CHAPTER IV

SOME SPACES OF LACUNARY SEQUENCES DEFINED BY THE MODULUS

4.1. DEFINITIONS AND NOTATIONS

All relevant definitions and notations are the same as in the preceding chapters except those given here.

Recently, Maddox [26] introduced the following spaces:

\[ w_0(f) = \{ x : \sigma_n(x) \to 0, \text{for some } n \} \]

\[ w(f) = \{ x : x - \sum_{\ell=0}^{\infty} e^{w_0(f), \text{ for some } l} \} \]

\[ w_\infty(f) = \{ x : \sup_n \sigma_n(x) < \infty \} \]

where \( \sigma_n(x) = \frac{1}{n} \sum_{k=1}^{n} f(|x_k|) \) and \( f \) is a modulus function. A modulus may be bounded or unbounded.

Das and Sahoo [8] introduced the following sequence spaces related to the concept of almost convergence.

\[ (w) = \{ x : \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n} t_k (x-s) = 0, \text{ uniformly in } s \} \]

\[ ([w]) = \{ x : \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n} |t_k (x-s)| = 0, \text{ uniformly in } s \} \]

A sequence \( \theta = (k_r) \), \( r = 0, 1, 2, \ldots \), where \( k_0 = 0 \), of increasing non-negative integers \( h_r = (k_r - k_{r-1})^\infty \), is said to be a lacunary sequence. The intervals determined by \( \theta \) are denoted by \( I_r = (k_{r-1}, k_r) \) and the ratio \( k_r/k_{r-1} \) will be denoted by \( q_r \).
A sequence $x$ is said to be \textit{lacunary-$[w]$-convergent} to the value $s$ (cf\cite{2}) if
\[
\limsup_{r \rightarrow \infty} \frac{1}{h} \sum_{k \in I_r} |t_{km}(x-s)| = 0.
\]

By $[w]_{\varnothing}$, we denote the set of all lacunary $[w]$-convergent sequences and $[w]_{\varnothing}$-$\lim x = s$ if $x \in [w]_{\varnothing}$.

4.2. INTRODUCTION:

In this chapter we introduce the sequence spaces of strong almost convergence and lacunary strong almost convergence respectively defined by the modulus and establish certain inclusion relations and show that these two spaces are the same for any bounded sequences.

We define the following spaces
\[
[w(f)]:=\{x: \frac{1}{n+1} \sum_{k=0}^{n} f(|t_{km}(x-s)|) \rightarrow o \ (n \rightarrow \infty), \ 	ext{uniformly in} \ m, \ \text{for some} \ s\},
\]
\[
[w(f)]_{\varnothing}:=\{x: \sup_{r} \frac{1}{h} \sum_{k \in I_r} f(|t_{km}(x-s)|) \rightarrow o, \ \ (n \rightarrow \infty), \ \text{for some} \ s\}.
\]

REMARK. If $f(x)=x$, then $[w(f)]=[w]$ and $[w(f)]_{\varnothing}=[w]_{\varnothing}$. 
4.3 MAIN RESULTS :

We prove the following results

THEOREM 4.1. Let \( \theta=(K_r) \) be a lacunary sequence with \( \lim \inf q_r > 1 \). Then \( \hat{[w(f)]} \subseteq \hat{[w(f)]}_\theta \) and \( \hat{[w(f)]}_\theta - \lim x = \hat{[w(f)]}_\theta - \lim x \).

THEOREM 4.2. Let \( \theta=(K_r) \) be a lacunary sequence with \( \lim \sup q_r < \omega \). Then \( \hat{[w(f)]}_\theta \subseteq \hat{[w(f)]} \) and \( \hat{[w(f)]}_\theta - \lim x = \hat{[w(f)]}_\theta - \lim x \).

THEOREM 4.3. Let \( 1 < \lim \inf q_r \leq \lim \sup q_r < \omega \).

Then

\[
\hat{[w(f)]} = \hat{[w(f)]}_\theta.
\]

THEOREM 4.4. Let \( x \in \hat{[w(f)]} \cap \hat{[w(f)]}_\theta \). Then \( \hat{[w(f)]}_\theta - \lim x = \hat{[w(f)]}_\theta - \lim x \) and \( \hat{[w(f)]}_\theta - \lim x \) is unique for any lacunary sequence \( \theta = (K_r) \).

THEOREM 4.5. Suppose, for a given \( \varepsilon > 0 \), there exists \( n_0 \) and \( m_0 \) such that

\[
(4.3.1) \quad \frac{1}{n} \sum_{k=0}^{n-1} f(|t_{km}(x-s)|) < \varepsilon
\]

for all \( n \geq n_0 \), \( m \geq m_0 \). Then \( x \in \hat{[w(f)]} \).

THEOREM 4.6. For every lacunary sequence \( \theta=(K_r) \), we have

\[
\hat{[w(f)]}_\theta \cap \hat{1}_\omega = \hat{[w(f)]}.
\]

4.4. PROOF OF MAIN RESULTS :

Proof of Theorem 4.1. Let \( \lim \inf q_r > 1 \). Then there exists \( \delta > 0 \) such that \( q_r > 1+\delta \) and hence...
Therefore

\[ \frac{h_r}{k_r} = 1 - \frac{k_r-1}{k_r} \geq 1 - \frac{1}{1+\delta} = \frac{\delta}{1+\delta}. \]

Therefore

\[ \frac{1}{k_r} \sum_{i=1}^{k_r} f(|t_{im}(x-s)|) \geq \frac{1}{k_r} \sum_{i \in I_r} f(|t_{im}(x-s)|) \]

\[ \geq \frac{\delta}{1+\delta} \cdot \frac{1}{h_r} \sum_{i \in I_r} f(|t_{im}(x-s)|), \]

and if \( x \in [w(f)] \) with \( [w(f)] - \lim x = s \), then it follows that \( x \in [w(f)]_\theta \) with \( [w(f)]_\theta - \lim x = s \).

This completes the proof of the theorem.

**Proof of Theorem 4.2.** Let \( x \in [w(f)]_\theta \) with \( [w(f)]_\theta - \lim x = s \).

Then, for \( \epsilon > 0 \), there exists \( j_o \) such that for every \( j \geq j_o \) and all \( m \)

\[ g_{jm} = \frac{1}{h_j} \sum_{i \in I_j} f(|t_{im}(x-s)|) < \epsilon, \]

that is, we can find some positive constant \( M \) such that

(4.4.1) \[ g_{jm} < M \]

for all \( j \) and \( m \). \( \lim \sup q_r < \infty \) (given) implies that there exists some positive number \( K \) such that

(4.4.2) \[ q_r < K \] for all \( r \geq 1. \]

Therefore, for \( K_{r-1} < n \leq K_r \), we have by (4.4.1) and
(4.4.2) that

$$\frac{1}{n+1} \sum_{i=0}^{n} f(|t_{im}(x-s)|) \leq \frac{1}{k_{r-1}} \sum_{i=0}^{k} f(|t_{im}(x-s)|)$$

$$= \frac{1}{k_{r-1}} \sum_{j=0}^{r} \sum_{i \in I_j} f(|t_{im}(x-s)|)$$

$$= \frac{1}{k_{r-1}} \left[ \sum_{j=0}^{j_0} + \sum_{j=j_0+1}^{r} \right] \sum_{i \in I_j} f(|t_{im}(x-s)|)$$

$$\leq \frac{1}{k_{r-1}} \left( \sup_{i \leq j \leq j_0} g_{pm} K_j + (K_{r-j_0} - K_{j_0}) \right) \frac{1}{k_{r-1}}$$

$$\leq M K_{j_0} / k_{r-1} + e K.$$ 

Since $k_{r-1} \to \infty$ as $n \to \infty$, we get

$$x \in [\hat{w}(f)] \text{ with } [\hat{w}(f)]-\lim x = s.$$ 

This completes the proof of the theorem.

**Proof of Theorem 4.3.** It follows directly from Theorem 4.1 and 4.2.

**Proof of Theorem 4.4.** Let $x \in [\hat{w}(f)] \cap [\hat{w}(f)]_{\theta}$, and

$$[\hat{w}(f)]-\lim x = s, [\hat{w}(f)]_{\theta}-\lim x = s'.$$

Suppose that $s \neq s'$. We see that

$$f(|s-s'|) \leq \frac{1}{h_r} \sum_{i \in I_r} f(|t_{im}(x-s)|) + \frac{1}{h_r} \sum_{i \in I_r} f(|t_{im}(x-s')|)$$

$$\leq \lim sup \frac{1}{h_r} \sum_{r} \sum_{i \in I_r} f(|t_{im}(x-s)|) = 0$$

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Hence there exists $r_0$ such that

$$
\frac{1}{hr} \sum_{i \in I_r} f(||t_{im}(x-s)||) > \frac{1}{2} f(||s-s'||).
$$

Since $[\hat{w}(f)]_{\text{-lim}x=s}$, it follows that

$$
\exists \lim_{r} \sup (h_r/k_r)f(||s-s'||) \geq \lim_{r} \inf (h_r/k_r)f(||s-s'||) \geq 0
$$

and so that $\lim q_r = 1$. Hence by Theorem 4.2, $[\hat{w}(f)]_0 \subset [\hat{w}(f)]$ and $[\hat{w}(f)]_{\text{-lim}x=s'} = [\hat{w}(f)]_{\text{-lim}x}.$

Further

$$
\frac{1}{n+1} \sum_{i=0}^{n} f(||t_{im}(x-s)||) + \frac{1}{n+1} \sum_{i=0}^{n} f(||t_{im}(x-s')||) \geq f(||s-s'||) \geq 0
$$

and taking the limit on both sides as $n \to \infty$, we have $f(||s-s'||) = 0$, i.e., $s = s'$ for any modulus $f$.

This completes the proof of the theorem.

**Proof of Theorem 4.5.** Let $\varepsilon > 0$ be given. Choose $n'_o, m'_o$ such that

$$
\frac{1}{n} \sum_{k=0}^{n-1} f(||t_{km}(x-s)||) < \frac{\varepsilon}{4} \text{ for } n \geq n'_o, m \geq m'_o.
$$

It is now enough to show that, there exists no $n''$ such that for $n \geq n''$, $0 \leq m \leq m'_o$, we have

$$
\frac{1}{n} \sum_{k=0}^{n-1} f(||t_{km}(x-s)||) < \varepsilon.
$$

Since $m'_o$ is fixed, put
\[ \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} f(|x_j - s|) = M. \]

Now, let \(0 \leq m \leq \sqrt{n} \) and \(n > m^2\), then

\[ (4.4.3) \quad \frac{1}{n} \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} f(t_{km}^n(x - s)) \leq \frac{1}{n} \sum_{k=0}^{m} \sum_{j=0}^{m} f(x_j - s) + \frac{1}{n} \sum_{k=m}^{m+k-1} \sum_{j=m}^{m+k-1} f(x_j - s) \]

\[ \leq \frac{M}{n} + \frac{1}{n} \sum_{k=0}^{m} \sum_{j=m}^{m+k-1} f(x_j - s) + \frac{1}{n} \sum_{k=m}^{m+k-1} \sum_{j=m}^{m+k-1} f(x_j - s) \]

Let \(k - m > n'\). Then for \(0 \leq m \leq m^{\sqrt{\frac{n}{\sqrt{n}}}\sqrt{n}}\), we have \(k + m - m^{\sqrt{\frac{n}{\sqrt{n}}}\sqrt{n}} > n'\).

From (4.3.1)

\[ (4.4.4) \quad \frac{1}{m^{\sqrt{\frac{n}{\sqrt{n}}}\sqrt{n}}} \sum_{k=0}^{m} \left( \frac{1}{k + m - m^{\sqrt{\frac{n}{\sqrt{n}}}\sqrt{n}}} \sum_{j=m^{\sqrt{\frac{n}{\sqrt{n}}}\sqrt{n}}}^{m+k-1} f(x_j - s) \right) \leq \frac{\varepsilon}{4}. \]

From (4.4.3) and (4.4.4);

\[ \frac{1}{n} \sum_{k=0}^{n-1} f|t_{km}^n(x - s)| \leq \frac{M}{n} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leq \varepsilon \]

for sufficiently large \(n\). Hence the result.

Proof of Theorem 4.6. Let \(x \in \{w(f)\}_\theta \cap 1. \), for \(\varepsilon > 0\), there exists \(r_0\) and \(q_0\) such that
\[(4.4.5) \quad \frac{1}{r} \sum_{k=0}^{h-1} \frac{1}{\mu} \sum_{j=q}^{q+\mu+1} f\left(|t_{kq}(x-s)|\right) < \frac{\varepsilon}{2}\]

for \(r \geq r_0\) and \(q \geq q_0\), \(q = k_{r-1} + 1 + i, i \geq 0\).

Now, let \(n \geq h\), \(m\) is an integer greater than equal to 1.

Then

\[(4.4.6) \quad \frac{1}{n} \sum_{k=0}^{n-1} f\left(|t_{kq}(x-s)|\right) \leq \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{\mu} \sum_{j=q}^{q+\mu+1} f(x_j-s)\]

\[+ \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{\mu} \sum_{j=q}^{q+\mu+1} f(x_j-s)\]

\[\leq \frac{1}{n} \sum_{m \to \mu} \sum_{k=0}^{\mu} \frac{1}{\mu} \sum_{j=q}^{q+\mu+1} f(x_j-s) \leq \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{\mu} \sum_{j=q}^{q+\mu+1} f(x_j-s)\]

Since \(x \in \ell_\infty\), for all \(j\), \(f(|x_j-s|) < M\). So from (4.4.5) and (4.4.6), we have

\[\frac{1}{n} \sum_{k=0}^{n-1} f\left(|t_{kq}(x-s)|\right) < \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{\mu} \sum_{j=q}^{q+\mu+1} f(x_j-s) \leq \frac{Mh_r}{r} + \frac{\varepsilon}{2}\]

For \(\frac{h}{n} \leq 1\), \(\frac{Mh_r}{n}\) can be made less than \(\varepsilon/2\) by taking \(n\) sufficiently large and since \(\frac{h}{m} \leq 1\), then

\[\frac{1}{n} \sum_{k=0}^{n-1} f\left(|t_{kq}(x-s)|\right) < \varepsilon\]

for \(r \geq r_0\), \(q \geq q_0\). Hence by Theorem 4.5, \([\hat{w}(f)]_{\theta} \cap \ell_\infty \subseteq [\hat{w}(f)]_{\theta}\)

It is trivial that \([\hat{w}(f)]_{\theta} \cap \ell_\infty \subseteq [\hat{w}(f)]_{\theta}\).

This completes the proof.