ and the crack location ‘x’ and stiffness of crack ‘K’ is given by Jialou and Liang (1993).

$$\frac{\Delta f_n}{f_n} = 2g_n(x)\frac{1}{K}$$

...(4.15)

Where,

- $x$ is non-dimensional crack location = $L_1/L$ or $\frac{(e+1)}{2}$
- $K = \frac{(K_1L)}{EI}$

...(4.16)

$\Delta f_n$ is difference between uncracked and cracked beam = $f_n - \tilde{f}_n$

$f_n$ and $\tilde{f}_n$ are the natural frequencies of uncracked and cracked beam

The $g_n(x)$ function in equation (4.15) can be expressed in terms of mode shape $\phi(x)$ for the intact beam which is given by Jialou and Liang et al. (1993).

$$g_n(x) = \frac{1}{4} \int_0^1 \left[\phi''_n(x)\right]^2 \mathrm{d}x$$

...(4.17)

From elementary beam theory, the mode shapes of beams with typical homogeneous boundary conditions can be easily calculated. For simply supported beam the mode shape is $\phi_n = \sin(n \pi x)$.

Now,

Put $\phi_n(x) = \sin n \pi x$ in equation (4.17)
\[ g_n(x) = \frac{1}{4} \left[ \frac{d^2}{dx^2} (\sin n\pi x) \right]^2 \int_0^1 \left[ \frac{d^2}{dx^2} (\sin n\pi x) \right]^2 dx \] ...

Consider first,

\[ I_1 = \left[ \frac{d^2}{dx^2} (\sin n\pi x) \right]^2 \]

\[ \therefore I_1 = \left[ \frac{d}{dx} (\cos n\pi x \cdot n\pi) \right]^2 \]

\[ \therefore I_1 = \left[ n\pi (\sin n\pi x \cdot n\pi) \right]^2 \]

\[ \therefore I_1 = n^4\pi^4 (\sin^2 n\pi x) \]

Now consider,

\[ I_2 = \int_0^1 \left[ \frac{d^2}{dx^2} (\sin n\pi x) \right]^2 dx \]

\[ \therefore I_2 = \int_0^1 n^4\pi^4 (\sin^2 n\pi x) dx \]

\[ \therefore I_2 = n^4\pi^4 \int_0^1 (\sin^2 n\pi x) dx \]
\[
\therefore I_2 = n^4 \pi^4 \int_0^1 \frac{(1 - \cos 2n\pi x)}{2} dx
\]

\[
\therefore I_2 = \frac{n^4 \pi^4}{2} \int_0^1 (1 - \cos 2n\pi x)dx
\]

\[
\therefore I_2 = \frac{n^4 \pi^4}{2} \left[ \int_0^1 dx - \int_0^1 (\cos 2n\pi x)dx \right]
\]

\[
\therefore I_2 = \frac{n^4 \pi^4}{2} \left[ \left[ x \right]_0^1 - \left[ \frac{\sin 2n\pi x}{2n\pi} \right]_0^1 \right]
\]

\[
\therefore I_2 = \frac{n^4 \pi^4}{2} \{1 - 0\}
\]

\[
\therefore I_2 = \frac{n^4 \pi^4}{2}
\]

Now,

\[
\therefore g_n(x) = \frac{1}{4} \left[ I_1 \right]_{I_2}
\]

\[
\therefore g_n(x) = \frac{\sin^2(n\pi x)}{2}
\]

Put this value of \( g_n(x) \) in equation (4.15) we get,

\[
\frac{\Delta f_n}{f_n} = \frac{\sin^2(n\pi x)}{K} \frac{1}{K}
\]
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Put \( K = (K_r L)/EI \) we get,

\[
\frac{\Delta f_n}{f_n} = \sin^2(n\pi x) \frac{EI}{K_r L} \quad \ldots (4.19)
\]

The equation (4.19) which relates the change in eigenfrequency, crack location and stiffness of the crack for simply supported beam.

Therefore for a simply supported beam, equation (4.19) gives the relationship between the changes in eigenfrequencies and the crack location and stiffness of crack.

Rizos et.al. (1990) given the spring stiffness \( K_r \) in the vicinity of the cracked section of a beam having width ‘b’, height ‘h’ and crack depth ‘a’. From the crack strain energy function,

\[
K_r = \frac{EI}{(5.346h)f(a/h)}
\]

Put this value in equation (4.19) we get,

\[
\frac{\Delta f_n}{f_n} = \sin^2(n\pi x) \frac{5.346h.f(a/h)}{L}
\]

\[
\therefore \frac{\Delta f_n}{f_n} = 5.346 \sin^2 \left[ n\pi \frac{(e+1)/2}{h} \right] \frac{h}{L} f(a/h) \quad \ldots (4.20)
\]

Where, \( f(a/h) = 1.8624(a/h)^2 - 3.95(a/h)^3 + 16.375(a/h)^4 - 37.226(a/h)^5 + 78.81(a/h)^6 - 126.9(a/h)^7 + 172(a/h)^8 - 143.97(a/h)^9 + 66.56(a/h)^10 \)
Put this value in equation (4.20) and neglecting higher order values, the equation becomes,

\[(a/h)^2 = \frac{\Delta f_n / f_n}{9.9563 \sin^2 \left[ n \pi \left( e + 1/2 \right) \right]} \frac{h}{L} \]  

…(4.21)

Using equation (4.21) we can easily find out the crack depth ratio \((a/h)\) if we know the natural frequencies of uncracked and cracked beam and crack location for simply supported beam.

4.5 IDENTIFICATION OF CRACK LOCATION IN CANTILEVER BEAM BY MEASUREMENT OF TRANSVERSE VIBRATION

The physical model, which will be considered, is a cantilever uniform Euler-Bernoulli beam as shown in Fig. 4.3. The length of the beam is ‘L’ and it has a crack at a distance ‘\(L_1\)’ from the fixed end. The beam has constant cross-section area ‘A’ and geometric moment of inertia ‘I’. Its material properties i.e. Young’s modulus ‘E’ and mass density ‘\(\rho\)’ are also constant.

Figure 4.3 Model of cracked cantilever beam
We know that the equation (4.10) is called as equation of harmonic motion of the beam which is given as,

\[ Y(x) = a_1 \sin \lambda x + a_2 \cos \lambda x + a_3 \sinh \lambda x + a_4 \cosh \lambda x \]

This equation of motion of Euler-Bernoulli is does not hold near the crack, due to abrupt change of the cross-section. The beam can be treated as two uniform beams, connected by a torsional spring at the crack location. The above equation is then valid for each segment of the beam separately, with the appropriate boundary condition. The left segment of the beam will be designated by subscript 1, the right one by 2. The end points are designated by A and C, and the crack section by B. This kind of modeling for the cracked beam has the advantage of using the exact solution throughout the beam, except for a narrow region near the crack, where the true stress-strain field is approximated by spring.

From equation (4.11a and 4.11b) for two beam segments we get,

\[ Y_1(x) = a_1 \sin \lambda x + a_2 \cos \lambda x + a_3 \sinh \lambda x + a_4 \cosh \lambda x \]
\[ Y_2(x) = a_5 \sin \lambda x + a_6 \cos \lambda x + a_7 \sinh \lambda x + a_8 \cosh \lambda x \]

Where ‘x’ is the origin for both segments from the fixed end A, and \( \lambda^2 = \rho A \omega^2/EI \). The coefficients \( a_i \) can be found by substituting this solution in boundary conditions. The boundary conditions for cantilever beam are as follows.

For the free vibrations of the beam, there is no external excitation and consequently no displacement and no moments at fixed end.

\[ Y_{1A} = Y''_{2C} = 0, \text{ and} \]
\[ Y'_{1A} = Y''''_{2C} = 0 \]
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With the non dimensional crack section flexibility denoted by $\Theta$, the angular displacement between the two beam segments can be related to the moment at this section by,

$$Y'_{2B} + \Theta L Y''_{2B} = Y'_{1B}$$

Substituting above boundary conditions in the equation (4.11a and 4.11b) and obtain a set of eight homogeneous linear algebraic equations for the eight unknown coefficients. The natural frequencies are then calculated by equating the set determinant to zero.

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
\cosh(e\beta) & \sinh(e\beta) & \cos(e\beta) & \sin(e\beta) \\
\sinh(e\beta) & \cosh(e\beta) & \sin(e\beta) & -\cos(e\beta) \\
\cosh(e\beta) & \sinh(e\beta) & -\cos(e\beta) & -\sin(e\beta) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\frac{\sin(e\beta)}{(\Theta\beta)} + \cos(e\beta) \\
\frac{\cos(e\beta)}{(\Theta\beta)} + \sin(e\beta) \\
-\sin(e\beta) - \frac{\cos(e\beta)}{(\Theta\beta)} \\
\frac{\cos(e\beta)}{(\Theta\beta)} - \sin(e\beta) \\
\end{bmatrix}
= 0
\]
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Defining a non-dimensional frequency parameter $\beta = \lambda L$ and a non-dimensional crack location $e = x = L1/L$, it can be shown the set of equations reduces to a single trigonometric equation,

$$-3 \cdot \cos(e\beta)^2 \cdot \cosh(\beta) \cdot \cos(\beta) + \cos(e\beta)^2 \cdot \cosh(\beta) \cdot \sin(\beta) + \cosh(\beta)^2 \cdot \cos(\beta) \cdot \cos(e\beta) - 2 \cdot \cos(e\beta) \cdot \Theta \cdot \cosh(\beta) \cdot \sin(\beta) \cdot \sin(e\beta) + 1 + \cosh(\beta) \cdot \cos(e\beta) - 3 \cdot \cos(e\beta)^2 - 3 \cdot \cos(e\beta)^2 + \cosh(\beta)^2 \cdot \cos(e\beta) - \cosh(\beta) \cdot \cos(\beta) \cdot \sin(\beta) \cdot \sin(e\beta) \cdot \cos(e\beta) - 2 \cdot \cosh(\beta) \cdot \sin(\beta) + \cosh(\beta)^2 \cdot \sin(\beta) \cdot \cos(\beta) = 0$$

...(4.22)

After differentiating (4.22) and putting $\Theta = 0$, $\beta = n \pi$, $n = \frac{1}{2}$ (for 1st natural mode),

$$2 \Delta \beta / \beta = \Delta f / f$$

and $\Delta \Theta = (f/\Delta f) \cdot \sin^2(e\beta)$, then further simplifying, we get,

$$\left(\frac{\Delta f}{f}\right)^2 = -4.9362 e^3 - 3.7554 e^2 + 0.2146 e - 1.5 + (15.5 e^3 - 23.25 e^2 + 9.8 e - 1.02) = 0$$

...(4.23)

The above equation (4.23) gives the relation between a crack location and the natural frequencies of a cantilever beam for bending vibrations. The value for ‘$e$’ can be calculated by using the ‘trial and error method’ to solve the above transcendental equation or by using ‘C’ program which is given in Appendix-II.
4.6 IDENTIFICATION OF CRACK SIZE IN CANTILEVER BEAM BY MEASUREMENT OF TRANSVERSE VIBRATION

Consider a typical beam structure that has been damaged by a discrete crack. Based on the consideration of the characteristic equations of the physical model shown in Fig. 4.3, the relationship between the change in eigenfrequency ratio $\Delta f_n/f_n$ and the crack location ‘x’ and stiffness of crack ‘K’ is given by Jialou and Liang (1993).

\[
\frac{\Delta f_n}{f_n} = 2g_n(x) \frac{1}{K}
\]  

...(4.24)

Where,

$x$ is non-dimensional crack location $= \frac{L_1}{L}$ or ‘e’

\[
K = \frac{(K_rL)}{EI}
\]  

...(4.25)

$\Delta f_n$ is difference between uncracked and cracked beam $= f_n - \tilde{f}_n$

$f_n$ and $\tilde{f}_n$ are the natural frequencies of uncracked and cracked beam

The $g_n(x)$ function in equation (4.15) can be expressed in terms of mode shape $\varphi(x)$ for the intact beam which is given by Jialou and Liang et al. (1993).

\[
g_n(x) = \frac{1}{4} \left[ \frac{\varphi_n^*(x)}{dx} \right]^2
\]  

...(4.26)
From elementary beam theory, the mode shapes of beams with typical homogeneous boundary conditions can be easily calculated. For cantilever beam the mode shape is \( \varphi_n = \sin\left(\frac{n}{2}\pi x\right) \).

Now,

Put \( \varphi_n(x) = \sin\left(\frac{n}{2}\pi x\right) \) in equation (4.26) we get,

\[
\therefore g_n(x) = \frac{1}{4} \frac{\left[ \frac{d^2}{dx^2} (\sin n\pi x) \right]^2}{\int_0^1 \left[ \frac{d^2}{dx^2} (\sin n\pi x) \right]^2 dx}
\]

Consider first,

\[
I_1 = \left[ \frac{d^2}{dx^2} (\sin \frac{n}{2}\pi x) \right]^2
\]

\[
\therefore I_1 = \left[ \frac{d}{dx} (\cos n\pi x) \right]^2
\]

\[
\therefore I_1 = \left[ n\pi (-\sin n\pi x) \right]^2
\]

\[
\therefore I_1 = n^2\pi^4 (\sin^2 n\pi x)
\]

Now consider,

\[
I_2 = \int_0^1 \left[ \frac{d^2}{dx^2} (\sin n\pi x) \right]^2 dx
\]

\[
\therefore I_2 = \int_0^1 n^4\pi^4 (\sin^2 n\pi x) dx
\]
\[ \therefore I_2 = n^4 \pi^4 \int_0^1 (\sin^2 n\pi x) \, dx \]

\[ \therefore I_2 = n^4 \pi^4 \int_0^1 \left( \frac{1 - \cos 2n\pi x}{2} \right) \, dx \]

\[ \therefore I_2 = \frac{n^4 \pi^4}{2} \left( \int_0^1 (1 - \cos 2n\pi x) \, dx \right) \]

\[ \therefore I_2 = \frac{n^4 \pi^4}{2} \left( \int_0^1 \left( \int_0^1 (\cos 2n\pi x) \, dx \right) \, dx \right) \]

\[ \therefore I_2 = \frac{n^4 \pi^4}{2} \left( \left[ \frac{n\pi x}{2} \right]_0^1 - \left[ \sin 2n\pi x \right]_0^1 \right) \]

\[ \therefore I_2 = \frac{n^4 \pi^4}{2} \{1 - 0\} \]

\[ \therefore I_2 = \frac{n^4 \pi^4}{2} \]

Now,

\[ \therefore g_n(x) = \frac{1}{4} \left[ \frac{I_1}{I_2} \right] \]

\[ \therefore g_n(x) = \frac{\sin^2 (n/2\pi x)}{2} \]

Put this value of \( g_n(x) \) in equation (4.24) we get,

\[ \frac{\Delta f_n}{f_n} = \sin^2 \left( \frac{n}{2} \pi x \right) \frac{1}{K} \]

Put \( K = (K_r L)/EI \) we get,

\[ \frac{\Delta f_n}{f_n} = \sin^2 \left( \frac{n}{2} \pi x \right) \frac{EI}{K_r L} \]

\[ \text{... (4.27)} \]
Chapter 4 Theoretical Analysis for Crack Assessment Using Transverse Vibration of Beam

The above equation, which relates the change in eigenfrequency, crack location and stiffness of the crack for cantilever beam.

Therefore for a cantilever beam, equation (4.27) gives the relationship between the changes in eigenfrequencies and the crack location and stiffness of crack.

Rizos et.al. (1990) given the spring stiffness $K_r$ in the vicinity of the cracked section of a beam having width ‘b’, height ‘h’ and crack depth ‘a’, from the crack strain energy function,

$$K_r = \frac{EI}{(5.346h)f(a/h)}$$  \hspace{1cm} \text{(4.28)}

We get,

$$\frac{\Delta f_n}{f_n} = \sin^2\left(\frac{n}{2}\pi \frac{x}{L}\right) \frac{5.346h.f(a/h)}{L}$$

$$\therefore \frac{\Delta f_n}{f_n} = 5.346 \sin^2\left[\frac{n}{2} \pi \frac{e}{L}\right] \frac{h}{f(a/h)}$$  \hspace{1cm} \text{(4.29)}

Where, $f(a/h) = 1.8624(a/h)^2 - 3.95(a/h)^3 + 16.375(a/h)^4 - 37.226(a/h)^5 + 78.81(a/h)^6 - 126.9(a/h)^7 + 172(a/h)^8 - 143.97(a/h)^9 + 66.56(a/h)^10$

Putting this value in above equation and neglecting higher order values, the equation becomes,

$$(a/h)^2 = \frac{\Delta f_n/f_n}{9.9563.\sin^2\left[\frac{n}{2} \pi e/2\right] \frac{h}{L}}$$  \hspace{1cm} \text{(4.30)}

Using equation (4.30) we can easily find out the crack depth ratio (a/h) if we know the natural frequencies of uncracked and cracked beam and crack location for cantilever beam.
1.7 SUMMARY

This chapter gives the theoretical formulation of the method based on transverse vibration of beam, which gives the relation between natural frequencies, crack location and crack size for simply supported and cantilever beam which are as follows.

For Simply Supported Beam:

Crack location from equation (4.14),

\[ e = \frac{1}{\pi} \cos^{-1}\left[ 1 - \frac{(\Delta f_2 / f_2)/(\Delta f_1 / f_1)}{2} \right] \]

Crack size from equation (4.21),

\[ (a/h)^2 = \frac{\Delta f_n / f_n}{9.9563 \sin^2 \left[ n \pi (e + 1)/2 \right] \frac{h}{L}} \]

For Cantilever Beam:

Crack location from equation (4.23),

\[ \left( \frac{\Delta f}{f} \right)^2 (-4.9362 e^3 - 3.7554 e^2 + 0.2146 e - 1.5) + (15.5 e^3 - 23.25 e^2 + 9.8 e - 1.02) = 0 \]

Crack size from equation (4.30),

\[ (a/h)^2 = \frac{\Delta f_n / f_n}{9.9563 \sin^2 \left[ n \pi e/2 \right] \frac{h}{L}} \]