Chapter 4

KIEFER’S LAW OF THE ITERATED LOGARITHM FOR VECTOR OF EXTREMES

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4.1 Introduction

In 1943, Gnedenko presented a rigorous foundation for the extreme value theory and provided necessary and sufficient conditions for the weak convergence of the extreme order statistics. Barndorff-Nielsen (1963) obtained a proof of the sufficiency of a condition for almost sure stability of the maximal order statistic. Barndorff-Nielsen (1963) established the following strong law, for $M_n = \max(X_1, X_2, \ldots, X_n)$, where $\{X_n\}$ is a sequence of i.i.d. r.v.’s with a common d.f. F.

1The part of findings of this chapter has been communicated as a research article, entitle ”Kiefer’s law of the iterated logarithm for vector of extremes “, to Journal of Extremes.
Theorem 4.1. (Strong law for \( \{M_n\} \)) Let \( r(F)=\infty \) and let \( b_n \) be a solution of
\[
 b_n = \inf\{x; n(1-F(x)) \leq 1\} . \quad \text{Then} \quad \frac{M_n}{b_n} \to 1 \quad \text{a.s. if and only if} \quad \sum_n (1-F(kb_n)) < \infty \quad \text{for all} \quad k > 1 .
\]

Note 4.1. Unlike in the case of SLLN’s for partial sums, existence of \( EX_1 \) is neither necessary, nor sufficient, for the above strong law.

If \( \{X_n\} \) is a sequence of i.i.d. uniform \((0,1)\) r.v.’s, then Barndorff-Nielsen (1961) established that

\[
 \limsup \frac{n(1-M_n)}{\ln \ln n} = 1 \quad \text{a.s} \quad (4.1)
\]

and

\[
 \liminf \frac{n(1-M_n)}{\ln \ln n} = 0 \quad \text{a.s} \quad (4.2)
\]

The purpose of the l.i.l. is to obtain sharp a.s. bounds for the growth of random sequences. The relation (4.1) gives precise a.s. lower bound for \( M_n \) as 
\[
 (1 - \frac{(1+\epsilon)\ln \ln n}{n}), \quad \text{for} \quad \epsilon > 0 .
\]

However, the result in (4.2) is not sharp, as one can show that \( \liminf \frac{n(1-M_n)}{\theta_n} = 0 \quad \text{a.s., for any} \quad \theta_n \to \infty . \) This is a consequence of the fact that \( n(1-M_n) \) converges to a unit exponential distribution.

In case of other continuous distributions, the above l.i.l. can be obtained by
transformation. For unit exponential, standard normal and Frechet with parameter $\alpha$, we respectively get

$$\lim \ (M_n - \ln n + \ln 3n) = \infty \quad \text{a.s.}$$

$$\lim \ \sqrt{2\ln n} (M_n - \sqrt{2\ln n}) - \ln 3n = \infty \quad \text{a.s.}$$

$$\lim \ \frac{M_n (\ln \ln n)^{1/2}}{n^{1/2}} = \infty \quad \text{a.s.}$$

It may be observed in all the three examples, the limit superior of $M_n$, normalized, is $\infty$. Pickands (1967) considered the problem of obtaining finite limit superior of $\{M_n\}$, properly normalized, which was later generalized by de Haan and Hordijk (1972). Suppose that the derivative of $F$, say $F'$, exists. Define $h(x) = \frac{1-F(x)}{F'(x)}$ and $g(x)=h(x)\ln \ln(\frac{1}{1-F(x)})$. Note that the behavior of $h(x)$ or $h'(x)$ for $x$ large gives von Mises (1936) conditions for weak convergence of $\{M_n\}$, properly normalized.

Let $(b_n)$ be a solution of $n(1-F(b_n)) \simeq 1$. de Haan and Hordijk (1972) proved the following results

**Theorem 4.2.** Suppose that $\lim \frac{g(x)}{x} = c$, $c \geq 0$ as $n \to \infty$. Then $\lim \inf \frac{M_n}{b_n} = 1$ and $\lim \sup \frac{M_n}{b_n} = e^c$ a.s.

**Theorem 4.3.** If $\lim g'(x) = 0$ as $n \to \infty$, then $\lim \inf \frac{M_n-b_n}{h(b_n)\ln \ln n} = 0$ and $\lim \sup \frac{M_n-b_n}{h(b_n)\ln \ln n} = 1$ a.s.

Denote a sequence of i.i.d. uniform $(0,1)$ r.v., by $\{X_n\}$ and by $\{Y_n\}$ a sequence of i.i.d. r.v. with some common continuous d.f. $F$. Suppose that both $\{X_n\}$
and \( \{Y_n\} \) are defined over a common probability space \((\Omega, \mathcal{F}, P)\). Let the right extremity of \( F \) be denoted by \( r(F) \). Let the \( j^{th} \) maxima of \((X_1, X_2, \ldots, X_n)\) and \((Y_1, Y_2, \ldots, Y_n)\), \( n \geq 1, 1 \leq j \leq n \), respectively be denoted by \( M_{j,n} \) and \( M'^*_{j,n} \). Let \( c \) and \( k \) (integer), with or without a suffix, denote positive constants. Kiefer (1971), established the l.i.l. for extreme and intermediate order statistics, assuming that \( F \) is uniform \((0,1)\). The result for extremes is given below. For other distributions, one can obtain the l.i.l. by transformation. Peter Hall (1979b) extended Kiefer’s results for a class of distributions with exponentially fast right tail.

**Theorem A** (Kiefer (1971))

\[
\limsup \frac{\ln(1 - M_{r,n}) + \ln n}{\ln \ln n} = 0 \quad \text{a.s.} \quad \text{and} \quad \liminf \frac{\ln(1 - M_{r,n}) + \ln n}{\ln \ln n} = -\frac{1}{r} \quad \text{a.s.}
\]

Since \( X \) is uniform \((0,1)\), implies that \((1-X)\) is also uniform \((0,1)\). By identifying \((1-M_{r,n})\) as the \( r^{th} \) lower extreme \( m_{r,n} \), the theorem follows from Result 1.6.

The above theorem can be equivalently stated as under, with power normalization.

**Theorem B**

\[
\limsup (n(1 - M_{r,n}))^{\frac{1}{(1+\alpha)}} = 1 \quad \text{a.s.} \\
\liminf (n(1 - M_{r,n}))^{\frac{1}{(1+\alpha)}} = e^{-\frac{1}{\alpha}} \quad \text{a.s.}
\]

**Note 4.2.** In the case of the l.i.l. for partial sums \( \{S_n\} \) of symmetric stable r.v.’s, Chover (1967), used power normalization and established that

\[
\limsup |n^{-\frac{1}{\pi}} S_n|^{\frac{1}{\pi+\alpha}} = e^{\frac{1}{\pi}} \quad \text{a.s.,}
\]
where $0 < \alpha < 2$, is the index of the stable distribution. Results with such a normalization have frequently appeared in literature.

As mentioned earlier, the l.i.l. result for other distributions can be obtained through transformation. Peter Hall (1979b) has extended Kiefer’s l.i.l. to a large class of distributions, which include, exponential, normal, gamma, Gumbel and so on. In fact, all d.f.’s F with the right tail $1 - F(y) = \exp(-y^\gamma L(y))$, where $\gamma$ is a real constant and $L(.)$ is a slowly varying (s.v.) function, are members of this class.

Peter Hall (1979b), defines $U(y) = -\ln(1 - F(y))$ and introduces $V(.)$ as its inverse function and obtains the following result from Kiefer (1971). Under a smoothness assumption in terms of $V(.)$, as

$$\lim_{y \to \infty} \frac{V(y(1 + a(y))) - V(y)}{a(y)v(y)} = \frac{1}{\gamma},$$

(4.3)

where $\gamma > 0$ is some constant and $a(.)$ is a real valued function such that $a(y) \to 0$ as $y \to \infty$, the l.i.l. is obtained.

**Theorem C** (Peter Hall (1979b))

$$\lim \sup \frac{\gamma \ln n}{\ln \ln n} \left( \frac{M_{r,n}^*}{V(\ln n)} - 1 \right) = \frac{1}{r} \quad a.s.,$$

$$\lim \inf \frac{\gamma \ln n}{\ln \ln n} \left( \frac{M_{r,n}^*}{V(\ln n)} - 1 \right) = 0 \quad a.s.$$

In the case of unit exponential, standard normal and gamma($\alpha$) distributions, $V(\ln n)$ will be respectively, $\log n$, $\sqrt{2 \ln n}$ and $\ln \left( \frac{n}{\sqrt{\alpha}} \right) + (\alpha - 1) \ln \ln n$. Also, when $r=1$, the l.i.l. results coincide with those of de Haan and Hordijk (1972).
As can be noted, distributions with regularly varying right tail are not considered in Hall (1979b). In Vasudeva and Savitha (1992), the l.i.l. is established for \( \{M_{r,n}^*\} \) (the right tail is regularly varying) when \( F \) belongs to domain of attraction of Frechet/Weibull law. For a r.v. with distribution function, \( 1-F(y) = y^{-\alpha}L(y) \), \( \alpha > 0 \) and \( L(.) \) a s.v. function, one can have the following result.

**Theorem D** Let \( 1-F(y) = y^{-\alpha}L(y), \alpha > 0 \) and \( L(.) \) a.s. function. Then

\[
\limsup \frac{\alpha(\ln M_{r,n}^* - \ln b_n)}{\ln n} = \frac{1}{r} \quad \text{a.s.} \quad \text{or} \quad \limsup \left( \frac{M_{r,n}^*}{b_n} \right)^{\alpha_{\ln n}} = e^{\frac{1}{r}} \quad \text{a.s.}
\]

and

\[
\liminf \frac{\alpha(\ln M_{r,n}^* - \ln b_n)}{\ln n} = 0 \quad \text{a.s.} \quad \text{or} \quad \liminf \left( \frac{M_{r,n}^*}{b_n} \right)^{\alpha_{\ln n}} = 1 \quad \text{a.s.,}
\]

where \( b_n \) is a solution of \( n(1-F(b_n)) = 1 \).

**Note 4.3.** Let \( U^*(y) = 1-F(y), y \to o \) and \( V^*(.) \) denote its inverse function. Note that \( V^*(z) = z^{-\frac{1}{r}}l(\frac{1}{z}), 0 < z < 1 \), where \( l(.) \) is a s.v. function. From the relation \( M_{r,n} = F(M_{r,n}^*) \) and from Theorem A, the above theorem follows.

**Remark 4.1.** For a sequence \( \{Z_n\} \) of random vector in \( R_k \), a point \( \Theta \in R_k \) is said to be an a.s. limit point if for any given \( \epsilon > 0 \), \( P(||Z_n - \Theta|| < \epsilon \text{ i.o.}) = 1 \).

In the next section, we give the limit set of extremes, when the underlying distribution is uniform distribution and in Section 4.3, we extent the same to the vector of extremes. In Section 4.4, the corresponding limit sets are obtained when (i) the d.f. \( F \) has exponentially fast right tail and (ii) the d.f. \( F \) has regularly varying right tail. The l.i.l. for spacing between the extremes order statistics in the cases of uniform, exponentially fast and regularly fast right tail is given in Section 4.5. In the last section we discuss some boundary crossing properties of Theorem B.
4.2 Almost sure limit set under Kiefer’s law

The following theorem obtains the limit set of extremes in the case of uniform population, under Kiefer’s Li.1.

**Theorem 4.4.** The set of all a.s., limit points of \((n(1 - M_{r,n}))^{\frac{1}{\ln n}}\) coincides with \([e^{-\frac{1}{4}}, 1] \).

**Proof:** Without loss of generality, we establish the result for \(r=3\). Let \(\eta_n = (n(1 - M_{3,n}))^{\frac{1}{\ln n}}\). It is sufficient to show that each point \(\theta \in [e^{-\frac{1}{4}}, 1]\) is an a.s. limit point of \(\{\eta_n\}\), in view of Theorem B. Equivalently, we show that for each point \(e^{-\theta'}, \theta' \in [0, \frac{1}{3}]\), there exist a subsequence \((n_k)\) such that for any given \(\epsilon > 0\),

\[
P(\eta_{n_k} - e^{-\theta'} < -\epsilon \text{ i.o.}) = 0, \tag{4.4}
\]
\[
P(\eta_{n_k} - e^{-\theta'} < \epsilon \text{ i.o.}) = 1. \tag{4.5}
\]

We have for any given \(\epsilon > 0\), there exist a \(\epsilon' > 0\), such that

\[
P(\eta_{n_k} - e^{-\theta'} < -\epsilon) = P\left(\left(n_k(1 - M_{3,n_k}) \right)^{\frac{1}{\ln n_k}} < e^{-\theta' - \epsilon'}\right) = P\left(M_{3,n_k} > 1 - \frac{(\ln n_k)^{-\theta' - \epsilon'}}{n_k}\right)
\]
\[
= 1 - \left(1 - \frac{1}{n_k(\ln n_k)^{-\theta' - \epsilon'}}\right)^{n_k} + n_k \left(1 - \frac{1}{n_k(\ln n_k)^{-\theta' - \epsilon'}}\right)^{n_k - 1}
\]
\[
+ \frac{n_k(n_k - 1)}{2} \left(1 - \frac{1}{n_k(\ln n_k)^{-\theta' - \epsilon'}}\right)^{n_k - 2} \tag{4.6}
\]

Note that,

\[
\left(1 - \frac{1}{n_k(\ln n_k)^{\theta' + \epsilon'}}\right)^{n_k} = \left[1 - \frac{1}{n_k(\ln n_k)^{\theta' + \epsilon'}}\right]^{n_k(\ln n_k)^{\theta' + \epsilon'}}
\]
Since,
\[
(1 - \frac{1}{n_k \ln(n_k)^{\theta'+\epsilon'}})^{n_k} \to e^{-1}, \quad n_k \to \infty,
\]
for \( n_k \) large, one gets for some \( c > 0 \),
\[
(1 - \frac{1}{n_k \ln(n_k)^{\theta'+\epsilon'}})^{n_k} \geq e^{-\frac{c}{\ln(n_k)^{\theta'+\epsilon'}}}.
\]

By similar arguments, for \( n_k \) large, one can show that
\[
P\left( \frac{1}{\ln(n_k)} < e^{-\left(\theta'+\epsilon'\right)} \right) \leq 1 - e^{-\frac{c}{\ln(n_k)^{\theta'+\epsilon'}}} \left(1 + \frac{1}{\ln(n_k)^{\theta'+\epsilon'}} + \frac{1}{2\ln(n_k)^{2(\theta'+\epsilon')}}\right)
\]

Expanding \( e^{-\frac{c}{\ln(n_k)^{\theta'+\epsilon'}}} \), as a Taylor’s series, up to four terms, and simplifying, one can show that for \( n_k \) large
\[
P\left( \frac{1}{\ln(n_k)} < e^{-\left(\theta'+\epsilon'\right)} \right) \leq \frac{c_1}{\ln(n_k)^{3(\theta'+\epsilon')}}
\]
for some \( c_1 > 0 \). Taking \( n_k = \left[e^{\left(\frac{1}{3\theta'}\right)}\right] \), one can find a \( k_0 \) such that for all \( k \geq k_0 \),
\[
P\left( \frac{1}{\ln(n_k)} < e^{-\left(\theta'+\epsilon'\right)} \right) \leq \frac{c_1}{k^{1+\epsilon'}}
\]
for some \( \epsilon'' > 0 \). By Borel-Cantelli Lemma, (4.4) is established. Define \( M_{3,n_k}^* \) as the third maximum of \( (X_{n_k-1+1}, X_{n_k-1+2}, X_{n_k-1+3}, \ldots, X_{n_k}) \) and \( \eta_{n_k}^* = (n_k(1 - M_{3,n_k}^*)) \). \( M_{3,n_k}^* < M_{3,n_k} \) implies that \( \eta_{n_k}^* > \eta_{n_k} \) or
\[
(\eta_{n_k} - e^{-\theta'} < e) \supset (\eta_{n_k}^* - e^{-\theta'} < e)
\] (4.8)
One can show that,

\[
P(\eta^*_n - e^{-\theta'} < e) = P \left( \left( n_k (1 - M_{3,n_k}^*) \right)^{-\frac{1}{\ln n_k}} < e^{-\theta' + \epsilon} \right) = P \left( M_{3,n_k}^* > 1 - \frac{(\ln n_k)}{n_k} \right)
\]

\[
= 1 - \left[ (1 - \frac{(\ln n_k)}{n_k})^{n_k-n_{k-1}} 
+ (n_k - n_{k-1}) \left( \frac{(\ln n_k)}{n_k} \right)^{n_k-n_{k-1}-1} 
+ \frac{(n_k - n_{k-1})(n_k - n_{k-1} - 1)}{2} \left( \frac{(\ln n_k)}{n_k} \right)^2 \left( 1 - \frac{(\ln n_k)}{n_k} \right)^{n_k-n_{k-1}-2} \right]
\]

for \( n_k = \left[ e^{(\frac{1}{\ln n} \theta')} \right] \), defined above, note that \( \frac{n_{k-1}}{n_k} \rightarrow 0 \) as \( k \rightarrow \infty \). Proceeding on the lines similar to the one used in establishing (4.4), one can find \( c_2 > 0 \) and a \( k_1 > 0 \) such that for all \( k \geq k_1 \).

\[
P \left( \left( n_k (1 - M_{3,n_k}^*) \right)^{-\frac{1}{\ln n_k}} < e^{-\theta' - \epsilon} \right) \geq \frac{c_2}{k^{1/2}}.
\]

Consequently,

\[
\sum_k P \left( \left( n_k (1 - M_{3,n_k}^*) \right)^{-\frac{1}{\ln n_k}} < e^{-\theta' - \epsilon} \right) = \infty.
\]

From the fact that \( \{M_{3,n_k}^*\} \) are mutually independent by Borel-Cantelli Lemma, one gets \( P(\eta^*_n - e^{-\theta'} < \epsilon \ i.o.) = 1 \). By (4.8), the second part of theorem is completed.

### 4.3 Limit set of vector of extremes

Let \( \{X_n\} \) be i.i.d. uniform \((0,1)\) r.v.’s. Recall that \( M_{j,n} \) denotes the \( j^{th} \) maximum of \((X_1, X_2, \ldots, X_n)\), \( j \geq 1 \). Define \( T_{j,n} = \frac{-\ln \left(1 - M_{j,n}\right) - \ln n}{\ln \ln n}, j=1,2 \) and
$T_n = (T_{1,n}, T_{2,n})$. From Theorem A, observe that $T_{j,n} \in [0, \frac{1}{3}]$ a.s., $j=1,2$. Note that $M_{1,n} \geq M_{2,n} \Leftrightarrow T_{1,n} \geq T_{2,n}$. One can trivially see that $T_n \in A$ a.s., where $A = \{(x,y); 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{2}, y \leq x\}$, which is the quadrilateral, OABC (Figure 4.1).

![Image of quadrilateral OABC](image)

**Figure 4.1:** Quadrilateral OABC, under limit set of extremes $T_n = (T_{1,n}, T_{2,n})$

Interestingly, the points $(x,y) \in A$ with $x + y > 1$ fail to be limit points, as proved in the Theorem below. The a.s. limit set of $\{T_n\}$ turns out to be a triangle, see, Figure 4.2.

**Theorem 4.5.** The set of all almost sure limit points of $\{T_n\}$ is given by the triangle described by

$$L_2 = \{(x,y); 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{2}, y \leq x, x+y \leq 1\}$$

**Proof:** The proof of Theorem 4.5, follows in two parts. First we establish that the points $(x,y) \in A$ with $x + y > 1$ fail to be limit points. Next we show all the points in $L_2$ are the a.s. limit points of $\{T_n\}$. To prove the first part, it is sufficient to show that, for $\epsilon > 0$, but sufficiently small,

$$P(T_{1,n} > \theta_1^* - \epsilon, T_{2,n} > \theta_2^* - \epsilon \ i.o.) = 0 \quad \forall \ (\theta_1^*, \theta_2^*) \in S,$$

(4.9)
where, $S = \{(x, y) : \frac{1}{2} \leq x < 1, 0 \leq y \leq \frac{1}{2}, x \geq y, x + y > 1\}$. Let,

$$A_n = (T_{1,n} > \theta_1^* - \epsilon, T_{2,n} > \theta_2^* - \epsilon) = (M_{1,n} > 1 - \frac{1}{n(\ln n)^{\frac{1}{2^2 - \epsilon}}}, M_{2,n} > 1 - \frac{1}{n(\ln n)^{\frac{1}{2^2 - \epsilon}}}).$$

For $n_k = [e^k]$, define

$$B_k = \left\{(M_{1,n} > 1 - \frac{1}{n_k(\ln n_k)^{\frac{1}{2^2 - \epsilon}}}, M_{2,n} > 1 - \frac{1}{n_k(\ln n_k)^{\frac{1}{2^2 - \epsilon}}}), \text{for atleast one } n_k \leq n \leq n_{k+1}\right\}$$

and

$$C_k = (M_{1,n_{k+1}} > 1 - \frac{1}{n_k(\ln n_k)^{\frac{1}{2^2 - \epsilon}}}, M_{2,n_{k+1}} > 1 - \frac{1}{n_k(\ln n_k)^{\frac{1}{2^2 - \epsilon}}}).$$

Note that $(A_n \text{ i.o.}) \subseteq (B_k \text{ i.o.}) = (C_k \text{ i.o.})$, which implies that

$$P(A_n \text{ i.o.}) \leq P(C_k \text{ i.o.}). \quad (4.10)$$
Taking $\alpha_{i,n_k} = 1 - \frac{1}{n_k(\ln n_k)^{\theta_i-\epsilon}}$, $i=1,2$, one can show that

$$P(C_k) = P(M_{1,n_k+1} > \alpha_{1,n_k}, M_{2,n_k+1} > \alpha_{2,n_k}) = \int_{\alpha_{1,n_k}}^{1} \int_{\alpha_{2,n_k}}^{1} n_k+1(n_k+1-1)z_{n_k+1}^{-2}dz_2dz_1$$

$$= 1 - \left(1 - \frac{1}{n_k(\ln n_k)^{\theta_1-\epsilon}}\right)^{n_k+1} - n_k+1\left(1 - \frac{1}{n_k(\ln n_k)^{\theta_2-\epsilon}}\right)^{n_k+1-1}\frac{1}{n_k(\ln n_k)^{\theta_2-\epsilon}}$$

Expanding the right side expression as Taylor’s series, one gets for some constants $c_1$, $c_2$ and $c > 0$,

$$P(C_k) = 1 - \left(1 - \frac{n_k+1}{n_k(\ln n_k)^{\theta_1-\epsilon}} + \frac{c_1}{(\ln n_k)^{2(\theta_1-\epsilon)}}\right) - \frac{n_k+1}{n_k(\ln n_k)^{\theta_1-\epsilon}}\left(1 - \frac{n_k+1}{n_k(\ln n_k)^{\theta_2-\epsilon}} + \frac{c_2}{(\ln n_k)^{2(\theta_2-\epsilon)}}\right) \leq \frac{c}{(\ln n_k)^{(\theta_1+\theta_2-2\epsilon)}}$$

By choosing $\epsilon$ sufficiently small, using the fact that $\theta_1^* + \theta_2^* > 1$, one can make $\theta^* = \theta_1^* + \theta_2^* - 2\epsilon > 1$. Hence, $\sum_{k=0}^{\infty} \frac{1}{k^{\theta^*}} < \infty$. By Borel-Cantelli Lemma, we have $P(C_k \ i.o.) = 0$. From (4.10), the first part of proof is complete.

To establish the second part, it is sufficient to show that, every point $(\theta_1, \theta_2) \in L_2$ is an a.s. limit point of $\{T_n\}$, where $L_2 = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{2}, x \geq y, x + y \leq 1\}$. For a point $(\theta_1, \theta_2) \in L_2$ with $\theta_1 + \theta_2 < 1$, define $m_k = \left[e^{\frac{\theta_1}{m_k}}\right]$ and note that $\theta_1 + \theta_2 < 1$ implies $\frac{m_{k+1}}{m_k} \to \infty$ as $k \to \infty$. One can see that $(\theta_1, \theta_2)$ is a limit point of $(T_{1,n}, T_{2,n})$, if for $\epsilon > 0$, $\epsilon' > 0$ with $\epsilon' > \epsilon$,

$$P(T_{1,m_k} > \theta_1 - \epsilon, T_{2,m_k} > \theta_2 - \epsilon \ i.o.) = 1 \quad (4.11)$$

$$P(T_{1,m_k} > \theta_1 - \epsilon, T_{2,m_k} > \theta_2 + \epsilon' \ i.o.) = 0 \quad (4.12)$$

$$P(T_{1,m_k} > \theta_1 + \epsilon', T_{2,m_k} > \theta_2 - \epsilon \ i.o.) = 0 \quad (4.13)$$

Define $M_{i,m_k} = \max(X_{m_k-1}, X_{m_k-2}, \ldots, X_{m_k})$, $M_{i,m_k} = 2nd \max(X_{m_k-1}, X_{m_k-2}, \ldots, X_{m_k})$, $T_{1,m_k} = \frac{-\ln(1-M_{1,m_k})}{\ln m_k}$ and $T_{2,m_k} = \frac{-\ln(1-M_{2,m_k})}{\ln m_k}$. Since $M_{i,m_k} \geq M_{i,m_k}$, $i=1,2$. 

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The relation

\[ P(T_{1,m_k} > \theta_1 - \epsilon, T_{2,m_k} > \theta_2 - \epsilon \text{ i.o.) } > P(T^*_{1,m_k} > \theta_1 - \epsilon, T^*_{2,m_k} > \theta_2 - \epsilon \text{ i.o.)} \]

(4.14)

holds obviously. Observe that \( \frac{m_{k}-m_{k-1}}{m_k} \to 1 \), as \( k \to \infty \). Let \( u_{i,k} = 1 - \frac{1}{m_k(\ln m_k)^{\theta_i - \epsilon}} \), \( i=1,2 \). Then for some \( c_2 > 0 \) and \( k \geq k_0 \), proceeding as above, we have

\[ P(T^*_{1,m_k} > \theta_1 - \epsilon, T^*_{2,m_k} > \theta_2 - \epsilon) = P(M^0_{1,m_k} > u_{1,k}, M^0_{2,m_k} > u_{2,k}) \]

\[ = \int_{u_{1,k}}^{1} \int_{u_{2,k}}^{z_1} (m_k - m_{k-1})(m_k - m_{k-1} - 1)z_2 m_{k-1}^{m_{k-1}-2}dz_2dz_1 \]

\[ = 1 - \left( 1 - \frac{1}{m_k(\ln m_k)^{\theta_1 - \epsilon}} \right)^{m_k-m_{k-1}} \]

\[ = \frac{c_2}{(\ln m_k)^{\theta_1+\theta_2-2\epsilon}}. \]

Substituting for \( m_k \), one gets for \( k \geq k_0 \),

\[ P(T^*_{1,m_k} > \theta_1 - \epsilon, T^*_{2,m_k} > \theta_2 - \epsilon) \geq \frac{c_2}{(\ln m_k)^{\theta_1+\theta_2-2\epsilon}} \]

\[ = \frac{c_2}{k^{1-\epsilon}}, \text{ where } \delta = \frac{2\epsilon}{\theta_1 + \theta_2}. \]

From the fact that \( \sum_{k=1}^{\infty} \frac{1}{k^{1-\epsilon}} = \infty \), by the independence of sequence \( (T^*_{1,m_k}, T^*_{2,m_k}) \), Borel-Cantelli Lemma implies that

\[ P(T^*_{1,m_k} > \theta_1 - \epsilon, T^*_{2,m_k} > \theta_2 - \epsilon \text{ i.o.) } = 1. \]

Now (4.11) is immediate from (4.14).
To establish (4.12), for \( k \) large and some \( c_3 > 0 \), observe that

\[
P(T_{1,m_k} > \theta_1 - \epsilon, T_{2,m_k} > \theta_2 + \epsilon') = P(M_{1,m_k} > 1 - \frac{1}{m_k(\ln m_k)^{\epsilon_1 - \epsilon}}, M_{2,m_k} > 1 - \frac{1}{m_k(\ln m_k)^{\epsilon_2 + \epsilon}})
\]

\[
= \int_{1}^{1 - \frac{1}{m_k(\ln m_k)^{\epsilon_1 - \epsilon}}} \int_{1}^{z_1} \frac{1}{m_k(m_k - 1)z_2^{m_k-2}} dz_2 dz_1
\]

\[
= 1 - (1 - \frac{1}{m_k(\ln m_k)^{\epsilon_1 - \epsilon}}) - m_k(1 - \frac{1}{m_k(\ln m_k)^{\epsilon_2 + \epsilon}})^{m_k-1} \frac{1}{m_k(\ln m_k)^{\epsilon_1 - \epsilon}}
\]

\[
\leq \frac{c_3}{(\ln m_k)^{\epsilon_1 + \epsilon_2 + \epsilon' - \epsilon}}.
\]

Since \( \epsilon' > \epsilon > 0 \), taking \( \epsilon^* = \epsilon' - \epsilon \), by substituting \( m_k = \left[ e^{\frac{1}{1+\epsilon}} \right] \), one gets,

\[
P(T_{1,m_k} > \theta_1 - \epsilon, T_{2,m_k} > \theta_2 + \epsilon') \leq \frac{c_3}{k^{1+\epsilon}}.
\]

From the fact that \( \sum_{k=1}^{\infty} \frac{1}{k^{1+\epsilon}} < \infty \), Borel-Cantelli Lemma implies (4.12).

Similarly, for \( n_k \) large and \( c_4 > 0 \)

\[
P(T_{1,m_k} > \theta_1 + \epsilon', T_{2,m_k} > \theta_2 - \epsilon) = P(M_{1,m_k} > 1 - \frac{1}{m_k(\ln m_k)^{\epsilon_1 + \epsilon'}}, M_{2,m_k} > 1 - \frac{1}{m_k(\ln m_k)^{\epsilon_2 - \epsilon}})
\]

\[
= \int_{1}^{1 - \frac{1}{m_k(\ln m_k)^{\epsilon_1 + \epsilon'}}} \int_{1}^{z_1} \frac{1}{m_k(m_k - 1)z_2^{m_k-2}} dz_2 dz_1
\]

\[
\leq \frac{c_4}{(\ln m_k)^{\epsilon_1 + \epsilon_2 + \epsilon' - \epsilon}}.
\]

Hence, for \( k \) large, \( P(T_{1,m_k} > \theta_1 + \epsilon', T_{2,m_k} > \theta_2 - \epsilon) \leq \frac{c_4}{k^{1+\epsilon}} \). Since \( \sum_{k=1}^{\infty} \frac{1}{k^{1+\epsilon}} < \infty \), Borel-Cantelli Lemma implies (4.13). Points \( (\theta_1, \theta_2) \) with \( \theta_1 + \theta_2 = 1 \), being boundary points of \( L_2 \), will be limit points of \( \{T_n\} \).

**Note 4.4.** The above result can be extended to the vector of \( k \) extremes (\( k \leq n \)). If \( T_{j,m} = -\frac{\ln(1-M_{j,n})-\ln n}{\ln n} \), \( j = 1, 2, 3, \ldots, k \) and if \( T_n = (T_{1,n}, T_{2,n}, \ldots, T_{k,n}) \). We
have the following theorem.

**Theorem 4.6.** The set of all a.s. limit points of \( T_n = (T_{1,n}, T_{2,n}, \ldots, T_{k,n}) \) is given by,

\[
L_k = \left\{ (x_1, x_2, \ldots, x_k) : 0 \leq x_j \leq \frac{1}{j}; j = 1, 2, \ldots, k; \sum_{j=1}^{k} x_j \leq 1; x_1 \geq x_2 \geq \ldots \geq x_k \right\}
\]

**Proof:** The proof follows on the lines similar to the previous theorem and hence is omitted.

**Note 4.5.** In the particular case when \( k = 3 \), Figures 4.3 and 4.4 demonstrate the a.s. limit points of \( T_n = (T_{1,n}, T_{2,n}, T_{3,n}) \) from two different directions.

![Figure 4.3: almost sure limit points of \( T_n = (T_{1,n}, T_{2,n}, T_{3,n}) \)](image)

**Remark 4.2.** It is interesting to note that the limit set of \( (T_{2,n}, T_{3,n}) \) is given by

\[
\left\{ (y, z) : 0 \leq y \leq \frac{1}{2}, 0 \leq z \leq \frac{1}{3}, z \leq y, 2y + z \leq 1 \right\},
\]

(see, Figure 4.5) and that of
Figure 4.4: almost sure limit points of $T_n = (T_{1,n}, T_{2,n}, T_{3,n})$

Figure 4.5: almost sure limit points of $T_n = (T_{2,n}, T_{3,n})$


(T_{1,n}, T_{3,n}) \text{ is given by } \left\{ (x, z) : 0 \leq x \leq 1, 0 \leq z \leq \frac{1}{3}, z \leq x, x + 2z \leq 1 \right\} \text{ (see, Figure 4.6).}

![Diagram](image.png)

**Figure 4.6:** almost sure limit points of $T_n = (T_{1,n}, T_{3,n})$

In the next section, by using transformation, we extend the results of this section to other distributions.

### 4.4 Limit set when either $\left( - \ln(1 - F) \right)$ or $1 - F$ is regularly varying

The a.s. limit set can be obtained in case of other distributions, by transformation. Here are some of the results. Suppose that $- \ln(1 - F(x))$ is regularly varying with index $\gamma$ and let $V(.)$ stand for the inverse. As in Hall (1979b), assume that $V(.)$ satisfies the condition

$$
\lim_{y \to \infty} \frac{V(y(1 + a(y)) - V(y)}{a(y)V(y)} = \frac{1}{\gamma}
$$

(4.15)

where $a(.)$ is a real valued function such that $a(y) \to 0$ as $y \to \infty$. 
Let \( \{X_n\} \) be a sequence of i.i.d. r.v.’s with a common d.f. \( F \) such that 
\(-\ln(1 - F(x))\) is regularly varying with index \( \gamma (> 0) \) and let \( M_{j,n}^* = j^{th} \) maxima 
of \( (X_1, X_2, \ldots, X_n) \), \( j = 1, 2 \). Under the condition (4.15) we have the following 
theorem.

**Theorem 4.7.** The set of all a.s. limit points of

\[
\frac{\gamma \ln n}{\ln \ln n} \left( \frac{M_{1,n}^*}{V(\ln n)} - 1, \frac{M_{2,n}^*}{V(\ln n)} - 1 \right)
\]
is \( L_2 = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{2}, y \leq x, x + y \leq 1 \} \).

**Proof:** Let \( \omega_{i,n} = \frac{\gamma \ln n}{\ln \ln n} \left( \frac{M_{i,n}^*}{V(\ln n)} - 1 \right) \), \( i=1,2 \) and \( \omega_n = (\omega_{1,n}, \omega_{2,n}) \), from Theorem C, 
\( \omega_n \subseteq [0, 1] \times [0, \frac{1}{2}] = A \). Since \( M_{1,n}^* \geq M_{2,n}^* \), trivially any point \((x, y) \in A \) such that 
\( x < y \) is not limit point of \( \{\omega_n\} \). The proof of the theorem has two steps. First 
we show that any point in

\[ S = \{(x, y) : \frac{1}{2} \leq x < 1, 0 \leq y \leq \frac{1}{2}, x \geq y, x + y > 1 \} \]
is not a limit point or equivalently we show that for \( \epsilon > 0 \), but sufficiently small,

\[
P(\omega_{1,n} > \theta_1^* - \epsilon, \omega_{2,n} > \theta_2^* - \epsilon \text{ i.o.}) = 0 \quad \forall (\theta_1^*, \theta_2^*) \in S \quad (4.16)
\]

In the second step we show that each point in \( L_2 \) is an a.s. limit point. Note that

\[
(T_{1,n} > \theta_1^* - \epsilon, T_{2,n} > \theta_2^* - \epsilon \text{ i.o.})
\]

\[
= \left( \frac{-\ln(1 - M_{1,n}) - \ln n}{\ln \ln n} > \theta_1^* - \epsilon, \frac{-\ln(1 - M_{2,n}) - \ln n}{\ln \ln n} > \theta_2^* - \epsilon \text{ i.o.} \right)
\]

\[
\Leftrightarrow \left( -\ln(1 - F(M_{1,n}^*)) > (\theta_1^* - \epsilon) \ln n + \ln n, -\ln(1 - F(M_{2,n}^*)) > (\theta_2^* - \epsilon) \ln n + \ln n \text{ i.o.} \right)
\]

\[
\Leftrightarrow \left( U(M_{1,n}^*) > \ln n(1 + \frac{(\theta_1^* - \epsilon) \ln n}{\ln n}), U(M_{2,n}^*) > \ln n(1 + \frac{(\theta_2^* - \epsilon) \ln n}{\ln n}) \text{ i.o.} \right)
\]

\[
\Leftrightarrow \left( M_{1,n}^* > V(\ln n(1 + \frac{(\theta_1^* - \epsilon) \ln n}{\ln n})), M_{2,n}^* > V(\ln n(1 + \frac{(\theta_2^* - \epsilon) \ln n}{\ln n})) \text{ i.o.} \right)
\]

\[
\Leftrightarrow \left( M_{1,n}^* - V(\ln n) > V(\ln n(1 + \frac{(\theta_1^* - \epsilon) \ln n}{\ln n})), M_{2,n}^* - V(\ln n) > V(\ln n(1 + \frac{(\theta_2^* - \epsilon) \ln n}{\ln n})) - V(\ln n) \text{ i.o.} \right) \quad (4.17)
\]
Using (4.3) in (4.17), one can get

\[
(T_{1,n} > \theta_1^* - \epsilon, T_{2,n} > \theta_2^* - \epsilon \text{ i.o.})
\]
\[
\Leftrightarrow (M_{1,n}^* - V(\ln n) > \frac{V(\ln n) (\theta_1^* - \epsilon) \ln \ln n}{\gamma \ln n}, M_{2,n}^* - V(\ln n) > \frac{V(\ln n) (\theta_2^* - \epsilon) \ln \ln n}{\gamma} \text{ i.o.})
\]
\[
\Leftrightarrow \left( \frac{\gamma \ln n (M_{1,n}^* - \epsilon)}{\ln \ln n} - 1 > (\theta_1^* - \epsilon), \frac{\gamma \ln n (M_{2,n}^* - \epsilon)}{\ln \ln n} - 1 > (\theta_2^* - \epsilon) \text{ i.o.} \right)
\]
\[
= (\omega_{1,n} > \theta_1^* - \epsilon, \omega_{2,n} > \theta_2^* - \epsilon \text{ i.o.}).
\]

Consequently,

\[
(T_{1,n} > \theta_1^* - \epsilon, T_{2,n} > \theta_2^* - \epsilon \text{ i.o.}) \Leftrightarrow (\omega_{1,n} > \theta_1^* - \epsilon, \omega_{2,n} > \theta_2^* - \epsilon \text{ i.o.}). \quad (4.18)
\]

From (4.9),

\[
P(\omega_{1,n} > \theta_1^* - \epsilon, \omega_{2,n} > \theta_2^* - \epsilon \text{ i.o.}) = 0 \quad \forall \ (\theta_1^*, \theta_2^*) \in S.
\]

Next, we show that any point in \(L_2\) is an a.s. limit point of \(\{\omega_n\}\) or equivalently by taking \(m_k = \left[ e^{\frac{1}{\sqrt{\ln \ln n}}} \right] \), we show that for any point \((\theta_1, \theta_2) \in L_2\) and for some \(\epsilon, \epsilon' > 0\) with \(\epsilon' > \epsilon\),

\[
P(\omega_{1,m_k} > \theta_1 - \epsilon, \omega_{2,m_k} > \theta_2 - \epsilon \text{ i.o.}) = 1 \quad (4.19)
\]
\[
P(\omega_{1,m_k} > \theta_1 - \epsilon, \omega_{2,m_k} > \theta_2 + \epsilon' \text{ i.o.}) = 0 \quad (4.20)
\]
\[
P(\omega_{1,m_k} > \theta_1 + \epsilon', \omega_{2,m_k} > \theta_2 - \epsilon \text{ i.o.}) = 0 \quad (4.21)
\]
Proceeding on the lines of argument used in establishing (4.16), one can have

\[
(T_{1,m_k} > \theta_1 - \epsilon, T_{2,m_k} > \theta_2 - \epsilon \ i.o.) \iff (\omega_{1,m_k} > \theta_1 - \epsilon, \omega_{2,m_k} > \theta_2 - \epsilon \ i.o.)
\]

\[
(T_{1,m_k} > \theta_1 - \epsilon, T_{2,m_k} > \theta_2 + \epsilon' \ i.o.) \iff (\omega_{1,m_k} > \theta_1 - \epsilon, \omega_{2,m_k} > \theta_2 + \epsilon' \ i.o.)
\]

\[
(T_{1,m_k} > \theta_1 + \epsilon', T_{2,m_k} > \theta_2 - \epsilon \ i.o.) \iff (\omega_{1,m_k} > \theta_1 + \epsilon', \omega_{2,m_k} > \theta_2 - \epsilon \ i.o.)
\]

From (4.11), (4.12) and (4.13) the proof of second part complete.

Note 4.6. The function $V(.)$ is given as, $V(x) = x^{\frac{1}{2}}l(\frac{1}{x})$ where $l(.)$ is a s.v. function. In particular, for unit exponential and standard normal distributions, $V(\ln n)$ will be $\ln n$ and $\sqrt{2\ln n}$ respectively (See, Hall (1979b)). Consequently,

(i) if $F$ is unit exponential distribution, the a.s. limit set of

\[
\left(\frac{M_{1,n}^* - \ln n}{\ln \ln n}, \frac{M_{2,n}^* - \ln n}{\ln \ln n}\right)
\]

is $L_2 = \left\{(x,y), 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{2}, y \leq x, x + y \leq 1 \right\}$.

(ii) if $F$ is standard normal distribution. the a.s. limit set of

\[
\left(\frac{\sqrt{2\ln n(M_{1,n}^* - \sqrt{2\ln n})}}{\ln \ln n}, \frac{\sqrt{2\ln n(M_{2,n}^* - \sqrt{2\ln n})}}{\ln \ln n}\right)
\]

coincides with $L_2$.

Remark 4.3. Gut (1990) has obtained the set of all a.s. limit points, under the setup of de Haan and Hordijk (1972). Our result in Theorem 4.7, extends the result of Gut (1990).

In Theorem 4.8 below, we extend Theorem 4.5 to a class of d.f.’s with regularly varying right tail (d.f.’s belonging to the domain of attraction of Frechet law, see, Galambos (1978)).
Theorem 4.8. Let $1 - F(y) = y^{-\alpha}L(y)$, where $\alpha > 0$ is a constant and $L(\cdot)$ is a s.v. function. Define $\tau_{i,n} = \frac{\alpha(n^{-\alpha} - \ln b_n)}{\ln n}$, $i=1,2$ and $\tau_n = (\tau_{1,n}, \tau_{2,n})$, where $b_n$ is a solution of $n(1 - F(b_n)) = 1$. Then the a.s. limit set of $\{\tau_n\}$ can be given by

$$L_2 = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{2}, y \leq x, x + y \leq 1\}$$

or equivalently the a.s. limit set of $\xi_n = (\xi_{1,n}, \xi_{2,n})$, where $\xi_{i,n} = \exp(\tau_{i,n})$, $i=1,2$ can be given by

$$L_2^* = \{(x^*, y^*) : 1 \leq x^* \leq e, 1 \leq y^* \leq e^{\frac{1}{2}}, y^* \leq x^*, x^*y^* \leq e\}.$$ 

Proof: We use Result 1.7 in proving Theorem 4.8. The proof of the theorem follows on the lines similar to Theorem 4.7. From Theorem D, note that the a.s. limit points of $\{\tau_n\}$ is contained in $[0,1] \times [0, \frac{1}{2}]$. In what follows, we first show that no point in $S = \{(x,y) : \frac{1}{2} \leq x < 1, 0 \leq y \leq \frac{1}{2}, x \geq y, x + y > 1\}$ is a limit point of $\{\tau_n\}$. Equivalently, for any given $\epsilon_i > 0$, we show that

$$P(\tau_{1,n} > \theta_1^* - \epsilon_1, \tau_{2,n} > \theta_2^* - \epsilon_1 \ i.o.) = 0 \quad \forall (\theta_1^*, \theta_2^*) \in S.$$ 

By considering the notation in Note 4.3, for $\epsilon > 0$, the event,

$$(T_{1,n} > \theta_1^* - \epsilon, T_{2,n} > \theta_2^* - \epsilon \ i.o.)$$

$$\Leftrightarrow \left( - \ln(1 - F(M_{1,n}^*)) > (\theta_1^* - \epsilon) \ln \ln n + \ln n, - \ln(1 - F(M_{2,n}^*)) > (\theta_2^* - \epsilon) \ln \ln n + \ln n \ i.o. \right)$$

$$\Leftrightarrow \left( U^*(M_{1,n}^*) < \exp(-(\theta_1^* - \epsilon) \ln \ln n - \ln n), U^*(M_{2,n}^*) < \exp(-(\theta_2^* - \epsilon) \ln \ln n - \ln n) \ i.o. \right)$$

$$\Leftrightarrow \left( M_{1,n}^* > V^*(\exp(-(\theta_1^* - \epsilon) \ln \ln n - \ln n)), M_{2,n}^* > V^*(\exp(-(\theta_2^* - \epsilon) \ln \ln n - \ln n)) \ i.o. \right).$$
\[ \iff \left( M_{1,n}^* > V^*\left(\frac{(\ln n)^{(\theta_1^*-\epsilon)}_n}{n}\right), M_{2,n}^* > V^*\left(\frac{(\ln n)^{(\theta_2^*-\epsilon)}_n}{n}\right) \right) \text{i.o.} \]

Since \( V^*(z) = z^{-\frac{1}{\alpha}} l\left(\frac{1}{z}\right) \), where \( l(.) \) is a s.v. function.

\[ (T_{1,n} > \theta_1^* - \epsilon, T_{2,n} > \theta_2^* - \epsilon \text{ i.o.}) \]
\[ \iff \left( M_{1,n}^* > (\ln n)^{(\theta_1^*-\epsilon)}_n b_n (\ln n)^{-\sigma_1(\theta_1^*-\epsilon)}_n, M_{2,n}^* > (\ln n)^{(\theta_2^*-\epsilon)}_n b_n (\ln n)^{-\sigma_2(\theta_2^*-\epsilon)}_n \text{ i.o.} \right) \]
\[ \iff \left( \frac{M_{1,n}^*}{b_n} > (\ln n)^{(\theta_1^*-\epsilon)}_n, \frac{M_{2,n}^*}{b_n} > (\ln n)^{(\theta_2^*-\epsilon)}_n \text{ i.o.}, \right) \text{ where } \epsilon_1 = \epsilon + \epsilon' \]
\[ \iff \left( \frac{\alpha (\ln M_{1,n} - \ln b_n)}{\ln \ln n} > \theta_1^* - \epsilon_1, \frac{\alpha (\ln M_{2,n} - \ln b_n)}{\ln \ln n} > \theta_2^* - \epsilon_1 \text{ i.o.} \right). \]

Consequently, for any given \( \epsilon > 0 \) one can get \( \epsilon_1 \), sufficiently small and positive, such that,

\[ (T_{1,n} > \theta_1^* - \epsilon, T_{2,n} > \theta_2^* - \epsilon \text{ i.o.}) \iff (\tau_{1,n} > \theta_1^* - \epsilon_1, \tau_{2,n} > \theta_2^* - \epsilon_1 \text{ i.o.}). \]

From (4.9), one gets

\[ P(\tau_{1,n} > \theta_1^* - \epsilon_1, \tau_{2,n} > \theta_2^* - \epsilon_1 \text{ i.o.}) = 0 \quad \forall \ (\theta_1^*, \theta_2^*) \in S. \]

which completes the first part.
On similar lines, one can show that every point \((\theta_1, \theta_2) \in L_2\) is an a.s. limit point of \(\{\tau_n\}\), or equivalently, for some \(\epsilon, \epsilon', \epsilon_1, \epsilon_1'\) positive and sufficiently small,

\[
(T_{1,m_k} > \theta_1 - \epsilon, T_{2,m_k} > \theta_2 - \epsilon \text{ i.o.}) \iff (\tau_{1,m_k} > \theta_1 - \epsilon_1, \tau_{2,m_k} > \theta_2 - \epsilon_1 \text{ i.o.})
\]

\[
(T_{1,m_k} > \theta_1 - \epsilon, T_{2,m_k} > \theta_2 + \epsilon' \text{ i.o.}) \iff (\tau_{1,m_k} > \theta_1 - \epsilon_1, \tau_{2,m_k} > \theta_2 + \epsilon_1' \text{ i.o.})
\]

\[
(T_{1,m_k} > \theta_1 + \epsilon', T_{2,m_k} > \theta_2 - \epsilon \text{ i.o.}) \iff (\tau_{1,m_k} > \theta_1 + \epsilon_1', \tau_{2,m_k} > \theta_2 - \epsilon_1 \text{ i.o.})
\]

From (4.11), (4.12) and (4.13), one gets,

\[
P(\tau_{1,m_k} > \theta_1 - \epsilon_1, \tau_{2,m_k} > \theta_2 - \epsilon_1 \text{ i.o.}) = 1 \tag{4.22}
\]

\[
P(\tau_{1,m_k} > \theta_1 - \epsilon_1, \tau_{2,m_k} > \theta_2 + \epsilon_1' \text{ i.o.}) = 0 \tag{4.23}
\]

\[
P(\tau_{1,m_k} > \theta_1 + \epsilon_1', \tau_{2,m_k} > \theta_2 - \epsilon_1 \text{ i.o.}) = 0 \tag{4.24}
\]

the proof of second part is complete.

As a consequence of above theorem, one can see that, if \(F\) is Frechet distribution with index \(\alpha\), the a.s. limit set of

\[
((n^{-\frac{1}{\alpha}}M_{1,n}^*)_{\frac{n}{\ln n}}, (n^{-\frac{1}{\alpha}}M_{2,n}^*)_{\frac{n}{\ln n}})
\]

is given by \(L_2^* = \{(x^*, y^*); 1 \leq x^* \leq e, 1 \leq y^* \leq e^{\frac{1}{2}}, y^* \leq x^*, x^*y^* \leq e\}\).

**Remark 4.4.** Considering the distribution with exponentially fast right tail, from Theorem 4.6 one can establish that the set of a.s. limit points of \((\omega_{1,n}, \omega_{2,n}, \omega_{3,n})\), where \(\omega_{i,n} = \frac{\gamma \ln n}{\ln \ln n} \left(\frac{M_{i,n}^*}{V(\ln n)} - 1\right)\), \(i=1,2,3\) can be given by,

\[
L_3 = \{(x, y, z); 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{2}, 0 \leq z \leq \frac{1}{3}, z \leq y \leq x, x + y + z \leq 1\}
\].
Similarly, for distributions with regularly varying right tail, \( L_3 \) can be given as set of a.s. limit points of \((\tau_{1,n}, \tau_{2,n}, \tau_{3,n})\), where \( \tau_{i,n} = \frac{a(n,M_{i,n}^- - \ln b_n)}{\ln \ln n} \), \( i=1,2,3 \).

### 4.5 Law of the iterated logarithm for spacing between the extremes order statistics

The difference between the order statistics \( M_{j,n} - M_{j+1,n} \), \( j \geq 1 \), is called the spacing. From Theorem 4.5, we observe that any point \((x, y) \in L_2\), where \( L_2 = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{2}, y \leq x, x + y \leq 1\}\), is an a.s. limit point of \( \{T_n\} \). i.e., for any given \( \epsilon, \epsilon' > 0 \)

\[
P(T_n \in (x - \epsilon, x + \epsilon) \times (y - \epsilon', y + \epsilon') \ i.o.) = 1,
\]

for all \((x, y) \in L_2\) or there exists a subset \( \Omega_0 \subseteq \Omega \) with \( P(\Omega_0) = 1 \) such that for any \( \omega \in \Omega_0 \), one can find subsequence \( n_k(\omega) \to \infty \), such that

\[
T_{n_k}(\omega) \in (x - \epsilon, x + \epsilon) \times (y - \epsilon', y + \epsilon') \ i.o. \quad (4.25)
\]

For \((x, y) \in L_2\) with \( x > y \), one can choose \( \epsilon \) and \( \epsilon' \), such that, \( x - \epsilon > y + \epsilon' \).

Consequently, \(\frac{1}{(\ln n)^{y+\epsilon'}} > \frac{1}{(\ln n)^{y-\epsilon}} \quad \text{and} \quad \frac{1}{(\ln n)^{y' - \epsilon}} > \frac{1}{(\ln n)^{y'' + \epsilon}}, \) which in turn implies that for \( n \) large,

\[
\frac{1}{(\ln n)^{y+\epsilon'}} - \frac{1}{(\ln n)^{y-\epsilon}} = \frac{1}{(\ln n)^{y+\epsilon'}}(1 - \frac{1}{(\ln n)^{y-\epsilon'}}) \approx \frac{1}{(\ln n)^{y+\epsilon'}} \quad (4.26)
\]

\[
\frac{1}{(\ln n)^{y' - \epsilon}} - \frac{1}{(\ln n)^{y'' + \epsilon}} = \frac{1}{(\ln n)^{y' - \epsilon}}(1 - \frac{1}{(\ln n)^{y'' + \epsilon}}) \approx \frac{1}{(\ln n)^{y' - \epsilon}} \quad (4.27)
\]
With \( n_k(\omega) = n_k \), from (4.25), we have,

\[
(x - \epsilon < \frac{-\ln (1 - M_{1,n_k}) - \ln n_k}{\ln \ln n_k} < x + \epsilon, \ y - \epsilon' < \frac{-\ln (1 - M_{2,n_k}) - \ln n_k}{\ln \ln n_k} < y + \epsilon' \text{ i.o.})
\]

\[
\Leftrightarrow \quad (e^{-x+\epsilon} \ln n_k < n_k(1 - M_{1,n_k}) < e^{-(x-\epsilon) \ln n_k}, \quad e^{-(y+\epsilon') \ln n_k} < n_k(1 - M_{2,n_k}) < e^{-(y-\epsilon') \ln n_k} \text{ i.o.})
\]

\[
\Leftrightarrow \quad \left( \frac{1}{(\ln n_k)^{x+\epsilon}} < n_k(1 - M_{1,n_k}) < \frac{1}{(\ln n_k)^{y-\epsilon'}}, \quad \frac{1}{(\ln n_k)^{x-\epsilon}} < n_k(1 - M_{2,n_k}) < \frac{1}{(\ln n_k)^{y+\epsilon'} \text{ i.o.}} \right)
\]

\[
\Leftrightarrow \quad \left( \frac{1}{(\ln n_k)^{y+\epsilon'}} < n_k(M_{1,n_k} - M_{2,n_k}) < \frac{1}{(\ln n_k)^{y-\epsilon'} \text{ i.o.}} \right)
\]

\[
\Rightarrow \quad (e^{-(y+\epsilon')} \ln n_k < n_k(M_{1,n_k} - M_{2,n_k}) < e^{-(y-\epsilon') \ln n_k} \text{ i.o.})
\]

\[
\Rightarrow \quad (e^{-(y+\epsilon')} < \left( n_k(M_{1,n_k} - M_{2,n_k}) \right)^{\frac{1}{\ln \ln n_k}} < e^{-(y-\epsilon') \text{ i.o.}})
\]

Consequently, for any \((x, y) \in L_2\),

\[
(x - \epsilon < \frac{-\ln (1 - M_{1,n_k}) - \ln n_k}{\ln \ln n_k} < x + \epsilon, \ y - \epsilon' < \frac{-\ln (1 - M_{2,n_k}) - \ln n_k}{\ln \ln n_k} < y + \epsilon' \text{ i.o.})
\]

\[
\Rightarrow \quad (e^{-(y+\epsilon')} < \left( n_k(M_{1,n_k} - M_{2,n_k}) \right)^{\frac{1}{\ln \ln n_k}} < e^{-(y-\epsilon') \text{ i.o.}})
\]

or equivalently, for \( y \in [0, \frac{1}{2}] \),

\[
P\left( e^{-(y+\epsilon')} < \left( n_k(M_{1,n_k} - M_{2,n_k}) \right)^{\frac{1}{\ln \ln n_k}} < e^{-(y-\epsilon') \text{ i.o.}} \right) = 1.
\]

Let \( \psi_n = \left( n(M_{1,n} - M_{2,n}) \right)^{-\frac{1}{\ln n_k}} \). Then one can state the following theorem.

**Theorem 4.9.**

\[
\lim \inf \psi_n = e^{-\frac{1}{2}} \text{ a.s. and } \lim \sup \psi_n = 1 \text{ a.s.}
\]

and every point \( \theta \in [e^{-\frac{1}{2}}, 1] \) is an a.s. limit point of \( \psi_n \).
Remark 4.5. Consider $\beta_n = \left( n(M_{2,n} - M_{3,n}) \right)^{\frac{1}{\ln n}}$. Then $\lim \inf \beta_n = e^{-\frac{1}{2}}$ a.s., $\lim \sup \beta_n = 1$ a.s. and any point $\theta \in [e^{-\frac{1}{2}}, 1]$ is an a.s. limit point of $\beta_n$.

Note 4.7. For distributions with exponentially fast right tail the result of spacing follows as a consequence of Theorem 4.7.

Theorem 4.10.

$$\lim \inf \frac{\gamma \ln n (M_{1,n}^* - M_{2,n}^*)}{\ln \ln n} = 0 \text{ a.s. and } \lim \sup \frac{\gamma \ln n (M_{1,n}^* - M_{2,n}^*)}{V(\ln n)} = 1 \text{ a.s.}$$

and every point $x \in [0, 1]$ is an a.s. limit point of $\frac{\gamma \ln n (M_{1,n}^* - M_{2,n}^*)}{\ln \ln n}$.

Remark 4.6. In particular cases,

(i) If $F$ is a unit exponential distribution, then

$$\lim \inf \frac{M_{1,n}^* - M_{2,n}^*}{\ln \ln n} = 0 \text{ a.s., } \lim \sup \frac{M_{1,n}^* - M_{2,n}^*}{\ln \ln n} = 1 \text{ a.s.}$$

and every point $x \in [0, 1]$ is an a.s. limit point.

(ii) If $F$ is a standard normal distribution, then

$$\lim \inf \frac{\sqrt{2 \ln n (M_{1,n}^* - M_{2,n}^*)}}{\ln \ln n} = 0 \text{ a.s., } \lim \sup \frac{\sqrt{2 \ln n (M_{1,n}^* - M_{2,n}^*)}}{\ln \ln n} = 1 \text{ a.s.}$$

and the set of a.s. limit points is $[0, 1]$.

Similarly, when $1 - F(x)$ is regularly varying, by Theorem 4.8, one can obtain the l.i.l. for $(M_{1,n}^* - M_{2,n}^*)$. The detail are omitted.
4.6 Boundary crossing problem associated with Law of the iterated logarithm

Let \( \{\eta_n\} \) be any sequence of r.v.'s and let \((\alpha_n)\) and \((\beta_n)\) be sequences of real constants such that \( \alpha_n \leq \eta_n \leq \beta_n \) a.s.. Then we have

\[
P(\eta_n < \alpha_n \ i.o) = P(\eta_n > \beta_n \ i.o) = 0.
\]

Define

\[
Z_{1,n} = \begin{cases} 
1 & \text{if } \eta_n > \beta_n \\
0 & \text{otherwise}
\end{cases}
\]

\[
Z_{2,n} = \begin{cases} 
1 & \text{if } \eta_n < \alpha_n \\
0 & \text{otherwise}
\end{cases}
\]

and \( N_1 = \sum_{n=0}^{\infty} Z_{1,n} \) and \( N_2 = \sum_{n=0}^{\infty} Z_{2,n} \). When \( \eta_n > \beta_n \), we say that \( \eta_n \) has crossed the upper boundary. Then \( N_1 \) denotes the number of crossing of upper boundary. Similarly, \( N_2 \) denotes the number of crossing of the lower boundary. Note that both \( N_1 \) and \( N_2 \) are proper r.v.'s. Observe that \( N_1 \) (or \( N_2 \)) is an infinite sum of dependent non identically distributed Bernoulli r.v.'s. We discuss the existence of \( EN_1 \) (or \( EN_2 \)) as a measure of efficiency of the bounds. \( EN_i < \infty \) \((=\infty)\) indicates that the respective bound is crude (precise), \( i=1,2 \). We study \( EN_i, i=1,2 \), for the bounds obtained in Theorem B.

From Theorem B, note that for any \( \epsilon > 0 \),

\[
P\left(\left(\frac{n(1 - M_{r,n})}{\ln n}\right)^{\frac{1}{\ln n}} > e^\epsilon \ i.o\right) = 0
\]

and

\[
P\left(\left(\frac{n(1 - M_{r,n})}{\ln n}\right)^{\frac{1}{\ln n}} < e^{-\frac{1+\epsilon}{r}} \ i.o\right) = 0
\]
Define

\[
Z_{1,n} = \begin{cases} 
1 & \text{if } n(1 - M_{r,n}) > (\ln n)^\varepsilon \\
0 & \text{Otherwise}
\end{cases}
\]

and

\[
Z_{2,n} = \begin{cases} 
1 & \text{if } n(1 - M_{r,n}) < \frac{1}{(\ln n)^{1+\varepsilon}} \\
0 & \text{Otherwise}
\end{cases}
\]

\[N_1 = \sum_{n=1}^{\infty} Z_{1,n}, N_2 = \sum_{n=1}^{\infty} Z_{2,n} \text{ and } N = N_1 + N_2.\] Note that \(N\) is a proper r.v. denoting the total number of boundary crossings and that \(EN = EN_1 + EN_2.\)

One can show that \(EN_1 = \sum_n EZ_{1,n},\) where

\[EZ_{1,n} = P(M_{r,n} < (1 - \frac{(\ln n)^\varepsilon}{n}))\]

With no loss of generality, take \(r=2.\) Then

\[P(M_{2,n} < (1 - \frac{(\ln n)^\varepsilon}{n})) \approx e^{-\frac{(\ln n)^\varepsilon}{n}}\]

and hence \(EN_1 = \infty.\) Similarly \(EN_2 = \sum_{n=1}^{\infty} EZ_{2,n},\) where

\[EZ_{2,n} = P(M_{r,n} > 1 - \frac{1}{n(\ln n)^{1+\varepsilon}}) \approx \frac{1}{(\ln n)^{1+\varepsilon}}.\]

Consequently, \(EN_2 = \infty.\) The fact the \(EN_1 = \infty\) and \(EN_2 = \infty\) indicate that both the upper and lower bounds are sharp bounds.