Chapter 1

Preliminaries

In this chapter we present all the necessary preliminary definitions with examples and results required for our work which are available in the standard literature. This chapter consists of three sections, namely - basics of rings and modules, basics of graph theory and basics of topology.

1.1 Basics of Rings and Modules

In this section we define the algebraic structures needed for the sequel and we present some basic results related to rings and modules. Throughout this section \( R \) denotes a ring and \( M \) denotes a left(right) \( R \)-module. A submodule is a substructure of a module. By \( N \leq M \) we mean "\( N \) is a submodule of the module \( M \)" and by \( N < M \) we mean "\( N \) is a proper submodule of the module \( M \)."

Most of the definitions and basic results mentioned in this section can be found in [14,
30, 37, 38, 48–50, 60].

We begin with the following definition.

**Definition 1.1.1.** Let $M$ be a module and $N$ be a submodule of $M$. Then $N$ is said to be an essential submodule of $M$ if every non-zero submodule of $M$ has non-zero intersection with $N$ and is denoted by $N \leq_e M$. In this case $M$ is called essential extension of $N$.

**Example 1.1.1.** $n\mathbb{Z} \leq_e \mathbb{Z}$ for all positive integer $n$. Also, $\mathbb{Z} \leq_e \mathbb{Q}$.

**Lemma 1.1.1.** [37] If $A \leq B \leq M$, then $A \leq_e M$ if and only if $A \leq_e B \leq_e M$.

**Proof.** First we assume that $A \leq_e B \leq_e M$. We need to show that $A \leq_e M$. Let us consider a nonzero submodule $C$ of $M$. Since $B \leq_e M$, we must have $C \cap B \neq 0$ which follows that $(C \cap B) \cap A \neq 0$ as $A \leq_e B$. This gives $C \cap A \neq 0$. Thus $A \leq_e M$.

Conversely, let us assume that $A \leq_e M$. Since any nonzero submodule of $M$ has nonzero intersection with $A$, the same can be said for nonzero submodules of $B$; hence $A \leq_e B$. Also since any nonzero submodule $C$ of $M$ satisfies $M \cap C \neq 0$, it must satisfy $C \cap B \neq 0$. Thus $B \leq_e M$. Hence we have $A \leq_e B \leq_e M$. □

**Lemma 1.1.2.** [37] If $A \leq_e B \leq M$ and $A_1 \leq_e B_1 \leq M$ then $A \cap A_1 \leq_e B \cap B_1$.

**Proof.** If $C$ is any nonzero submodule of $B \cap B_1$, then since $A \leq_e B$ we have $C \cap A \neq 0$. Since $A_1 \leq_e B_1$, we obtain $(C \cap A) \cap A_1 \neq 0$ and hence we must have $A \cap A_1 \leq_e B \cap B_1$. □

**Lemma 1.1.3.** [37] If $f : B \to M$ and $A \leq_e M$ then $f^{-1}A \leq_e B$.
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Proof. If not, then $B$ has a nonzero submodule $C$ such that $C \cap f^{-1}A = 0$. In particular, $C \cap (\ker f) = 0$; hence $f$ maps $C$ isomorphically onto $fC$, so that $fC$ is a nonzero submodule of $M$. However since $C \cap f^{-1}A = 0$ we obtain $fC \cap A = 0$, which is impossible. \qed

Definition 1.1.2. A simple module is a nonzero module $M$ in which the only submodules are 0 and $M$.

Equivalently, an $R$-module $M$ is simple if and only if $M \cong \frac{R}{I}$ for some maximal ideal $I$ of $R$.

Definition 1.1.3. For any module $M$, the sum of all simple submodules of $M$ is called the socle of $M$ and is denoted by $\text{Soc}M$.

Equivalently $\text{Soc}M$ is the intersection of all essential submodules of $M$.

Example 1.1.2. We know that every abelian group $A$ is a $\mathbb{Z}$-module. The socle of $A$ consists of 0 together with all elements of $A$ of finite square free order.

Definition 1.1.4. A module $M$ is called semisimple if $\text{Soc}M = M$.

In other words, a module $M$ is semisimple if it is the sum of all its simple submodules.

It is obvious that 0 is a semisimple module.

Example 1.1.3. Any vector space $V$ over a field $\mathbb{F}$ is a semisimple $\mathbb{F}$-module, since all the one-dimensional subspaces of $V$ are simple $\mathbb{F}$-submodules.

Lemma 1.1.4. [37] For any module $M$, the following conditions are equivalent:

(1) $M$ is semisimple.

(2) $M$ has no proper essential submodules.
(3) Every submodule of $M$ is a direct summand of $M$.

**Definition 1.1.5.** Let $N$ be a submodule of a module $M$. A relative complement for $N$ in $M$ is any submodule $L$ of $M$ which is maximal with respect to the property $N \cap L = 0$.

**Lemma 1.1.5.** [37] Let $N \leq M$. If $L$ is any relative complement for $N$ in $M$, then $N \oplus L \leq_e M$.

**Definition 1.1.6.** A submodule $N$ of a module $M$ is said to be a closed submodule of $M$ if $N$ has no proper essential extension inside $M$. In other words, $N$ is a closed submodule of $M$ if $N \leq_e L \leq M$ gives $L = N$ for any submodule $L$ of $M$.

**Example 1.1.4.** For the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, the submodule $(1,0)\mathbb{Z}$ is closed as it is a direct summand of $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

**Lemma 1.1.6.** [37] If $B \leq M$, then the following conditions are equivalent:

(1) $B$ is a closed submodule of $M$.

(2) $B$ is a relative complement for some $A \leq M$.

(3) If $A$ is any relative complement for $B$ in $M$, then $B$ is a relative complement for $A$ in $M$.

(4) If $B \leq K \leq_e M$, then $\frac{K}{B} \leq_e \frac{M}{B}$.

**Definition 1.1.7.** A uniform module is a non-zero module $M$ such that any two non-zero submodules of $M$ have non-zero intersection. Equivalently, a module $M$ is uniform if and only if $M \neq 0$ and every non-zero submodule of $M$ is essential in $M$.

**Example 1.1.5.** Every simple module is uniform. Also, $\mathbb{Q}$ is a uniform $\mathbb{Z}$-module.
**Definition 1.1.8.** A module $M$ has Goldie dimension $n$ (written $G\dim M = n$) if there is an essential submodule $V \leq_e M$ that is a direct sum of $n$ uniform submodules. If on the other hand, no such integer $n$ exists, we write $G\dim M = +\infty$.

**Definition 1.1.9.** A module $M$ is called finite dimensional if $G\dim M < +\infty$. Otherwise it is infinite dimensional.

**Example 1.1.6.** Since $\mathbb{Q}$ is a uniform $\mathbb{Z}$-module, so $G\dim \mathbb{Q} = 1$. But $\mathbb{Q} / \mathbb{Z}$ does not have finite Goldie dimension.

**Definition 1.1.10.** A ring $R$ is called right finite dimensional if it is finite dimensional as a right $R$-module. Left finite dimensional rings can be defined similarly.

**Lemma 1.1.7.** [37] The following holds:

1. If $N$ is a closed submodule of a module $M$, then $G\dim M = G\dim N + G\dim \frac{M}{N}$.

2. If $M_1, M_2, \ldots, M_n$ be modules, then $G\dim (M_1 \oplus M_2 \oplus \ldots \oplus M_n) = G\dim M_1 + G\dim M_2 + \ldots + G\dim M_n$.

3. Let $N \leq M$ and assume that $N$ is finite dimensional. Then $G\dim M = G\dim N$ if and only if $N \leq_e M$.

**Definition 1.1.11.** A sequence $0 \rightarrow A \rightarrow^f B \rightarrow^g C \rightarrow 0$ of modules and homomorphisms is said to be exact if $\text{Im } f = \text{Ker } g$.

**Lemma 1.1.8.** [49] Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence of modules. Then $G\dim B \leq G\dim A + G\dim C$. 
**Definition 1.1.12.** An R-module $M$ satisfies the ascending chain condition (ACC) on its submodules if every ascending chain $M_1 \leq M_2 \leq M_3 \leq \ldots \leq M_n \ldots$ of submodules of $M$ is ultimately stationary, i.e., if there exists a positive integer $N$ such that $M_n = M_{n+1} = \ldots$ for all $n \geq N$.

Equivalently, $M$ satisfies the ACC on its submodules if and only if every non-empty family of submodules of $M$ has a maximal element.

**Definition 1.1.13.** An R-module $M$ is said to be a Noetherian module if it satisfies ACC on its submodules.

**Example 1.1.7.** Any finite dimensional vector space $V$ over a field $\mathbb{F}$ is a Noetherian $\mathbb{F}$-module. Also, $\mathbb{Z}$ is a Noetherian module over itself.

**Proposition 1.1.1.** [37] Given an R-module $M$, the following conditions are equivalent:

1. $M$ is Noetherian.
2. All submodules of $M$ are finitely generated.
3. The finitely generated submodules of $M$ satisfy the ACC.

**Proposition 1.1.2.** [37] Let $N$ be a submodule of an R-module $M$. Then $M$ is Noetherian if and only if $N$ and $\frac{M}{N}$ are both Noetherian.

**Definition 1.1.14.** An R-module $M$ satisfies the descending chain condition (DCC) on its submodules if every descending chain $M_1 \geq M_2 \geq M_3 \geq \ldots \geq M_n \ldots$ of submodules of $M$ is ultimately stationary, i.e., if there exists a positive integer $N$ such that $M_n = M_{n+1} = \ldots$ for all $n \geq N$. 
Equivalently, $M$ satisfies the DCC on its submodules if and only if every non-empty family of submodules of $M$ has a minimal element.

**Definition 1.1.15.** An $R$-module $M$ is said to be an Artinian module if it satisfies DCC on its submodules.

**Example 1.1.8.** Any finite dimensional vector space $V$ over a field $F$ is an Artinian $F$-module. Also, finite abelian groups are Artinian $\mathbb{Z}$-module.

**Proposition 1.1.3.** [37] Let $N$ be a submodule of an $R$-module $M$. Then $M$ is Artinian if and only if $N$ and $\frac{M}{N}$ are both Artinian.

**Definition 1.1.16.** A composition series of a non-zero $R$-module $M$ is a finite descending chain of submodules of $M$ starting with $M$ and ending with $0$, say $M = M_0 > M_1 > M_2 > \ldots > M_m = 0$ such that the successive quotients $\frac{M_i}{M_{i+1}}$ are simple for all $i, 0 \leq i \leq m - 1$.

The integer $m$ is called the length of the composition series.

**Definition 1.1.17.** An $R$-module $M$ is said to be a module of finite length if it is either zero or has some composition series.

**Lemma 1.1.9.** [60] A module is of finite length $\iff$ it is both Artinian and Noetherian.

**Definition 1.1.18.** A ring $R$ is said to be commutative if $ab = ba$ for all $a, b \in R$.

**Definition 1.1.19.** A ring $R$ is said to be a ring with unity if there exists an element $1 \in R$ such that $1r = r = r1$ for all $r \in R$.

**Example 1.1.9.** The rings $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}_n$ (n being any positive integer) are all commutative rings with unity.
**Definition 1.1.20.** Let $R$ be a ring with unity $1$. An element $a \in R$ is said to be a unit or invertible if there exists an element $b \in R$ such that $ab = ba = 1$.

**Example 1.1.10.** Every non-zero element of the rings $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ is a unit. However, the only units in the ring $\mathbb{Z}$ are $1$ and $-1$.

**Definition 1.1.21.** An element $x$ in a ring $R$ is said to be nilpotent if there exists a positive integer $n$ such that $x^n = 0$.

In any ring $R$, $0$ is always a nilpotent element called the trivial nilpotent element.

**Example 1.1.11.** In the ring $\mathbb{Z}_9$ the nilpotent elements are $0$, $3$ and $6$.

**Definition 1.1.22.** A right(left) $R$-module $M$ is said to be unitary if $m_1 = m$ ($1m = m$) for all $m \in M$ and $1 \in R$.

**Example 1.1.12.** Any module over a ring with unity is unitary. Every abelian group is a unitary $\mathbb{Z}$-module.

**Definition 1.1.23.** For the $R$-modules $M$ and $N$, a mapping $f : M \to N$ is said to be a module homomorphism if

- $f(x+y) = f(x) + f(y)$ and
- $f(rx) = rf(x)$ for all $x, y \in M$ and $r \in R$.

If $f$ is also one-one, then it is said to be a module monomorphism.

A one-one and onto module homomorphism is called a module isomorphism.

**Definition 1.1.24.** An element $a \in R$ is said to be a zero divisor if there exists an element $b \in R^*$ such that $ab = 0$ or $ba = 0$.

The set of zero divisors of $R$ is denoted by $Z(R)$ and $Z^*(R) = Z(R) \setminus \{0\}$. 
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**Example 1.1.13.** Let $R = \mathbb{Z}_6$ be the ring of integers modulo 6. This ring has zero divisors $0, 2, 3$ and $4$.

Thus we have, $Z(R) = \{0, 2, 3, 4\}$ and $Z^*(R) = \{2, 3, 4\}$.

**Definition 1.1.25.** A ring $R$ is said to be an integral domain if there are no non-trivial zero divisors in $R$.

In other words, a ring $R$ is called an integral domain if for any two elements $a, b \in R$ such that $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$.

**Example 1.1.14.** The rings $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}_p$ ($p$ being a prime) are integral domains whereas the rings $\mathbb{Z}_n$ ($n$ being a composite) are not integral domains.

**Definition 1.1.26.** A commutative ring $R$ with unity is called a field if every non-zero element of $R$ has multiplicative inverse.

In other words, A commutative ring $R$ with unity is called a field if the set of non-zero elements of $R$ forms an Abelian group under multiplication in $R$.

**Example 1.1.15.** The rings $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}_n$ ($n$ being any positive integer) are fields but the ring $\mathbb{Z}$ is not a field.

**Definition 1.1.27.** Let $R$ be any ring. The characteristic of $R$, denoted by $\text{Char}(R)$ is defined as the least positive integer $n$ such that $na = 0$ for all $a \in R$.

If no such $n$ exists, then we say $\text{Char}(R) = 0$.

**Example 1.1.16.** In the ring of integers $\mathbb{Z}$, there exists no positive integer $n$ such that $na = 0$ for all $a \in \mathbb{Z}$. Therefore, we have $\text{Char}(\mathbb{Z}) = 0$. By the same argument we have $\text{Char}(\mathbb{Q}) = \text{Char}(\mathbb{R}) = \text{Char}(\mathbb{C}) = 0$. 
On the other hand, in the ring $\mathbb{Z}_n$ of integer modulo $n$, we have $n$ is the least positive integer such that $n\bar{x} = 0$ for all $\bar{x} \in \mathbb{Z}_n$. Thus, we have $\text{Char}(\mathbb{Z}_n) = n$.

**Lemma 1.1.10.** [60] The characteristic of an integral domain is either 0 or a prime number.

**Definition 1.1.28.** A prime ideal in a ring $R$ is a proper two-sided ideal $P$ such that whenever $a, b \in R$ with $aRb \subseteq P$, either $a \in P$ or $b \in P$.

**Example 1.1.17.** In the ring of integers $\mathbb{Z}$ for any prime $p$, $p\mathbb{Z}$ is a prime ideal.

**Definition 1.1.29.** Let $M$ be an $R$-module. Two submodules $N$ and $L$ of $M$ are said to be comparable if $N \subseteq L$ or $L \subseteq N$.

**Definition 1.1.30.** A right(left) $R$-module $M$ is said to be faithful if for any non-zero $r \in R$ there is an element $m \in M$ such that $mr \neq 0$ ($rm \neq 0$).

**Definition 1.1.31.** A right(left) $R$-module $F$ is said to be a flat module if given any monomorphism $A \longrightarrow B$ of left(right) $R$-modules, the induced homomorphism $F \otimes_R A \longrightarrow F \otimes_R B$ ($A \otimes_R F \longrightarrow B \otimes_R F$) is also a monomorphism.

**Example 1.1.18.** The module $\mathbb{Q}$ over the ring $\mathbb{Z}$ is a flat module.

**Definition 1.1.32.** A right(left) $R$-module $F$ is said to be a faithfully flat module if for left(right) $R$-modules $A$, $B$ and $C$, a sequence $A \longrightarrow B \longrightarrow C$ is exact if and only if $F \otimes_R A \longrightarrow F \otimes_R B \longrightarrow F \otimes_R C$ ($A \otimes_R F \longrightarrow B \otimes_R F \longrightarrow C \otimes_R F$) is exact.

Alternatively, a right(left) $R$-module $F$ is said to be a faithfully flat module if $F$ is flat and for any left(right) $R$-module $M$, $F \otimes_R M = 0 \implies M = 0$ ($M \otimes_R F = 0 \implies M = 0$).
Example 1.1.19. The module \( \mathbb{Q} \) over the ring \( \mathbb{Z} \) is not faithfully flat since \( \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = 0 \).

In general, a \( \mathbb{Z} \)-module \( F \) is faithfully flat if and only if \( F \) is torsionfree and \( Fp \neq F \) for any prime number \( p \).

Also, any non-zero module over a simple artinian ring is faithfully flat.

Definition 1.1.33. Let \( M \) be an \( R \)-module and \( N \) be a submodule of \( M \). Then \( (N : M) \) is defined as \( (N : M) = \{ r \in R | rM \subseteq N \} \).

Also, for every \( a \in R \), \( (N : a) \) is defined as \( (N : a) = \{ m \in M | am \in N \} \).

Definition 1.1.34. Let \( R \) be a ring with unity and \( M \) be a unitary \( R \)-module. Then \( M \) is said to be finitely generated over \( R \) if there is a finite subset \( X = \{ x_1, x_2, ..., x_n \} \) of \( M \) such that \( M = \{ \sum_{i=1}^{n} a_i x_i | a_i \in R \} \).

1.2 Basics of Graph Theory

In this section we discuss some definitions and properties of graph and different graph theoretic structures needed for the sequel and we present some basic results related to these. Throughout this section all graphs are undirected simple graph.

Most of the definitions and basic results mentioned in this section can be found in [21, 28, 44, 75].

We begin with the following definition.

Definition 1.2.1. A graph \( G \) consists of a non-empty finite set \( V(G) \) of objects called vertices together with a set \( E(G) \), possibly empty, of unordered pairs of distinct vertices of \( G \) called edges.
The cardinality of $V(G)$ is called order and the cardinality of $E(G)$ is called the size of the graph $G$.

**Definition 1.2.2.** A graph $G$ is said to be a simple graph if it has neither loops nor multiple edges.

![Fig 1.2.1 A simple graph](image)

**Definition 1.2.3.** Two graphs $G$ and $H$ are said to be isomorphic if there exists a one-to-one correspondence between their vertex sets which preserves adjacency and it is denoted by $G \cong H$.

![Fig 1.2.2 $G \cong H$](image)

**Definition 1.2.4.** The cartesian product of two graphs $G$ and $H$, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices $(a, b), (a', b') \in V(G) \times V(H)$ are
adjacent if and only if (i) \(a = a'\) and \(b\) is adjacent to \(b'\), or (ii) \(b = b'\) and \(a\) is adjacent to \(a'\).

![Fig 1.2.3 Cartesian product of two graphs \(G\) and \(H\).](image)

**Definition 1.2.5.** A subgraph of \(G\) is a graph having all of its vertices and edges in \(G\).

A spanning subgraph of \(G\) contains all vertices of it.

For any set \(S\) of vertices of \(G\), the induced subgraph \(<S>\) is the maximal subgraph of \(G\) with vertex set \(S\). Thus two points of \(S\) are adjacent in \(<S>\) if and only if they are adjacent in \(G\).

![Fig 1.2.4 \(G_1\) is an induced subgraph of \(G\) but \(G_2\) is not. \(G_2\) is a spanning subgraph of \(G\) but \(G_1\) is not.](image)

**Definition 1.2.6.** Two (induced) subgraphs \(G_1\) and \(G_2\) of a graph \(G\) are said to be disjoint if \(G_1\) and \(G_2\) have no common vertices and no vertex of \(G_1\) (respectively, \(G_2\)) is adjacent (in \(G\)) to any vertex not in \(G_2\) (respectively, \(G_1\)).
Definition 1.2.7. The degree of a vertex $v$ in a graph $G$ is the number of edges incident with $v$. The degree of a vertex $v$ is denoted by $\text{deg}(v)$.

The vertex $v$ is isolated if $\text{deg}(v) = 0$.

Example 1.2.1. In the labelled graph shown in figure 1.2.6 we have $\text{deg}(v_1) = 1$, $\text{deg}(v_2) = 4$, $\text{deg}(v_3) = 2$, $\text{deg}(v_4) = 2$ and $\text{deg}(v_5) = 3$. Hence this graph has no isolated vertex.

Lemma 1.2.1. [44] The sum of the degrees of the vertices of a graph $G$ is twice the number of edges, i.e., if $G$ is a graph with $p$ vertices and $q$ edges, then $\sum_{i=1}^{p} \text{deg}(v_i) = 2q$.

Lemma 1.2.2. [44] In any graph, the number of vertices of odd degree is even.
Example 1.2.2. In the labelled graph shown in figure 1.2.6 we have 6 edges and the sum of the degrees of the vertices is 12 which is twice the number of edges.

Also, we have 2 odd degree vertices in this graph which are \( v_1 \) and \( v_5 \).

Definition 1.2.8. The minimal and the maximal degree of a graph \( G \) denoted by \( \delta(G) \) and \( \Delta(G) \) respectively are defined as

\[
\delta(G) = \min \{\deg(v) | v \in V(G)\}
\]

and

\[
\Delta(G) = \max \{\deg(v) | v \in V(G)\}
\]

Example 1.2.3. In the labelled graph shown in figure 1.2.6 we have the minimal degree is 1 and the maximal degree is 4.

Definition 1.2.9. A graph \( G \) is said to be regular of degree \( r \) if every vertex has degree \( r \).

![Fig 1.2.7 A 3-regular graph](image)

Definition 1.2.10. A walk in a graph \( G \) is an alternating sequence of vertices and edges, \( v_0x_1v_1...x_nv_n \) in which each edge \( x_i \) is \( v_{i-1}v_i \). The length of such a walk is \( n \), the number of occurrences of edge in it.

A closed walk has the same first and last vertices.

A trail is a walk in which all edges are distinct.

A path is a walk in which all vertices are distinct.
**Definition 1.2.11.** A cycle or circuit in a graph $G$ is a closed walk with all vertices distinct (except the first and last). A cycle of $n$ vertices is denoted by $C_n$. An acyclic graph does not contain a cycle.

**Example 1.2.4.** In the labelled graph shown in figure 1.2.6 we have $v_1v_2v_5v_2v_3$ is a walk which is not a trail and $v_1v_2v_5v_4v_2v_3$ is a trail which is not a path. Also, we have $v_1v_2v_5v_4$ is a path and $v_2v_4v_5v_2$ is a cycle.

**Definition 1.2.12.** A graph $G$ is said to be connected if there is a path between every two distinct vertices. A graph which is not connected is called a disconnected graph.

A totally disconnected graph is one that does not contain any edge.

![Graphs](image)

Fig 1.2.8 $G$ is a connected graph, $G_1$ is a disconnected graph and $G_2$ is a totally disconnected graph.

**Definition 1.2.13.** For distinct vertices $x$ and $y$ in a graph $G$, the distance between $x$ and $y$ denoted by $d(x,y)$ is defined as the length of the shortest path between $x$ and $y$ and if there is no such path we define $d(x,y) = \infty$. The eccentricity $e(v)$ of a vertex $v$ in a connected graph $G$ is $\max d(u,v)$ for all $u$ in
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$V(G)$.

A vertex with minimum eccentricity is called a center of $G$.

**Definition 1.2.14.** The diameter of a graph $G$ denoted by $\text{diam}(G)$ is defined as

$$\text{diam}(G) = \sup\{d(x,y) | x, y \in G\}.$$  

Equivalently, the maximum eccentricity of $G$ is the diameter of $G$.

**Definition 1.2.15.** The girth of a graph $G$, denoted by $\text{gr}(G)$ is the length of the shortest cycle in $G$ and $\text{gr}(G) = \infty$ if $G$ has no cycle.

**Definition 1.2.16.** If in a graph $G$ any two vertices are adjacent, then it is called a complete graph and is denoted by $K^\alpha$ or $K_\alpha$ where $\alpha$ is the number of vertices of the graph.

![Fig 1.2.9 The complete graphs $K_5$ and $K_6$](image)

**Definition 1.2.17.** A complete subgraph of a graph $G$ is called a clique.

A maximum clique of $G$ is a clique with largest number of vertices and the number of vertices of a maximum clique is called the clique number of $G$, denoted by $\omega(G)$.

**Definition 1.2.18.** A graph $G$ is said to be a bipartite graph or bigraph if its vertex set $V$ can be partitioned into two disjoint subsets $V_1$ and $V_2$ with every edge of $G$ joining $V_1$ and $V_2$. 

Fig 1.2.10 A bipartite graph $G$

If $|V_1| = \alpha$ and $|V_2| = \beta$ and every vertex of $V_1$ is adjacent to every vertex of $V_2$, then $G$ is called a complete bipartite graph and is denoted by $K^{\alpha, \beta}$ or $K_{\alpha, \beta}$.

Fig 1.2.11 The complete bipartite graphs $K_{2,3}$ and $K_{3,3}$

**Lemma 1.2.3.** [44] A graph is bipartite if and only if all its cycles are even.

**Definition 1.2.19.** An Eulerian path is a trail in a graph which visits every edge exactly once. Similarly, an Eulerian cycle of a graph is an Eulerian path which starts and ends on the same vertex.

A graph $G$ is called Eulerian if it has an Eulerian cycle.

Fig 1.2.12 An Eulerian graph
Lemma 1.2.4. [44] The following statements are equivalent for a connected graph G:

1. G is Eulerian.
2. Every vertex of G has even degree.
3. The set of edges of G can be partitioned into cycles.

Definition 1.2.20. A Hamiltonian cycle is a spanning cycle in a graph. A graph G is called Hamiltonian if it has a Hamiltonian cycle.

Fig 1.2.13 A Hamiltonian graph

Definition 1.2.21. The connectivity or point-connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. Thus the connectivity of a disconnected graph is 0, while the connectivity of a connected graph with a cutpoint is 1. Also, $\kappa(K^p) = p - 1$.

Definition 1.2.22. The line-connectivity $\lambda(G)$ of a graph G is the minimum number of edges whose removal results in a disconnected or trivial graph. Thus the line-connectivity of a disconnected graph is 0, while that of a connected graph with a bridge is 1. Also, $\lambda(K^1) = 0$.

Lemma 1.2.5. [44] For any graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$. 
Definition 1.2.23. Given a graph $G$, its line graph $L(G)$ is a graph such that every vertex of $L(G)$ represents an edge of $G$, and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in $G$.

So, the set of vertices of $L(G)$ is exactly the set of edges of $G$, and $L(G)$ represents the adjacencies between edges of $G$.

![Fig 1.2.14 A graph $G$ and its line graph $L(G)$](image)

Lemma 1.2.6. [44] A connected graph is isomorphic to its line graph if and only if it is a cycle.

1.3 Basics of Topology

In this section we discuss some definitions and properties of topological spaces and different topological structures needed for the sequel and we present some basic results related to these.

Most of the definitions and basic results mentioned in this section can be found in [46, 59, 71].

We begin with the following definition.
Definition 1.3.1. A topology on a set $X$ is a collection $\mathcal{S}$ of subsets of $X$ having the following properties:

1. $\emptyset \in \mathcal{S}$ and $X \in \mathcal{S}$.
2. The union of the elements of any subcollection of $\mathcal{S}$ is in $\mathcal{S}$.
3. The intersection of the elements of any finite subcollection of $\mathcal{S}$ is in $\mathcal{S}$.

A topological space is an ordered pair $(X, \mathcal{S})$ consisting of a set $X$ and a topology $\mathcal{S}$ on $X$. The elements of $X$ are called its points.

Definition 1.3.2. If $(X, \mathcal{S})$ is a topological space, then a subset $U$ of $X$ is said to be an open set of $X$ if $U \in \mathcal{S}$.

we have, by definition $\emptyset$ and $X$ are both open sets.

Definition 1.3.3. A subset $A$ of a topological space $(X, \mathcal{S})$ is said to be a closed set of $X$ if the set $X - A$ is open in $X$.

we have, by definition $\emptyset$ and $X$ are both closed sets.

Example 1.3.1. Let us consider the set $X = \{a, b, c\}$. Then the collection $\mathcal{S} = \{\emptyset, \{b\}, \{a,b\}, \{b,c\}, X\}$ of subsets of $X$ is a topology on $X$ and hence $(X, \mathcal{S})$ is a topological space.

Clearly, in this topological space open sets are $\emptyset, \{b\}, \{a,b\}, \{b,c\}$ and $X$. Also, closed sets are $\emptyset, \{a\}, \{c\}, \{c,a\}$ and $X$.

Definition 1.3.4. Let $X$ be any set. A base or basis for a topology on $X$ is a collection $\mathcal{B}$ of subsets of $X$ (called basis elements) such that

1. For each $x \in X$, there is at least one basis element $B$ containing $x$. 
(2) If \( x \) belongs to the intersection of two basis elements \( B_1 \) and \( B_2 \), then there is a basis element \( B_3 \) containing \( x \) such that \( B_3 \subseteq B_1 \cap B_2 \).

**Example 1.3.2.** Let us consider the set \( X = \{a,b,c,d\} \) and the topology
\[
\mathcal{S} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}, X\} \text{ on } X.
\]
Then the collection \( \mathcal{B} = \{\{a\}, \{b\}, \{c,d\}\} \) is a base for the topology \( \mathcal{S} \) on \( X \).

**Lemma 1.3.1.** [59] Let \( X \) be a set; let \( \mathcal{B} \) be a base for a topology \( \mathcal{S} \) on \( X \). Then \( \mathcal{S} \) equals the collection of all unions of elements of \( \mathcal{B} \).

**Lemma 1.3.2.** [59] Let \( X \) be a topological space. Suppose that \( \mathcal{B} \) is a collection of open sets of \( X \) such that for each open set \( U \) of \( X \) and each \( x \in U \), there is an element \( B \) of \( \mathcal{B} \) such that \( x \in B \subseteq U \). \( \mathcal{B} \) is a base for the topology of \( X \).

**Definition 1.3.5.** Let \( X \) be a topological space with topology \( \mathcal{S} \). If \( Y \) is a subset of \( X \), then the collection \( \mathcal{S}_Y = \{Y \cap U | U \in \mathcal{S}\} \) is a topology on \( Y \), called the subspace or relative topology.

With this topology, \( Y \) is called a subspace of \( X \).

**Example 1.3.3.** Let us consider the set \( X = \{a,b,c,d,e\} \) and the topology
\[
\mathcal{S} = \{\emptyset, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}, X\} \text{ on } X.
\]
Let \( Y = \{a,d,e\} \). Then \( \mathcal{S}_Y = \{\emptyset, \{a\}, \{d\}, \{a,d\}, \{d,e\}, Y\} \) is a subspace topology on \( Y \) and \( Y \) is a subspace of \( X \).

**Lemma 1.3.3.** [59] If \( \mathcal{B} \) is a base for the topology of \( X \) then the collection \( \mathcal{B}_Y = \{B \cap Y | B \in \mathcal{B}\} \) is a base for the subspace topology on \( Y \).

**Lemma 1.3.4.** [59] Let \( Y \) be a subspace of \( X \). If \( U \) is open in \( Y \) and \( Y \) is open in \( X \), then \( U \) is open in \( X \).
Definition 1.3.6. Given a topological space \((X, \mathcal{S})\), a neighbourhood of a point in \(X\) is an open set which contains the point.

In other words, given a topological space \((X, \mathcal{S})\) and \(x \in X\), a subset \(N\) of \(X\) is called a neighbourhood of \(x\) if \(\exists\) an open set \(U\) such that \(x \in U \subseteq N\).

Example 1.3.4. Let us consider the set \(X = \{a, b, c, d\}\) and the topology \(\mathcal{S} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}\) on \(X\).

Then \(\{a, b, c\}\) is a neighbourhood of \(a\) as \(\exists\) an open set \(\{a, b\}\) such that \(a \in \{a, b\} \subset \{a, b, c\}\).

Definition 1.3.7. A topological space \((X, \mathcal{S})\) is said to be a \(T_0\)-space if for every pair of distinct points \(x, y \in X\), there is a neighbourhood of atleast one to which the other does not belong.

In other words, a topological space \((X, \mathcal{S})\) is said to be a \(T_0\)-space if \(\forall x, y \in X, x \neq y\), either \(\exists\) open set \(G \in \mathcal{S}\) such that \(x \in G, y \notin G\) or \(\exists\) open set \(H \in \mathcal{S}\) such that \(x \notin H, y \in H\).

Example 1.3.5. Every discrete topological space is a \(T_0\)-space and so is any indiscrete space containing not more than one point. Also, if \(X\) is an infinite set such that its open sets are empty set, then the set \(X\) and the subsets of which complements are finite form a \(T_0\)-space.

Definition 1.3.8. A topological space \((X, \mathcal{S})\) is said to be a \(T_1\)-space if given a distinct pair of points \(x, y \in X\), there is a neighbourhood of each to which the other does not belong.

In other words, a topological space \((X, \mathcal{S})\) is said to be a \(T_1\)-space if for distinct \(x, y \in X\), \(\exists\) two open sets, one containing \(x\) but not \(y\) and another containing \(y\) but not \(x\).

That is, \(x, y \in X\) and \(x \neq y\) \(\Rightarrow \exists\) open sets \(G, H \in \mathcal{S}\) such that \(x \in G, y \notin G\) and \(y \in H, x \notin H\).
Example 1.3.6. Every metric space is a $T_1$-space. Also, a topological space with co-finite topology is a $T_1$-space.

Lemma 1.3.5. [71] A topological space is a $T_1$-space $\iff$ each point is a closed set.

Remark 1.3.9. It is easy to see that every $T_1$-space is also a $T_0$-space. But the converse is not true.

For example, let us consider $X = \{x \in \mathbb{R} | 0 \leq x < 1\}$ and $G_k = \{x \in \mathbb{R} | 0 \leq x < k, 0 < k \leq 1\}$. Let $\mathcal{I} = \{G_k|0 < k \leq 1\}$. Then $\mathcal{I}$ is a topology on $X$. Clearly, $(X, \mathcal{I})$ is a $T_0$-space but not a $T_1$-space.

Definition 1.3.10. A topological space $(X, \mathcal{I})$ is said to be a $T_2$-space or a Hausdorff space iff for every distinct pair of points $x, y \in X$, $\exists$ disjoint neighbourhoods of $x$ and $y$, that is, if $\exists$ neighbourhoods $N$ of $x$ and $M$ of $y$ such that $N \cap M = \emptyset$.

In other words, a topological space $(X, \mathcal{I})$ is said to be a $T_2$-space or a Hausdorff space iff $\forall x, y \in X, x \neq y, \exists$ open sets $N, M \in \mathcal{I}$ such that $x \in N$, $y \in M$ but $N \cap M = \emptyset$.

Example 1.3.7. Every discrete topological space is a $T_2$-space. Also, every metric space is a $T_2$-space.

Lemma 1.3.6. [59] Every finite point set in a Hausdorff space is closed.

Lemma 1.3.7. [71] Every compact subspace of a Hausdorff space is closed.

Remark 1.3.11. It is easy to see that every $T_2$-space is also a $T_1$-space. But the converse is not true.

For example, a topological space with co-finite topology is a $T_1$-space but not a $T_2$-space.

Definition 1.3.12. Let $X$ be a topological space. A separation of $X$ is a pair $A, B$ of disjoint non-empty open subsets of $X$ such that $X = A \cup B$. 
A topological space $X$ is said to be connected if there does not exist a separation of $X$. In other words, a topological space $X$ is connected if $X$ cannot be represented as $X = A \cup B$ where $A$ and $B$ are two disjoint non-empty open subsets of $X$.

**Example 1.3.8.** We have $[0, 1]$ is a connected subspace of $\mathbb{R}$. Also, a indiscrete topological space is always connected.

**Definition 1.3.13.** A topological space $X$ is said to be disconnected if it is not connected. In other words, a topological space $X$ is disconnected if $X$ can be represented as $X = A \cup B$ where $A$ and $B$ are two disjoint non-empty open subsets of $X$.

Then the representation $X = A \cup B$ is called a disconnection of $X$.

**Example 1.3.9.** We have $(-1, 0) \cup (2, 7)$ is a disconnected subspace of $\mathbb{R}$. Also, the set of rationals $\mathbb{Q}$ is a disconnected space.

**Lemma 1.3.8.** ([71]) A subspace of the real line $\mathbb{R}$ is connected $\iff$ it is an interval.

In particular, $\mathbb{R}$ is connected.

**Lemma 1.3.9.** ([71]) Any continuous image of a connected space is connected.

**Lemma 1.3.10.** ([71]) A topological space $X$ is disconnected $\iff$ there exists a continuous mapping of $X$ onto the discrete two-point space $\{0, 1\}$.

**Definition 1.3.14.** A totally disconnected space is a topological space $X$ in which every pair of distinct points can be separated by a disconnection of $X$. This means that for every pair of points $x$ and $y$ in $X$ such that $x \neq y$, there exists a disconnection $X = A \cup B$ with $x \in A$ and $y \in B$. 

Example 1.3.10. The set of rationals $\mathbb{Q}$ with the relative usual topology is a totally disconnected space. Also, the set of irrationals with the relative usual topology is a totally disconnected space.

Lemma 1.3.11. [71] The components of a totally disconnected space are its points.

Definition 1.3.15. Let $X$ be a topological space. A class $\{G_i\}$ of open subsets of $X$ is said to be an open cover of $X$ if each point in $X$ belongs to at least one $G_i$, that is, if $\bigcup_i G_i = X$.

A subclass of an open cover which is itself an open cover is called a subcover.

A topological space $X$ is said to be compact if every open cover in it has a finite subcover.

Example 1.3.11. Any finite topological space is compact. Also, $[0, 1]$ is a compact subspace of $\mathbb{R}$ whereas $\mathbb{R}$ is not compact as the open cover $G = \{(n, n+2) | n \in \mathbb{Z}\}$ of it has no finite subcover.

Lemma 1.3.12. [71] Any closed subspace of a compact space is compact.

Lemma 1.3.13. [71] Any continuous image of a compact space is compact.

Lemma 1.3.14. [71] Every compact subspace of a Hausdorff space is closed.