CHAPTER - 5

\( \tau_1 \tau_2^* \) HYPER CONNECTED SPACE

In this chapter, we introduce \( \tau_1 \tau_2^* \) hyper connected space and we prove that X is \( \tau_1 \tau_2^* \) hyper connected space if and only if (X, \( \tau_1 \)) is hyper connected, (X, \( \tau_2 \)) is hyper connected and the intersection of any non empty \( \tau_1 \) open set and any non empty \( \tau_2 \) open set is non empty. Also we introduce the attractive space and we prove that attractive space is \( \tau_1 \tau_2^* \) hyper connected space. At the end of this chapter, we study the properties of image of \( \tau_1 \tau_2^* \) hyper connected space under star continuous function.

5.1 \( \tau_1 \tau_2^* \) hyper connected.

**Definition: 5.1.1**

A bigeneralized topological space (X, \( \tau_1, \tau_2 \)) is said to be \( \tau_1 \tau_2^* \) hyper connected space if every non empty \( \tau_1 \tau_2^* \) open sets is \( \tau_1 \tau_2^* \) dense in X.

Equivalently any non-empty \( \tau_1 \tau_2^* \) open sets intersects.

**Theorem: 5.1.2**

If X is \( \tau_1 \tau_2^* \) hyper connected space then (X, \( \tau_1 \)) is hyper connected.

**Proof:**

Let X be \( \tau_1 \tau_2^* \) hyper connected and let A be \( \tau_1 \) open set in X. Then A is \( \tau_1 \tau_2^* \) open set. Since X be \( \tau_1 \tau_2^* \) hyper connected, \( \tau_1 \tau_2^* \text{ cl}(A) = X \). This implies \( \tau_1 \text{ cl}A \cap \tau_2 \text{ cl}A = X \). Therefore, \( \tau_1 \text{ cl}A = X \). This implies A is dense in (X, \( \tau_1 \)). Hence (X, \( \tau_1 \)) is hyper connected.
Result: 5.1.3

The converse is not true. If $(X, \tau_1)$ is hyper connected then $(X, \tau_1, \tau_2)$ is not $\tau_1 \tau_2^*$ hyper connected. For example,

Let $X = \{a,b,c,d\}$

$\tau_1 = \{\emptyset, X, \{a,b\}, \{a,b,c\}, \{b,c,d\}, \{a,c\}\}$ and

$\tau_2 = \{\emptyset, X, \{a,b\}, \{c\}, \{a,b,c\}, \{a,d\}, \{a,b,d\}, \{a,c,d\}\}$. The $\tau_1 \tau_2^*$ open sets are $\emptyset, X, \{a,b\}, \{a,b,c\}, \{b,c,d\}, \{a,c\}, \{c\}, \{a,d\}, \{a,b,d\}$ and $\{a,c,d\}$. Clearly $(X, \tau_1)$ is hyper connected. But $(X, \tau_1, \tau_2)$ is not $\tau_1 \tau_2^*$ hyper connected space.

Theorem: 5.1.4

If $X$ is $\tau_1 \tau_2^*$ hyper connected space then $(X, \tau_2)$ is hyper connected space.

Proof:

Similar to theorem 5.1.2.

Result: 5.1.5

The converse is not true.

Theorem: 5.1.6

If $X$ is $\tau_1 \tau_2^*$ hyper connected then $(X, \tau_1)$ is hyper connected and $(X, \tau_2)$ is hyper connected.

Proof:

If follows from previous theorem 5.1.2 and theorem 5.1.4.
Result: 5.1.7

The converse is not true. If \((X, \tau_1)\) is hyper connected and \((X, \tau_2)\) is hyper connected then \((X, \tau_1, \tau_2)\) is not \(\tau_1\tau_2^*\) hyper connected. For example:

Let \(X = \{a,b,c,d\}\)

\[ \tau_1 = \{\emptyset, X, \{a,b\}, \{b,c\}, \{a,b,c\}, \{b,d\}, \{b,c,d\}\} \]

\[ \tau_2 = \{\emptyset, X, \{c,d\}, \{a,b,c\}\} \]

The \(\tau_1\tau_2^*\) open sets are \(\emptyset, X, \{a,b\}, \{b,c\}, \{a,b,c\}, \{b,d\}, \{a,b,d\}, \{b,c,d\}\)

Clearly \((X, \tau_1)\) is hyper connected and \((X, \tau_2)\) is hyper connected. But \(\tau_1\tau_2^*\) closure of \(\{a,b\}\) is \(\{a,b\}\). Hence \((X, \tau_1, \tau_2)\) is not \(\tau_1\tau_2^*\) hyper connected.

Now to find a situation where the converse is true.

Theorem: 5.1.8

If \((X, \tau_1)\) is hyper connected, \((X, \tau_2)\) is hyper connected and the intersection of any non empty \(\tau_1\) open set and any non empty \(\tau_2\) open set is non empty then \((X, \tau_1, \tau_2)\) is \(\tau_1\tau_2^*\) hyper connected space.

Proof:

Let \(A\) and \(B\) be two non empty \(\tau_1\tau_2^*\) open sets. Then \(A = \tau_1\) int \(A\) \(\cup\) \(\tau_2\) int \(A\) and \(B = \tau_1\) int \(B\) \(\cup\) \(\tau_2\) int \(B\). Now, \(A\cap B = (\tau_1\) int \(A\) \(\cup\) \(\tau_2\) int \(A\)) \(\cap\) \((\tau_1\) int \(B\) \(\cup\) \(\tau_2\) int \(B\)) Hence, \(A\cap B = (\tau_1\) int \(A\) \(\cap\) \(\tau_1\) int \(B\)) \(\cup\) \((\tau_1\) int \(A\) \(\cap\) \(\tau_2\) int \(B\)) \(\cup\) \((\tau_2\) int \(A\) \(\cap\) \(\tau_1\) int \(B\)) \(\cup\) \((\tau_2\) int \(A\) \(\cap\) \(\tau_2\) int \(B\)) Since \(A\) is non-empty, at least one of the set \(\tau_1\) int \(A\) or \(\tau_2\) int \(A\) is non empty. Also, since \(B\) is non empty, atleast one of the set \(\tau_1\) int \(B\) or \(\tau_2\) int \(B\) is non-empty.
Case (i) Suppose $\tau_1 \text{ int } A$ and $\tau_1 \text{ int } B$ are non-empty. Since $(X, \tau_1)$ is hyper connected,

\[ \tau_1 \text{ int } A \cap \tau_1 \text{ int } B \neq \emptyset \]

Case (ii) Suppose $\tau_1 \text{ int } A$ and $\tau_2 \text{ int } B$ are non empty. By hypothesis the intersection of any non empty $\tau_1$ open set and any non empty $\tau_2$ open set is non empty. Therefore, $\tau_1 \text{ int } A \cap \tau_2 \text{ int } B \neq \emptyset$

Case (iii) Suppose $\tau_2 \text{ int } A$ and $\tau_1 \text{ int } B$ are non empty. By above case (ii)

\[ \tau_2 \text{ int } A \cap \tau_1 \text{ int } B \neq \emptyset \]

Case (iv) Suppose $\tau_2 \text{ int } A$ and $\tau_2 \text{ int } B$ are non empty. Since $(X, \tau_2)$ is hyper connected,

\[ \tau_2 \text{ int } A \cap \tau_2 \text{ int } B \neq \emptyset \]

In each case, $A \cap B \neq \emptyset$. Therefore, $(X, \tau_1, \tau_2)$ is $\tau_1\tau_2^*$ hyper connected space.

**Theorem: 5.1.9**

If $(X, \tau_1, \tau_2)$ is $\tau_1\tau_2^*$ hyper connected space then $(X, \tau_1)$ is hyper connected, $(X, \tau_2)$ is hyper connected and the intersection of any non empty $\tau_1$ open set and any non empty $\tau_2$ open set is non empty.

**Proof:**

Let $(X, \tau_1, \tau_2)$ be $\tau_1\tau_2^*$ hyper connected. By theorem 5.1.6, $(X, \tau_1)$ is hyper connected and $(X, \tau_2)$ is hyper connected. Let $A$ be non empty $\tau_1$ open and $B$ be non empty $\tau_2$ open set.

Then $A$ and $B$ are non empty $\tau_1\tau_2^*$ open sets. Since $X$ is $\tau_1\tau_2^*$ hyper connected, $A \cap B \neq \emptyset$
**Theorem: 5.1.10**

A bigeneralized topological space \((X, \tau_1, \tau_2)\) is \(\tau_1 \tau_2^*\) hyper connected space if and only if \((X, \tau_1)\) is hyper connected, \((X, \tau_2)\) is hyper connected and the intersection of any non empty \(\tau_1\) open set and any non empty \(\tau_2\) open set is non empty.

**Proof:**

It follows from previous theorem 5.1.8 and theorem 5.1.9.

**Theorem: 5.1.11**

Let \(X\) be a \(\tau_1 \tau_2^*\) hyper connected space and \(A\) be a proper \(\tau_1 \tau_2^*\) open set. Then every point \(x \notin A\) is a \(\tau_1 \tau_2^*\) limit point of \(A\).

**Proof:**

Let \(x \notin A\) and \(U\) be a \(\tau_1 \tau_2^*\) open set containing \(x\). Since \(X\) is \(\tau_1 \tau_2^*\) hyper connected and \(A\) is \(\tau_1 \tau_2^*\) open, \(U \cap A \neq \emptyset\). Hence \(x\) is a \(\tau_1 \tau_2^*\) limit point of \(A\).

**Theorem: 5.1.12**

In a \(\tau_1 \tau_2^*\) hyper connected space, there is no proper set which is both \(\tau_1 \tau_2^*\) open and \(\tau_1 \tau_2^*\) closed.

**Proof:**

Let \((X, \tau_1, \tau_2)\) be a \(\tau_1 \tau_2^*\) hyper connected space. Suppose \(A\) be both \(\tau_1 \tau_2^*\) open and \(\tau_1 \tau_2^*\) closed set. Then \(A^c\) is \(\tau_1 \tau_2^*\) open. Also \(A \cap A^c = \emptyset\). Which is contradiction to \(X\) is \(\tau_1 \tau_2^*\) hyper connected space.
5.2 Attractive space

Definition: 5.2.1

Let \((X, \tau_1, \tau_2)\) be a \(\tau_1 \tau_2^*\) hyper connected space. A point \(x \in X\) is called a point of attraction if the intersection of all the non empty \(\tau_1 \tau_2^*\) open set contains \(x\).

The set of all points of attraction is called the set of attraction.

\(X\) is called an attractive space if \(X\) has a point of attraction.

Theorem: 5.2.2

Let \((X, \tau_1, \tau_2)\) be a bigeneralized topological space. If \(X\) is attractive space then it is \(\tau_1 \tau_2^*\) hyper connected.

Proof:

Since \((X, \tau_1, \tau_2)\) is attractive space, then \(X\) has a point of attraction. Let \(x \in X\) be the point of attraction. Then the intersection of all the non-empty \(\tau_1 \tau_2^*\) open set contains \(x\). Therefore, every \(\tau_1 \tau_2^*\) open sets are intersects. Hence \(X\) is \(\tau_1 \tau_2^*\) hyper connected space.

Remark: 5.2.3

The converse is not true. If \(X\) is a \(\tau_1 \tau_2^*\) hyper connected space then \(X\) is not a attractive space.

For example,

Let \(X = \{a,b,c,d\}\)

\(\tau_1 = \{\emptyset, X, \{a,b\}, \{b,d\}, \{a,b,d\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}\}\)

\(\tau_2 = \{\emptyset, X, \{a,c,d\}, \{b,c,d\}\}\)

\(\tau_1 \tau_2^*\) open sets are \(\emptyset, X, \{a, b\}, \{b, d\}, \{a, b, d\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, \{a, c, d\}\). Clearly \(X\) is \(\tau_1 \tau_2^*\) hyper connected space and, \(X\) is not attractive space.
Theorem: 5.2.4

Let $X$ be a $\tau_1 \tau_2$ hyper connected space. If $A$ is $\tau_1 \tau_2$ open then $\tau_1 \tau_2 \text{ Bd} (A) = X - A$

Proof:

Let $A$ be $\tau_1 \tau_2$ open set. Then $X - A$ is $\tau_1 \tau_2$ closed set. That is, $\tau_1 \text{ cl} (X - A) \cap \tau_2 \text{ cl} (X - A) = X - A$ Since $X$ is $\tau_1 \tau_2$ hyper connected, $\tau_1 \tau_2 \text{ cl} (A) = \tau_1 \text{ cl } A \cap \tau_2 \text{ cl } A = X$. Now, $\tau_1 \tau_2 \text{ Bd} (A) = (\tau_1 \text{ cl } A \cap \tau_2 \text{ cl } A) \cap (\tau_1 \text{ cl } (X - A) \cap \tau_2 \text{ cl } (X - A)) = X \cap (X - A) = X - A$

Theorem: 5.2.5

Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \mu_1, \mu_2)$ be onto star continuous. If $X$ is $\tau_1 \tau_2$ hyper connected then $Y$ is $\mu_1 \mu_2$ hyper connected.

Proof:

Let $B_1, B_2$ be two non-empty $\mu_1 \mu_2$ open sets of $Y$. Since $f$ is onto, $f^{-1} (B_1)$ and $f^{-1} (B_2)$ are non empty. Since $f$ is star continuous, $f^{-1} (B_1)$ and $f^{-1} (B_2)$ are $\tau_1 \tau_2$ open sets of $X$. Since $X$ is $\tau_1 \tau_2$ hyper connected, $f^{-1} (B_1) \cap f^{-1} (B_2) \neq \emptyset$ This implies $f^{-1} (B_1 \cap B_2) \neq \emptyset$ Hence $B_1 \cap B_2 \neq \emptyset$ Therefore, $Y$ is $\mu_1 \mu_2$ hyper connected.

Remark: 5.2.6

In the above theorem, $f : X \rightarrow Y$ is onto is necessary. Suppose $f : (X, \tau_1, \tau_2) \rightarrow (Y, \mu_1, \mu_2)$ is star continuous but not onto. If $X$ is $\tau_1 \tau_2$ hyper connected then $Y$ is not $\mu_1 \mu_2$ hyper connected.
For example, Let $X = \{a, b, c\}$ and $Y = \{a, b, c\}$. Take $\tau_1 = \tau_2 = \{\phi, X, \{a, b\}, \{b, c\}\}$ and $\mu_1 = \mu_2 = \{\phi, X, \{a, b\}, \{c\}\}$. Define $f : (X, \tau_1, \tau_2) \rightarrow (Y, \mu_1, \mu_2)$ by $f(a) = a; f(b) = b$ and $f(c) = a$. Clearly $f$ is star continuous but not onto. Also $X$ is $\tau_1 \tau_2$ * hyper connected and $Y$ is not $\mu_1 \mu_2$ * hyperconnected.

**Theorem: 5.2.7**

Let $(X, \tau_1, \tau_2)$ be a bigeneralized topological space and $\tau_2 = \{\phi, X\}$. Then $X$ is $\tau_1 \tau_2$ * hyper connected if and only if $(X, \tau_1)$ is hyper connected.

**Proof:**

If $X$ is $\tau_1 \tau_2$ * hyper connected then by theorem 5.1.2 $(X, \tau_1)$ is hyper connected. Suppose $(X, \tau_1)$ is hyper connected. Since $\tau_2$ open sets are $\phi$ and $X$, the $\tau_1 \tau_2$ * open sets are only the $\tau_1$ open sets. Since any two non empty $\tau_1$ open sets are intersect, any two non empty $\tau_1 \tau_2$ * open sets are intersect. Hence $(X, \tau_1, \tau_2)$ is $\tau_1 \tau_2$ * hyper connected.