CHAPTER-I

INTRODUCTION

1.1. Summability Theory can be considered as the study of linear transformations on sequence spaces originated from the attempts of mathematicians to assign limits to divergent sequences by setting \( x = 1 \) in the identity \( \frac{1}{1+x} = \sum (-x)^n \) which attributed the sum \( 1/2 \) to the oscillatory series \( 1 - 1 + 1 - \ldots \). Frobenious (1880) introduced the method of summability by arithmetic means, which was generalized by Cesàro (1890) as \((C, k)\) method of summability.

Towards the end of nineteenth century, the theory of sequence spaces and transformations on them attracted many mathematicians who were chiefly motivated by problems on summability theory, Fourier series, Power series and system of equations with infinitely many variables. In 1911, German mathematician, Otto Toeplitz (1881-1940) [45], studied summability method as a class of transformations of complex sequences by complex infinite matrices. But about 1930, functional analytic methods were apparently first employed by Mazur, Banach and Hahn which were followed by Lorentz, Zeller, Borwein, Russel and others. The basic results of this theory may be conveniently found in various books, e.g. Hardy [16], Cooke [6], Maddox [24], Peyerimhoff...
Presenting definitions and notations that are involved in the present work, the author proposes to give a brief resume of the hitherto obtained results against the background of which the problems studied in the present thesis suggest themselves.

1.2. NOTATIONS:

Here we state a few conventions which will be used throughout the thesis and will not be emphasized in the following chapters.

1.2.1. THE SYMBOLS \( \mathbb{N}, \mathbb{R} \) AND \( \mathbb{C} \):

\( \mathbb{N}, \mathbb{R} \) and \( \mathbb{C} \) will be used to denote, respectively the set of positive integers, set of real numbers and the set of complex numbers.

1.2.2. SUMMATION CONVENTION:

By \( \sum_{\alpha}^{\beta} f(n) \), we mean the sum of all values of \( f(n) \) for which \( \alpha \leq n \leq \beta \), if \( \beta < \alpha \), this is zero. Summations are over 0,1,2,3,---, when there is no indication to the contrary.

If \( (x_k) = (x_1, x_2, \ldots) \) is a sequence of terms, then by \( \sum_{k=1}^{\infty} x_k \), we mean \( \sum_{k=1}^{\infty} x_k \), and sometimes we shall write as \( \sum_{k=1}^{\infty} x_k \), without limits, in case where no possible confusion can arise.
1.2.3. LIMIT, sup AND inf :

By "lim", "sup" and "inf" we mean "lim", "sup", ,
n n n
n=\infty n=0,1,2...
"inf", , respectively.
n=0,1,2...

1.2.4. CONSTANT K :

Throughout, K denotes an absolute constant, not necess-
arily the same at each occurrence.

1.2.5. SEQUENCES x AND p :

x=(x_k) denotes any sequence and p=(p_k), a sequence of
strictly positive real numbers with sup p_k<\infty. e^{(k)} and e,

\begin{align*}
&\text{e}^{(k)}=(0,0,0,\ldots,0,1 \text{ (kth place)},0,0,\ldots \text{ for all } k\in\mathbb{N})
&\text{and } e=(1,1,1,\ldots).
\end{align*}

1.2.6. SEQUENCE SPACES :

The following classical complex sequences are well

\begin{align*}
s&:=\{x: x_n \in \mathbb{R} \text{ (or } \mathbb{C})\}, \text{ the space all sequences, real or } \\
c&:=\{x: \lim x_n = 0, \ell \in \mathbb{C}\}, \text{ space of convergent sequences } \\
c_0&:=\{x: \lim x_n = 0\}, \text{ space of null sequences } \\
l_\infty&=c_0, c_0 \text{ being the Banach spaces with norm :}
\end{align*}
\[ |x| = \sup \limits_{n} |x_k|. \]

We denote by \( s(X) \), \( l^\infty(X) \), \( c(X) \) and \( c_0(X) \) the vector spaces of all \( X \)-valued sequences, of all bounded sequences in \( X \), of all convergent sequences in \( X \) and of all null sequences in \( X \), respectively, where \( X \) is a Banach space over the field \( K \).

In case \( X = \mathbb{R} \) or \( \mathbb{C} \), these spaces reduce to \( s \), \( l^\infty \), \( c \) and \( c_0 \) respectively.

We have

\[ \phi \subset c_0 \subset c \subset l^\infty, \]

the inclusion being strict.

\[ l_p := \{ x : \sum \limits_k |x_k|^p < \infty, 0 < p < \infty \}; \]

\[ l_1 := \{ x : \sum \limits_k |x_k| < \infty \}, \text{ (in case of } p = 1 \) \]

and

\[ \text{BV} := \{ x : \sum \limits_k |x_k - x_{k-1}| < \infty, x_0 = 0 \} \]

denote respectively, the Banach spaces of \( l_p \), absolutely convergent sequences and sequence of bounded variation with usual norms:

\[ \|x\|_p = (\sum \limits_k |x_k|^p)^{1/p}, \text{ (in case of } l_p \) \]

and

\[ \|x\| = \sum \limits_k |x_k - x_{k-1}|, \text{ (in case of } x \in \text{BV}). \]
1.2.7. SYMBOLS \( \subseteq, \subset \) AND \( \sim \):

Given two methods of summability, \( P \) and \( Q \), we write \( P \subseteq Q \), or \( Q \supseteq P \) for '\( P \) is included in \( Q \)' or '\( Q \) includes \( P \)' to mean that every sequence summable by \( P \) is also summable by \( Q \).

If \( P \subseteq Q \) and \( Q \supseteq P \), the two methods \( P \) and \( Q \) are said to be equivalent and we write \( P \sim Q \).

If \( P \subseteq Q \) and there exists a sequence which is summable by \( Q \) but not summable by \( P \), then we write \( P \subset Q \).

1.2.8. CLASS OF MATRICES:

Let \( X \) and \( Y \) be two non-empty subsets of the space \( s \).

Let \( A = (a_{nk}) \), \((n,k=1,2,3, \ldots)\) be an infinite matrix with elements of real or complex numbers. We write

\[
A_n(x) = \sum_{k} a_{nk} x_k
\]

Then, \( Ax = (A_n(x)) \) is called the \( A \)-transform of \( x \). Also

\[
\lim_{n \to \infty} Ax = \lim_{n \to \infty} A_n(x),
\]

whenever it exists. If \( x \in X \) implies \( Ax \in Y \), we say that \( A \) defines a matrix transformation from \( X \) into \( Y \), denoted by \( A : X \to Y \). By \( (X,Y) \), we mean the class of matrices \( A \) such that \( A : X \to Y \). By \( (X,Y,P) \) or \( (X,Y)_{\text{reg}} \), we mean the subset of \( (X,Y) \) for which limits or sums are preserved.

1.3. We recall here some well-known classes of matrices.
Silverman-Teoplitz or regular matrices [16]:

**Theorem 1.1.** \( A \in (c, c; P) \) if, and only if

1. \( \sup_{n,k} \sum |a_{nk}|^p < \infty \),
2. \( a_{nk} \to 0 \), \( (n \to \infty, k \text{ fixed}) \),
3. \( \sum_k a_{nk} \to 1 \), \( (n \to \infty) \).

Kojima-Schur or conservative matrices [16]:

**Theorem 1.2.** \( A \in (c, c) \) if, and only if

1. \( \sup_{n,k} \sum |a_{nk}|^p < \infty \),
2. for each \( p \), there exists numbers \( a_p \) (depending only on \( p \)) such that

\[
\lim_{n \to \infty} \sum_{k=p}^{n} a_{nk} = a_p.
\]

The following theorems, which was proved by Schur in 1921, is distinguished from the previous theorems by the fact that the condition on the matrix is of a rather different nature.

Schur or coercive matrices [43]:

**Theorem 1.3.** \( A \in (l^\infty, c) \) if, and only if

1. \( \lim_k a_{nk} \) exists for every \( k \).
and
2. \( \sum_k |a_{nk}| \) converges uniformly in \( n \).
CHARACTERISTIC OF A CONSERVATIVE MATRIX:

Let \( A \in (c,c) \). Then \( A \) is called a conservative (or convergence preserving matrix) and

\[
\chi(A) = \lim_{n} \sum_{k} a_{nk} - \sum_{k} (\lim_{n} a_{nk})
\]

is called the characteristic of \( A \). The numbers \( \lim_{n} a_{nk}, k=1,2,3,\ldots \) and \( \lim_{n} \sum_{k} a_{nk} \) are referred as the characteristic numbers of \( A \).

CO-REGULAR AND CO-NULL-MATRICES:

Let \( A \in (c,c) \). Then \( A \) is co-regular if, and only if \( \chi(A) \neq 0 \), and \( A \) is co-null otherwise.

Thus, the Toeplitz matrices form a subset of the co-regular matrices, which in turn form a subset of the conservative matrices.

1.4. BANACH LIMITS AND ALMOST CONVERGENCE:

In 1948, Lorentz [22] introduced a new method of summation which assigns a general limit to certain bounded sequences. This method is narrowly connected with the limits of S. Banach. The sequences which are summable by this method are called almost convergent sequences.

A linear functional, \( L(x_n) \), which satisfies the condition (see Pettersen [36])

\[
L(x_n) \leq P(x_n),
\]

for all bounded sequences \( (x_n) \), is called Banach limit,
where

\[ P(x_n) = \inf_{n_1, n_2, \ldots, n_k} \limsup \left( \frac{1}{k} \sum_{p=1}^{k} x_{n+p-j} \right) \]

where \( k \) is a positive integer and \( n_1, n_2, \ldots, n_k \) is an arbitrary subset of integers.

A Banach limit, \( L(x_n) \), satisfies the following conditions:

(i) \( L(ax_n) = aL(x_n) \) for all real \( a \),

(ii) \( L(x_n + y_n) = L(x_n) + L(y_n) \),

(iii) \( L(x_{n+1}) = L(x_n) \),

(iv) \( L(1) = 1 \), where \( e = (1, 1, 1, \ldots) \)

(v) \( x_n \geq 0, n = (1, 2, \ldots) \) implies \( L(x_n) \geq 0 \).

A sequence \( x \in l_\infty \) is said to be almost convergent to \( l \) if each Banach limit of \( x \) is \( l \).

The class \( \hat{c} \) of almost convergent sequences was introduced by Lorents [22], who proved that a sequence \( x = (x_n) \) is almost convergent if, and only if

\[ \lim_{p \to \infty} \frac{x_n + x_{n+1} + \cdots + x_{n+p-1}}{p} = l, \]

uniformly in \( n \).

A convergent sequence is almost convergent and its limit and its generalized limit are identical and we have
the inclusion being strict.

By \([c]\), we denote the space of all strongly almost convergent sequences \([25]\), that is,

\[
[c] := \{ x : \lim_{n \to \infty} \frac{1}{n+m} \sum_{k=n+1}^{n+m} |x_k - L| = 0 \text{ uniformly in } n \text{ and } x \in l^\infty, l \in \mathbb{C} \}
\]

It is immediate that \([c] \subset \hat{c}\), and that the inclusion is strict. Also, \([c]\) is a closed subspace of \(l_\infty\). We have the following inclusion relation:

\[c \subset \hat{c} \subset c \subset l_\infty\].

The most remarkable property is that most of the commonly used matrix methods contains this method.

Using the concept of almost convergence, J.P. King \([17]\) introduced a slightly more general class of matrices than the conservative or regular matrices.

**ALMOST CONSERVATIVE AND ALMOST REGULAR MATRICES** :

The matrix \(A\) is said to be almost conservative if \(x \in c\) implies that the \(A\)-transform of \(x\) is almost conservative. \(A\) is said to be almost regular if the \(A\)-transform of \(x\) is almost convergent to the limit of \(x\) for each \(x \in c\).

King proved the following:
THEOREM 1.4. (see King [17]): \( A \in (c, c) \) if, and only if

\[
(i) \sup \left\{ \frac{1}{p} \sum_{j=0}^{n+p-1} \sum_{k} a_{jk} \mid p \in \mathbb{N}, n=0,1,2,\ldots \right\} < \infty, \]

\[
(ii) \text{there exists } \alpha_k \in \mathbb{C}, k=0,1,2,\ldots
\]

\[
\lim_{p \to \infty} \frac{1}{p} \sum_{j=0}^{n+p-1} \sum_{k} a_{jk} = \alpha_k, \quad \text{uniformly in } n,
\]

\[
(iii) \text{there exists } \alpha \in \mathbb{C}, \text{such that}
\]

\[
\lim_{p \to \infty} \frac{1}{p} \sum_{j=0}^{n+p-1} \sum_{k} a_{jk} = \alpha, \quad \text{uniformly in } n.
\]

The matrix \( A \) is almost regular that is, \( A \in (c, c; \mathcal{P}) \) if, and only if conditions (i),(ii) with \( \alpha_k = 0 \) for each \( k \) and (iii) with \( \alpha = 1 \) hold.

\((C,1)\text{-SUMMABILITY}:\)

A sequence \( (x_n) \) is said to be \((C,1)\)-summable (i.e. in the sense of Cesàro) to the value \( L \), we write \( x_n \to L(C,1) \), if

\[
t_n^\infty = \lim_{n \to \infty} \frac{x_0 + \ldots + x_n}{n+1}.
\]

We denote by

\[
(C,1):=\{x=(x_k): \text{for some } L, \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} x_k = L\},
\]

the set of all Cesàro summable sequences.
Also, we write
\[ [C,1] := \{x=(x_k) : \text{for some } L, \lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} |x_k - L| = 0 \}, \]
for the set of sequences which are strongly Cesàro summable to \( L \), i.e., \( x \to L[C,1] \).

The space of strongly Cesàro summable complex sequences of order 1 and index \( p \), where \( 0 < p < \infty \) will be denoted by \( w_p \), and we say that \( (x_k) \in w_p \) if and only if there exists \( \ell \in \mathbb{C} \), such that
\[
\frac{1}{n} \sum_{k=1}^{n} |x_k - \ell|^p \to 0 \text{ as } n \to \infty.
\]

1967, Maddox [23] defined the spaces of strongly A-summable sequences \( x=(x_k) \in s(X) \) with index \( p > 0 \) to \( x \in X \) (Banach space) if
\[
\lim_{n} \sum_{k} a_k |x_k - x_0|^p = 0.
\]

The set of all strongly A-summable sequences in \( X \) is denoted by \( w^p_A(X) \).

1.5. STATISTICAL CONVERGENCE :

The concept of statistical convergence was first introduced by Fast [9] and studied by Schonberg [42] as a summability method. Active researches on this topic were started after the papers of Salat [38], Fridy [11] and Maddox [28]. Strong summability and statistical convergence
were introduced separately and, until recently, followed independent lines of development by Connor [4]. Most recently, Connor [4] showed that if a sequence is strongly $p$-cesaro summable or $w_p$-convergent to $L$, $0 < p < \infty$, then the sequence must be statistically convergent to $L$ and that a bounded statistically convergent sequence must be $w_p$-convergent. It has also been shown that the statistically convergent sequences do not form a locally convex FK-space.

Independently Maddox [28] introduced the statistical convergence in locally convex spaces. Recent developments can be found in [4, 5, 12, 13, 14, 15, 19, 21, 41].

1.6 In the present thesis, we investigate into the theory of sequence spaces and present some new sequence spaces. We also propose to discuss the concept of statistical convergence.

In chapter two, we define some sublinear functionals and related sequence spaces involving $\sigma$-means that generalize the sequence spaces due to Das and Sahoo [8]. We also extend the concept of $\sigma$-bounded variation sequences to that of $\sigma$-bounded variation sequences with an index. Chapter three, has been devoted to the concept of strongly $\sigma$-convergent sequences defined by a modulus and $\sigma$-statistical convergence. Maddox [26] has introduced the sequence spaces defin-
ed by using a modulus function which generalize known spaces \( w_0, w \) and \( w_\infty \) of strongly summable sequences, (Maddox [26],[27]). In this chapter, we introduce some sequence spaces which arise from the notion of strongly \( \sigma \)-convergent sequences defined by a modulus function \( f \), we also study the concept of \( \sigma \)-statistical convergence.

In chapter four, we determine the sequence spaces \( [w(f)] \) and \( \left[ w(f) \right]_\theta \) of strong almost convergence and lacunary strong almost convergence respectively defined by the modulus and establish certain inclusion relations between them. These newly introduced spaces were also shown to be same for any bounded sequences. The concept of weighted statistical convergence has been introduced which we propose to discuss in chapter fifth.

The last chapter is mainly devoted to discuss Almost \( A \)-statistical convergence in Banach space. We also investigate its relations with strong almost \( A \)-summability defined for a sequence of moduli.

Towards the end, we include a comprehensive bibliography of books and various publications which have been referred to in the present thesis.