Chapter 3

Total Graph of a Module with respect to Singular Submodule

In this chapter we introduce and investigate the total graph \( T(\Gamma(M)) \) of a module \( M \) with respect to singular submodule \( Z(M) \). This chapter has two sections. The first section contains the preliminaries needed for the subsequent section and the second section contains our main results.

Throughout this chapter \( M \) denotes a unitary module over a commutative ring \( R \) with unity and \( T(\Gamma(M)) \) denotes the total graph of \( M \) with respect to singular submodule \( Z(M) \). We investigate some properties of the total graph \( T(\Gamma(M)) \) and its induced subgraphs \( Z(\Gamma(M)) \) and \( Z(\Gamma(M)) \). In some aspects, we have noticed some sort of finiteness.
3.1 Preliminaries

Let $M$ be a unitary module over a commutative ring $R$ with unity. Let $Z(M)$ be the singular submodule of $M$ and $Z(M) = M - Z(M)$.

We begin with the following definition.

**Definition 3.1.1.** Let $E(R)$ be the set of all essential right ideals of the ring $R$. Also, if $I$ is a right ideal of $R$ and $r \in R$, then we use $r^{-1}I$ to denote the right ideal \( \{x \in R | rx \in I\} \).

**Proposition 3.1.1.** [37] The following hold:

1. $R \in E(R)$.
2. If $I \leq J \leq R$ and $I \in E(R)$, then $J \in E(R)$.
3. If $I, J \in E(R)$, then $I \cap J \in E(R)$.
4. If $I \in E(R)$ and $r \in R$, then $r^{-1}I \in E(R)$.

**Proof.** (1) is obvious and (2) to (3) follow directly from lemmas 1.1.1, 1.1.2 and 1.1.3.

**Definition 3.1.2.** Let $Z(M)$ be the set of those $x \in M$ for which the right ideal \( \{r \in R | xr = 0\} \) belongs to $E(R)$, i.e., $Z(M) = \{x \in M | xI = 0 \text{ for some } I \in E(R)\}$. Then $Z(M)$ is a submodule of $M$, called the singular submodule of $M$.

**Lemma 3.1.1.** Let $M$ be a module over a commutative ring $R$. Then $Z(M)$ is a submodule of $M$. 

Theorem 3.1.1. Every torsion group is a singular Z-module and every torsion-free group is a nonsingular Z-module. Also, Z is a nonsingular module over itself.

Proposition 3.1.2. [37] A module C is singular if and only if there exists a short exact sequence

\[ 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \]

such that f is an essential monomorphism.

Proof. First let us assume that we have such an exact sequence. Given any \( b \in B \), we have a map \( k : R \rightarrow B \) defined by \( k(r) = br \). According to lemma 1.1.3 we have \( k^{-1}(fA) \leq_e R_R \), i.e., the right ideal \( I = \{ r \in R | br \in fA \} \) belongs to \( E(R) \). Now \( bI \leq fA = kerg \); hence \( (gb)I = 0 \) and so \( gb \in Z(C) \). Since \( g \) is an epimorphism, we thus obtain \( Z(C) = C \).

Therefore, \( C \) is singular.

Conversely, let us assume that \( C \) is singular and let us choose a short exact sequence

\[ 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \]

such that \( B \) is free. If \( \{ b_\alpha \} \) is a basis for \( B \), then for each \( \alpha \) we have \( (gb_\alpha)I_\alpha = 0 \) for some \( I_\alpha \in E(R) \); hence \( b_\alpha I_\alpha \leq A \). Since \( I_\alpha \leq_e R_R \) for all \( \alpha \), we get \( b_\alpha I_\alpha \leq_e b_\alpha R \) for all \( \alpha \); hence we have \( \oplus b_\alpha I_\alpha \leq_e \oplus b_\alpha R = B \). Inasmuch as \( \oplus b_\alpha I_\alpha \leq A \), we obtain \( A \leq_e B \), and thus the inclusion map \( A \rightarrow B \) is an essential monomorphism. \( \square \)
Proposition 3.1.3. [37] Let $B$ be nonsingular and let $A \leq B$. Then $\frac{B}{A}$ is singular if and only if $A \leq_e B$.

Proof. If $\frac{B}{A}$ is singular and $x$ is a nonzero element of $B$, then $\exists I = 0$ for some $I \in E(R)$, that is, $xI \leq A$. Inasmuch as $B$ is nonsingular, we have $xI \neq 0$ and $xR \cap A \neq 0$. Therefore $A \leq_e B$.

Conversely, let us assume that $A \leq_e B$. Since $A \leq_e B$, so we have a short exact sequence $0 \to A \to \to B \to \frac{B}{A} \to 0$ such that the inclusion map $A \to B$ is an essential monomorphism. Hence by proposition 3.1.2 we have $\frac{B}{A}$ is singular. \qed

Definition 3.1.4. The total graph of $M$ with respect to $Z(M)$, denoted by $T(\Gamma(M))$ is defined as the (undirected) graph with all elements of $M$ as vertices, and for distinct $x, y \in M$, the vertices $x$ and $y$ are adjacent, written $x \text{ adj } y$ if and only if $x + y \in Z(M)$.

Definition 3.1.5. $Z(\Gamma(M))$ is the (induced) subgraph of $T(\Gamma(M))$ with vertex set $Z(M)$ and $\overline{Z}(\Gamma(M))$ is the (induced) subgraph of $T(\Gamma(M))$ with vertex set $\overline{Z}(M)$.

Example 3.1.2. Let $M = \mathbb{Z}_4$ denotes the module of integers modulo 4 and $R = \mathbb{Z}_8$ be the ring of integers modulo 8. Then the essential ideals of $R$ are $I = \{0, 2, 4, 6\}$ and $R$ itself. We have $Z(M) = \{0, 2\}$ and therefore $\overline{Z}(M) = \{1, 3\}$.

Let us now observe the graph $T(\Gamma(M))$ and its induced subgraphs $Z(\Gamma(M))$ and $\overline{Z}(\Gamma(M))$.

It is very easy to conclude that $Z(\Gamma(M))$ is complete and also disjoint from $\overline{Z}(\Gamma(M))$.\
3.2 Main Results

In this section we present our main results.

We start this section with monomorphic character of module which depicts the corresponding graphical character. We observe that the monomorphic character of module carries the graphical character.

**Lemma 3.2.1.** Let \( f : M_1 \rightarrow M_2 \) be a module monomorphism. If \( x \text{ adj } y \) then \( f(x) \text{ adj } f(y) \), for all \( x, y \in M_1 \).

**Proof.** Let \( x \text{ adj } y \). Then there exists an essential ideal \( I \) of \( R \) such that \( (x + y)I = 0 \). Then it is easy see that \( (f(x) + f(y))I = 0 \). This completes the proof. \( \square \)

**Theorem 3.2.1.** Let \( f : M_1 \rightarrow M_2 \) be a module monomorphism. If \( T(\Gamma(M_1)) \) is a complete graph, then so is \( T(\Gamma(f(M_1))) \).

**Proof.** Suppose that \( T(\Gamma(M_1)) \) is a complete graph. To show that \( T(\Gamma(f(M_1))) \) is also a complete graph. For this, we assume \( y_1, y_2 \in f(M_1) \). Then there exists elements \( x_1 \) and
$x_2$ in $M_1$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$ respectively. Since $T(\Gamma(M_1))$ is complete, so we have $x_1 \text{ adj } x_2$. Then by Lemma 3.2.1 we get, $y_1 \text{ adj } y_2$. Thus $T(\Gamma(f(M_1)))$ is also a complete graph. □

**Theorem 3.2.2.** Let $f : M_1 \rightarrow M_2$ be a module isomorphism. Then $f$ is also an isomorphism from $T(\Gamma(M_1))$ onto $T(\Gamma(M_2))$.

**Proof.** We need only to show that adjacency relation is preserved. For this, we assume that $x \text{ adj } y$ for all $x, y \in M_1$. Then there exists an essential ideal $I$ of $R$ such that $(x+y)I = 0$. It can be easily obtained that $f(x) \text{ adj } f(y)$. Hence the result. □

**Theorem 3.2.3.** For any $x, y \in Z(M)$, $x \text{ adj } y$ if and only if every element of $x + Z(M)$ is adjacent to every element of $y + Z(M)$.

**Proof.** Let $a = x + z_1 \in x + Z(M)$ and $b = y + z_2 \in y + Z(M)$. If $x \text{ adj } y$, then $x + y \in Z(M)$. This gives $((a - z_1) + (b - z_2)) \in Z(M)$, that is, $(a + b) - (z_1 + z_2) \in Z(M)$. Since $Z(M)$ is a submodule of $M$, so $a + b \in Z(M)$ which implies $a \text{ adj } b$. Conversely, if $a \text{ adj } b$ then $a + b \in Z(M)$, that is, $(x + z_1) + (y + z_2) \in Z(M)$. Therefore $x + y \in Z(M)$. Hence $x \text{ adj } y$. □

**Theorem 3.2.4.** The following hold:

1. $Z(\Gamma(M))$ is a complete (induced) subgraph of $T(\Gamma(M))$ and $Z(\Gamma(M))$ is disjoint from $Z(\Gamma(M))$.

2. If $N$ is a submodule of $M$, then $T(\Gamma(N))$ is the (induced) subgraph of $T(\Gamma(M))$. 
Proof. (1) Since by definition \(Z(\Gamma(M))\) is the (induced) subgraph of \(T(\Gamma(M))\) with vertex set \(Z(M)\) so we need only to proof that \(Z(\Gamma(M))\) is complete.

For that let \(x, y \in Z(M)\) be any two distinct vertices of \(Z(\Gamma(M))\). As \(Z(M)\) is a submodule of \(M\), we have \(x + y \in Z(M)\). Thus, \(x \ adj y \) in \(Z(\Gamma(M))\). Therefore, \(Z(\Gamma(M))\) is complete.

Also, by definition the vertex sets \(Z(M)\) and \(\overline{Z}(M)\) are disjoint. Hence, \(Z(\Gamma(M))\) is disjoint from \(\overline{Z}(\Gamma(M))\).

(2) It is clear from the definitions.

\(\square\)

**Theorem 3.2.5.** The following hold:

(1) Assume that \(G\) is an induced subgraph of \(\overline{Z}(\Gamma(M))\) and let \(x\) and \(y\) be two distinct vertices of \(G\) that are connected by a path in \(G\). Then there exists a path in \(G\) of length 2 between \(x\) and \(y\). In particular, if \(\overline{Z}(\Gamma(M))\) is connected, then \(\text{diam}(\overline{Z}(\Gamma(M))) \leq 2\).

(2) Let \(x\) and \(y\) be distinct elements of \(\overline{Z}(\Gamma(M))\) that are connected by a path. If \(x + y \notin Z(M)\), then \(x - (-x) - y\) and \(x - (-y) - y\) are paths of length 2 between \(x\) and \(y\) in \(\overline{Z}(\Gamma(M))\).

Proof. (1) It is enough to show that if \(x_1, x_2, x_3,\) and \(x_4\) are distinct vertices of \(G\) and there is a path \(x_1 - x_2 - x_3 - x_4\) from \(x_1\) to \(x_4\), then \(x_1\) and \(x_4\) are adjacent. So \(x_1 + x_2, x_2 + x_3, x_3 + x_4 \in Z(M)\) gives \(x_1 + x_4 = (x_1 + x_2) - (x_2 + x_3) + (x_3 + x_4) \in Z(M)\), since \(Z(M)\) is a submodule of \(M\). Thus \(x_1 \ adj x_4\). So, if \(\overline{Z}(\Gamma(M))\) is connected, then \(\text{diam}(\overline{Z}(\Gamma(M))) \leq 2\).
(2) Since \( x + y \in \mathbb{Z}(\Gamma(M)) \) and so \( x + y \notin Z(M) \), there exists \( z \in \mathbb{Z}(\Gamma(M)) \) such that \( x - z - y \) is a path of length 2 by part (1) above. Thus \( x, z + y \in Z(M) \), and hence \( x - y = (x + z) - (z + y) \in Z(M) \). Also, since \( x + y \notin Z(M) \), we must have \( x \neq -x \) and \( y \neq -x \). Thus \( x - (-x) - y \) and \( x - (-y) - y \) are paths of length 2 between \( x \) and \( y \) in \( \mathbb{Z}(\Gamma(M)) \).

\[ \square \]

**Theorem 3.2.6.** The following statements are equivalent.

(1) \( \mathbb{Z}(\Gamma(M)) \) is connected.

(2) Either \( x + y \in Z(M) \) or \( x - y \in Z(M) \) for all \( x, y \in \mathbb{Z}(M) \).

(3) Either \( x + y \in Z(M) \) or \( x + 2y \in Z(M) \) for all \( x, y \in \mathbb{Z}(M) \).

In particular, either \( 2x \in Z(M) \) or \( 3x \in Z(M) \) (but not both) for all \( x \in \mathbb{Z}(M) \).

**Proof.** (1) \( \Rightarrow \) (2)

Let \( x, y \in \mathbb{Z}(M) \) be such that \( x + y \notin Z(M) \). If \( x = y \), then \( x, y \in \mathbb{Z}(M) \). Otherwise, \( x - (-y) - y \) is a path from \( x \) and \( y \) by Theorem 3.2.5(2), and hence \( x - y \in Z(M) \).

(2) \( \Rightarrow \) (3)

Let \( x, y \in \mathbb{Z}(M) \), and suppose that \( x + y \notin Z(M) \). By assumption, since \( (x + y) - y = x \notin Z(M) \), we conclude that \( x + 2y = (x + y) + y \in Z(M) \). In particular, if \( x \notin \mathbb{Z}(M) \), then either \( 2x \in Z(M) \) or \( 3x \in Z(M) \). Both \( 2x \) and \( 3x \) cannot be in \( Z(M) \) since then \( x = 3x - 2x \in Z(M) \), a contradiction.
(3) ⇒ (1)

Let \( x, y \in \mathbb{Z}(M) \) be distinct elements of \( M \) such that \( x + y \notin \mathbb{Z}(M) \). By hypothesis, since \( x + 2y \in \mathbb{Z}(M) \), we get \( 2y \notin \mathbb{Z}(M) \). Thus \( 3y \in \mathbb{Z}(M) \) by hypothesis. Since \( x + y \notin \mathbb{Z}(M) \) and \( 3y \in \mathbb{Z}(M) \), we conclude that \( x \neq 2y \), and hence \( x - 2y - y \) is a path from \( x \) to \( y \) in \( \mathbb{Z}(M) \).

\( \square \)

**Example 3.2.1.** Let \( R = \mathbb{Z}_4 \) denote the ring of integers modulo 4 and \( M = \mathbb{Z}_8 \) be the ring of integers modulo 8. Then \( M \) is an \( R \)-module with the usual operations, and \( \mathbb{Z}(M) = \{0, 2, 4, 6\} \). Thus \( \mathbb{Z}(M) = \{1, 3, 5, 7\} \). By Theorem 3.2.6, we conclude that \( \mathbb{Z}(\Gamma(M)) \) is connected which can be observed from the following graph.

![Graph of Z(\Gamma(M))](image)

**Fig 3.2.1** The induced subgraph \( \mathbb{Z}(\Gamma(M)) \)

**Theorem 3.2.7.** Let \( |\mathbb{Z}(M)| = \alpha \) and \( |M/\mathbb{Z}(M)| = \beta \).

1. If \( 2 \in \mathbb{Z}(R) \) then \( \mathbb{Z}(\Gamma(M)) \) is the union \( \beta - 1 \) disjoint \( K^{\alpha} \)'s.

2. If \( 2 \notin \mathbb{Z}(R) \) then \( \mathbb{Z}(\Gamma(M)) \) is the union of \( (\beta - 1)/2 \) disjoint \( K^{\alpha, \alpha} \)'s.
Proof. (1) It is obvious that \( x + Z(M) \subseteq \bar{Z}(M) \) for every \( x \notin Z(M) \). Let \( x_1, x_2 \in x + Z(M) \), where \( x_1, x_2 \in Z(M) \). Since \( Z(M) \) is a submodule of \( M \), so \( x_1 + x_2 = 2x_1 + x_2 \in Z(M) \). Thus the coset \( x + Z(M) \) is a complete subgraph of \( \bar{Z}(M) \). Again any two distinct cosets form disjoint subgraphs of \( \bar{Z}(M) \). If not, suppose \( x + x_1 \) is adjacent to \( y + x_2 \) for some \( x, y \in \bar{Z}(M) \) and \( x_1, x_2 \in Z(M) \) then \( x + y = (x + y) - 2y \in Z(M) \) since \( Z(M) \) is submodule of \( M \) and \( 2y \in Z(M) \). From this we get \( x + Z(M) = y + Z(M) \), a contradiction. Hence \( \bar{Z}(\Gamma(M)) \) is a union of \( \beta - 1 \) disjoint (induced) subgraphs \( x + Z(M) \), each of which is a \( K^\alpha \), where \( \alpha = |Z(M)| = |x + Z(M)| \).

(2) Let \( x \in \bar{Z}(M) \) and \( 2 \notin Z(R) \). Then no two distinct elements of \( x + Z(M) \) are adjacent, because, if \( x + x_1 \) is adjacent to \( x + x_2 \), \( x_1, x_2 \in Z(M) \); \( 2x \in Z(M) \). This implies that for some essential ideal \( I \) of \( R \) we have \( 2xI = 0 \). Now, we have for every non-zero ideal \( K \) of \( R \), \( I \cap K \neq 0 \), i.e. there exists a non-zero \( x \in R \) with \( x \in I \cap K \). From this we get \( x + x = 2x \in I \) and \( 2x \in K \). But \( 2 \notin Z(R) \), therefore \( 2x \neq 0 \). Thus \( 2x \) is a non-zero element with \( 2x \in 2I \cap K \) leading onto \( 2I \) is an essential ideal of \( R \). This will imply that \( x \in Z(M) \), as \( x(2I) = 0 \), which is a contradiction. Also, since \( 2x \notin Z(M) \), two cosets \( x + Z(M) \) and \( -x + Z(M) \) are disjoint. Moreover, it is easy to observe that every element of \( x + Z(M) \) is adjacent to every element of \( -x + Z(M) \). Thus \( (x + Z(M)) \cup (-x + Z(M)) \) is a complete bipartite (induced) subgraph of \( \bar{Z}(\Gamma(M)) \). Again, if \( x + x_1 \) is adjacent to \( y + x_2 \) for some \( x, y \in \bar{Z}(M) \) and \( x_1, x_2 \in Z(M) \), then \( x + y \in Z(M) - 0 \), and so \( x + Z(M) = -y + Z(M) \). Hence \( \bar{Z}(\Gamma) \) is the union of \( (\beta - 1)/2 \) disjoint (induced) subgraphs \( x + Z(M) = (-y + Z(M)) \), each of which is a \( K^{\alpha, \alpha} \), where \( \alpha = |Z(M)| = |x + Z(M)| \). \( \square \)

**Theorem 3.2.8.** Let \( M - Z(M) \neq \emptyset \).
(1) If \( Z(\Gamma(M)) \) is complete then either \( |M/Z(M)| = 2 \) or \( |M/Z(M)| = |M| = 3 \).

(2) If \( Z(\Gamma(M)) \) is connected then either \( |M/Z(M)| = 2 \) or \( |M/Z(M)| = 3 \).

(3) If \( Z(\Gamma(M)) \) (and hence \( Z(\Gamma(M)) \) and \( T(\Gamma(M)) \)) is totally disconnected then either \( Z(M) = 0 \) and \( 2 \in Z(R) \).

Proof. Suppose that \( |M/Z(M)| = \beta \) and \( |Z(M)| = \alpha \).

(1) First we assume \( Z(\Gamma(M)) \) is complete. This implies that \( Z(\Gamma(M)) \) is a single \( K^\alpha \) or \( K^{1,1} \), by Theorem 3.2.6. If \( 2 \in Z(R) \), then \( \beta - 1 = 1 \), that is, \( \beta = 2 \) and thus \( |M/Z(M)| = 2 \).

Again, if \( 2 \notin Z(R) \) then \( \alpha = 1 \) and \( (\beta - 1)/2 = 1 \). Hence \( Z(M) = 0 \) and \( \beta = 3 \); thus \( 3 = \beta = |M/Z(M)| = |M| \).

(2) Suppose that \( Z(\Gamma(M)) \) is connected. This implies that \( Z(\Gamma(M)) \) is a single \( K^\alpha \) or \( K^{\alpha,\alpha} \), by Theorem 3.2.6. If \( 2 \in Z(R) \), then \( \beta - 1 = 1 \), that is, \( \beta = 2 \) and thus \( |M/Z(M)| = 2 \).

Again, if \( 2 \notin Z(R) \) then \( (\beta - 1)/2 = 1 \), that is, \( \beta = 3 \) and thus \( |M/Z(M)| = 3 \).

(3) \( Z(\Gamma(M)) \) is totally disconnected if and only if it is a disjoint union of \( K^{1,1} \)'s. Thus by Theorem 3.2.6 we have \( |Z(M)| = 1 \) and \( |M/Z(M)| = 1 \), and hence the result. \( \Box \)

**Theorem 3.2.9.** Let \( x \) be a vertex of the graph \( T(\Gamma(M)) \). Then

\[
\text{deg}(x) = \begin{cases} 
|Z(M)| - 1, & \text{if } 2 \in Z(R) \text{ and } x \in Z(M) \\
|Z(M)|, & \text{otherwise.}
\end{cases}
\]

Proof. If \( x_i \in Z(M) \), the vertex \( x \in M \) is adjacent to vertices \( x_i - x \). Then \( \text{deg}(x) = |Z(M)| - 1 \) if and only if \( x = x_i - x \) for some \( x_i \in Z(M) \), i.e., if and only if \( 2x \in Z(M) \).
If \(2x \notin Z(M)\), then \(deg(x) = |Z(M)|\). If \(2 \in Z(R)\), then \(2x \in Z(M)\) for all \(x \in M\), thus \(deg(x) = |Z(M)| - 1\), i.e., all vertices of the graph \(T(\Gamma(M))\) are of degree \(|Z(M)| - 1\).

Again, if \(2 \notin Z(R)\), then two cases arise.

Case-1: If \(x \in Z(M)\), then \(deg(x) = |Z(M)| - 1\).

Case-2: If \(x \notin Z(M)\), then \(deg(x) = |Z(M)|\).

It follows that \(deg(x) = \begin{cases} |Z(M)| - 1, & \text{if } 2 \in Z(R) \text{ and } x \in Z(M) \\ |Z(M)|, & \text{otherwise.} \end{cases}\)

**Theorem 3.2.10.** Let \(M_1\) and \(M_2\) be two finite modules over a finite ring \(R\). Then the following hold:

1. If \(T(\Gamma(M_1))\) is a Hamiltonian graph, then so is \(T(\Gamma(M_1 \times M_2))\).
2. If \(\Gamma(M_1)\) is a Hamiltonian graph, then so is \(\Gamma(M_1 \times M_2))\).

**Proof.** (1) Let \(M_1 = \{m_1, m_2, \ldots, m_s\}\) and \(M_2 = \{m'_1, m'_2, \ldots, m'_t\}\) be such that the sequence \(m_1, m_2, \ldots, m_s\) is a Hamiltonian cycle. Then \(m_1 + m_s \in Z(M_1)\). Thus we get the Hamiltonian cycle in \(T(\Gamma(M_1 \times M_2))\) as

\((m_1, m'_1), (m_2, m'_1), \ldots, (m_s, m'_1), (m_1, m'_2), \ldots, (m_s, m'_2), \ldots, (m_1, m'_t), \ldots, (m_s, m'_t)\).

(2) Suppose that \(\Gamma(M_1) = \{m_1, m_2, \ldots, m_s\}\) and \(\Gamma(M_2) = \{m'_1, m'_2, \ldots, m'_t\}\). The above Hamiltonian cycle is also a Hamiltonian cycle for \(\Gamma(M_1 \times M_2)\).

**Theorem 3.2.11.** Let \(M = M_1 \times M_2\) be finite module. Then \(\kappa(T(\Gamma(M))) \geq |M_1| + |M_2| - 4\).

**Proof.** Let \((x, y)\) and \((x', y')\) be two distinct elements of \(M\). If \(x \neq x', y' \neq \pm y, \lambda \notin \{y, -y, y', -y'\}\), then consider the paths \((x, y), (-x, \lambda), (-x', -\lambda), (-x', y')\) for \(\lambda \in R_2\).
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If \( \eta \in R_1 \) and \((x, y) \neq (\eta, -y) \) and \((x', y') \neq (-\eta, -y') \), then consider the paths \((x, y), (\eta, -y), (-\eta, -y'), (x' , y')\).

If \((x, y) \neq (\eta, -y) \) and \((x', y') = (-\eta, -y') \), then consider the paths \((x, y), (\eta, -y), (x', y')\).

If \((x, y) = (\eta, -y) \) and \((x', y') \neq (-\eta, -y') \), then consider the paths \((x, y), (-\eta, -y'), (x', y')\).

If \((x, y) = (\eta, -y) \) and \((x', y') = (-\eta, -y') \) for some \( \eta \), then consider the paths \((x, y), (x', y')\) and \((x, y), (\eta, -y), (x', y')\) for some \( \eta \neq x \). So there are at least \(|R_1| + |R_2| - 4\) disjoint paths from \((x, y)\) to \((x', y')\).

Let \( x \neq x', y' \neq y \) and \( y' = -y \). Then the paths \((x, y), (-x, \lambda), (-x', -\lambda), (x', -y)\) for \( \lambda \in R_2 - \{\pm b\} \) and the paths \((x, y), (\eta, -y), (-\eta, y), (x', -y)\) for \( \eta \in R_1 - \{-x, x'\} \) are \(|R_1| + |R_2| - 4\) disjoint paths. Let \( x \neq x', y' = y \). Consider the paths \((x, y), (-x, \lambda), (-x', -\lambda), (x', -y)\) for \( \lambda \in R_2 - \{y, -y\} \) and the paths \((x, y), (\eta, -y), (x', y)\) for \( \eta \in R_1 - \{x, x'\} \). If \( x = x' \), since \((x, y)\) and \((x', y')\) are distinct, then \( y \neq y' \) and the proof is the same as the case \( x \neq x' \) and \( y = y' \). \( \Box \)