Chapter 5

Causes of instability

The main objective of our thesis is to find out the effect of electromagnetic fields on the onset of hydrodynamic instability. We use the term 'hydrodynamic' in a broad sense by including fluids of all types. In this chapter we review the physical causes triggering an instability. For, only if we understand what provokes an instability can we conjure an arrangement of fields that can possibly arrest it. We will use the simplest models to describe an instability so that the key physics is brought forward relegating the constitutive peculiarities of the fluids to the background. Many instabilities are a result of the property of the materials to flow, that is its 'fluidness'. Whether or not they evolve is usually not dependent on their constitutive properties, although we saw an example (in section 4.2.2) that shows up only if viscosity is a function of the rate of strain. We will use the insights gained in this chapter to propose field configurations to control instabilities in the later chapters.
5.1 Buoyancy (Rayleigh-Bénard Convection)

Consider a layer of fluid heated from below. Heating causes the fluid to expand and therefore reduce its density. A presence of a heat source creates a layer of low density, hot fluid below high density, cool fluid. If the temperature gradient in the fluid is small, the density difference too is small and heat diffuses through the bulk of the fluid by conduction alone. However, at higher temperature gradients, the low density fluid at the bottom can overcome viscosity by its buoyancy and rise to the top. As hot fluid rises, the colder fluid replaces it setting up convective currents. We will demonstrate the energy parcels method [9] (section 4.3.1) to derive an upper limit for the temperature gradient that a fluid can sustain without triggering convection. We will assume that the fluid is inviscid and that heating from below creates a temperature gradient $T(z)$, where $z$ is the height.

We will exchange a fluid element at height $z$ with the one at $z + \xi$. Let $v$ be the specific volume\(^1\) of the fluid element. It is a function of pressure, $p$ and entropy per unit mass, $s$. The temperature gradient is not yet destabilizing if a fluid element at $z$ when exchanged with another one at $z + \xi$ returns to its original position. This will happen only if the fluid element at $z$ continues to be heavier than the one at $z + \xi$ even after reaching there. Mathematically, it means that the specific volume $v(p', s')$ at $z + \xi$, that is the volume per unit mass, is greater than $v(p', s)$, the specific volume of the fluid element originally at $z$. We assume that entropy of the fluid element did not change during exchange because the process was adiabatic. Thus the condition for

\(^1\)Volume per unit mass
stability is

$$v(p', s') - v(p', s) = \frac{\partial v}{\partial s} \delta s = \frac{\partial v}{\partial s} \frac{ds}{dz} \xi > 0 \quad (5.1.1)$$

But,

$$\frac{\partial v}{\partial s} = T \frac{\partial v}{\partial T} \frac{1}{c_p} \quad (5.1.2)$$

where $c_p$ is the fluid's specific heat at constant pressure. Since both $T$ and $c_p$ are positive, the condition for stability, using equations (5.1.1) and (5.1.2) is

$$\frac{\partial v}{\partial T} \frac{ds}{dz} > 0 \quad (5.1.3)$$

Since most substance expand on heating, and therefore exhibit convection, $v_{T} > 0$ and the condition for stability is just

$$\frac{ds}{dz} > 0 \quad (5.1.4)$$

Since $s$ is a function of $T$ and $p$, equation (5.1.4) gives,

$$\frac{\partial s}{\partial T} \frac{dT}{dz} > -\frac{\partial s}{\partial p} \frac{dp}{dz} \quad (5.1.5)$$

Specific heat at constant pressure is defined as

$$c_p = \frac{\delta Q}{\delta T} = T \frac{\partial s}{\partial T} \quad (5.1.6)$$

where $\delta Q$ denotes the (inexact) differential of $Q$. Equation (5.1.5) therefore becomes

$$c_p \frac{dT}{T} \frac{dz}{dz} > -\frac{\partial s}{\partial p} \frac{dp}{dz} \quad (5.1.7)$$
Maxwell’s thermodynamic relation gives

\[-\frac{\partial s}{\partial p} = \frac{\partial v}{\partial T}\]  

(5.1.8)

The thermal expansion coefficient \(\alpha\) is defined as

\[\alpha = \frac{1}{v} \frac{\partial v}{\partial T}\]  

(5.1.9)

Using equations (5.1.8) and (5.1.9), the condition for the temperature gradient to be stabilizing is

\[\frac{dT}{dz} > \frac{\alpha v T}{c_p} \frac{dp}{dz}\]  

(5.1.10)

We will now consider a special cases of equation (5.1.10). For a fluid in mechanical equilibrium in a uniform gravitational field, the equation of motion becomes \(\nabla p = \rho g\) or

\[\frac{dp}{dz} = -\rho g = -\frac{g}{v}\]  

(5.1.11)

Equation (5.1.10) therefore becomes

\[\frac{dT}{dz} > -\frac{\alpha T g}{c_p}\]  

(5.1.12)

If the fluid is an ideal gas, \(\alpha T = 1\) and

\[\frac{dT}{dz} > -\frac{g}{c_p}\]  

(5.1.13)

Equation (5.1.13) is Schwarzschild criterion for convective stability of stellar atmosphere. Since \(g\) and \(c_p\) are positive, any positive temperature gradient,
with the top hotter than the bottom, is stable. This conclusion also follows from the stability of a layer of less dense layer on top of a relatively denser bottom. One can user linear analysis to get surprising amount of details [1] for the Rayleigh-Bénard convection. The problem was first analysed by Lord Rayleigh in 1916 [2]. It was studied experimentally in 1900's by H. Bénard [3].

5.2 Stratification (Rayleigh-Taylor Instability)

Since liquids are largely incompressible, an arrangement of a heavy liquid on top of a lighter one is certainly possible. However it comes as no surprise that if such an arrangement is slightly disturbed, the heavy liquid will displace the lighter one. This problem was first analysed by Lord Rayleigh in 1883 [5]. We will demonstrate linear analysis (section 4.3.2) to study this problem. To keep matters simple, we assume the fluids to be inviscid and instead of assuming two fluids of different densities, we will consider, following Lord Rayleigh [5], the case of a stratified fluid. The equations of the fluid are

\[ \rho \frac{du^{(0)}}{dt} = -\nabla p^{(0)} + \rho g \]  \hspace{1cm} (5.2.1)

\[ \nabla \cdot u^{(0)} = 0 \] \hspace{1cm} (5.2.2)

Let the arrangement be disturbed by a small perturbation in pressure \( p^{(1)} \) resulting in a similar perturbation in density \( \rho^{(1)} \) and velocity \( u^{(1)} \). Thus after perturbation, pressure is \( p^{(0)} + p^{(1)} \), density is \( \rho^{(0)} + \rho^{(1)} \) and velocity is \( u^{(0)} + u^{(1)} \). Substituting these quantities in equations (5.2.1) and (5.2.2)
and subtracting the resulting equations from (5.2.1) and (5.2.2) we get

\[ \rho \frac{d\mathbf{u}^{(1)}}{dt} = -\nabla p^{(1)} + \rho^{(1)} \mathbf{g} \]  
\[ \nabla \cdot \mathbf{u}^{(1)} = 0 \]  

(5.2.3) \hspace{4cm} (5.2.4)

We have ignored higher than linear order terms in perturbed quantities after subtracting the equations. Since the density of every particle remains unchanged,

\[ \frac{d(\rho^{(0)} + \rho^{(1)})}{dt} = 0 \]  

(5.2.5)

This equation simplifies to

\[ \frac{\partial \rho^{(1)}}{\partial t} + \mathbf{u}^{(0)} \cdot \nabla \rho^{(1)} + \mathbf{u}^{(1)} \cdot \nabla \rho^{(0)} = 0 \]  

(5.2.6)

If we now assume that the stratified fluid was initially at rest, that is putting \( \mathbf{u}^{(0)} = 0 \), we get

\[ \frac{\partial \rho^{(1)}}{\partial t} + \mathbf{v}^{(1)} \cdot \nabla \rho^{(0)} = 0 \]  

(5.2.7)

Lord Rayleigh’s analysis began with normal mode analysis of equations (5.2.3), (5.2.4) and (5.2.7). Assume normal modes \( \exp(i lx), \exp(i mx) \) and \( \exp(i nt) \), where \( l, m \) and \( n \) are complex numbers. If \( \mathbf{u}^{(1)} = (u^{(1)}, v^{(1)}, w^{(1)}) \), equations
(5.2.3), (5.2.4) and (5.2.7) give,

\[
lp^{(1)} = -n\rho^{(0)}u^{(1)} \quad (5.2.8) \\
m_p^{(1)} = -n\rho^{(0)}v^{(1)} \quad (5.2.9) \\
p_{,z}^{(1)} = -\rho^{(1)}g - in\rho^{(0)}w^{(1)} \quad (5.2.10)
\]

\[
ilu^{(1)} + imv^{(1)} + w^{(1)}_{,z} = 0 \quad (5.2.11)
\]

\[
in\rho^{(1)} + w^{(1)}d\rho^{(0)} \quad (5.2.12)
\]

where we have used the fact that \( \rho^{(0)} \) is a function of \( z \) alone while deriving the last equation. From equations (5.2.8), (5.2.9) and (5.2.11), we get

\[
i(l^2 + m^2)p^{(1)} - n\rho^{(0)}\frac{\partial w^{(1)}}{\partial z} = 0 \quad (5.2.13)
\]

Substituting equation (5.2.13) in (5.2.10), we get

\[
i(l^2 + m^2)(\rho^{(1)}g + in\rho^{(0)}w^{(1)}) + n\rho \frac{\partial}{\partial z} \left( n\rho^{(0)}\frac{\partial w^{(1)}}{\partial z} \right) = 0 \quad (5.2.14)
\]

Eliminating \( \rho' \) between equations (5.2.12) and (5.2.14) we get

\[
\frac{\partial}{\partial z} \left( \rho^{(0)} \frac{\partial w^{(1)}}{\partial z} \right) - (l^2 + m^2) \left( \frac{g}{n^2} \frac{d\rho^{(0)}}{dz} + \rho^{(0)} \right) w^{(1)} = 0 \quad (5.2.15)
\]

Let \( k^2 = l^2 + m^2 \) and assume that the stratification is in the form of two different fluids of uniform density \( \rho_1^{(0)} \) and \( \rho_2^{(0)} \) respectively. In that case, equation (5.2.15) simplifies to

\[
\frac{\partial^2 w^{(1)}}{\partial z^2} - k^2 w^{(1)} = 0 \quad (5.2.16)
\]
the solution of which is \( w^{(1)} = A \exp(kz) + B \exp(-kz) \). Since the perturbations have to vanish away from the interface, \( A \) should be zero for the upper fluid and \( B \) should be zero for the lower fluid. Continuity of velocity implies that the solution is \( B \exp(-kz) \) in upper fluid and \( B \exp(kz) \) in the lower fluid. We get another boundary condition by integrating equation (5.2.14) across the interface.

\[
\left( \frac{\rho^0 \partial w^{(1)}}{\partial z} \right)_2 - \left( \frac{\rho^0 \partial w^{(1)}}{\partial z} \right)_1 - \frac{gk^2}{n^2} (\rho^0_2 - \rho^0_1) = 0 \quad (5.2.17)
\]

Substituting appropriate form for \( w^{(1)} \) in (5.2.17) we get

\[
n^2 = gk \left( \frac{\rho^0_1 - \rho^0_2}{\rho^0_1 + \rho^0_2} \right) \quad (5.2.18)
\]

Equation (5.2.18) is the dispersion relation for the instability of two superposed fluids. If the upper fluid is light, that is \( \rho^0_2 < \rho^0_1 \), \( n^2 > 0 \) and the instability is in form of waves. On the contrary, if the upper fluid is heavy, \( n \) is imaginary and the disturbance grows without bounds, or the arrangement is unstable. We will propose ways of modifying the specific weights (\( \rho g \)) of the fluids by applying non-uniform electric and magnetic fields in chapter 8 and making an otherwise unstable arrangement stable.

### 5.3 Shear (Kelvin-Helmholtz instability)

It is easy to understand that an arrangement of a heavy fluid on top of a lighter one is unstable because even the slightest of perturbations will disturb the arrangement by bringing down the heavy fluid. This is, as we saw in
section 5.2, the cause of Rayleigh-Taylor instability. On the contrary, it is not obvious why a light fluid moving over a heavier one should result in an unstable interface. Further, we know that there will be no instability for small enough relative speeds of the two fluids. We try to understand the reasons for instability by looking at the ways of energy transfer from the faster fluid to the slower one.

We present two analyses, one emphasizing the role of surface tension and the other emphasizing the role of density in the onset of Kelvin-Helmholtz instability (KHI).

To identify the physical parameters of a fluid that affect the onset of KHI most, we focus on the energy exchange involved in the phenomenon as opposed to the detailed dynamics. With our simple approach we do not get such details as the rate of growth of most unstable mode, the precise nature of the random field creating initial waves or the dispersion relation of waves. At the cost of giving up these details, which are also difficult to measure experimentally, we get a deeper insight into the physics of the phenomenon.

1. Flow of wind over still water: Let us consider the most common manifestation of KHI - creation of ripples over the surface of a pond due to breeze. When wind flows over water, random fluctuations in wind velocity result in variation of pressure on water surface. Owing to its surface tension, the water surface is like a stretched membrane. The pressure fluctuations on water surface have an effect similar to gently tapping a membrane, which leads to vibrations on the surface. The waves function as sites for drawing energy from the air. We find out
the velocity of wind needed to excite waves by looking at the energy needed to create them.

In absence of breeze, the surface of a pond is planar. This is because the surface tension tends to pull the water molecules at the top inward, minimizing the area of the surface. Let us now estimate the energy needed to create a wave on the pond’s surface. Let the pond be rectangular, with length $L$ and width $W$. The depth of the pond is not important in this problem as long as it is much greater than the height of the ripple. If the ripple has $m$ crests along the length and if it is uniform across, then the displacement of the surface can be expressed as

$$y_m = a \sin \left( \frac{2\pi mx}{L} \right)$$  \hspace{1cm} (5.3.1)

where $a$ is the amplitude of the wave, assumed to be small. The functional form in (5.3.1) guarantees that the displacement is zero at the end points of the pond. The arc-length $L_m$ of this wave is

$$L_m = \int_0^L \sqrt{1 + \frac{4\pi^2 m^2 a^2}{L^2} \cos^2 \left( \frac{2\pi mx}{L} \right)} \, dx$$  \hspace{1cm} (5.3.2)

The sequence $\{L_m\}$ is monotone increasing. If $A_m = WL_m$ is the surface area of a wave with $m$ crests, $A_m \geq A_{m-1}$ for all $m \geq 1$. If $\gamma$ is the surface tension of water-air interface, then the energy needed to create the wave with $m$ crests is $E_m = \gamma(A_m - A_0)$, where $A_0 = WL$.

The only agency to supply it is the breeze.

The greater the number of crests $m$, smaller is the wavelength $L/m$
and greater is the energy needed to create such a wave. This is the reason why surface tension stabilizes short waves [8] and prevents KHI from being Hadamard unstable even in inviscid fluids.

We assume that the air trapped below the level of crests in the wave transfers its entire energy to water through viscous dissipation. The volume of air trapped below the level of crests is \( aLW \). Therefore, the kinetic energy available to ruffle the water surface is \( \frac{1}{2} \rho_a U_a^2 aLW \), \( \rho_a \) being the density of air and \( U_a \) being its velocity. If we add the energy density due to elevation of water surface \( \frac{1}{2} \rho ga^2 \) then the minimum wind velocity to create a ripple of \( m \) crests can be found from the relation

\[
\frac{1}{2} \rho_a U_a^2 a = \gamma \left( \frac{1}{L} \int_0^L \sqrt{1 + \frac{4\pi^2 m^2 a^2}{L^2} \cos^2 \left( \frac{2\pi mx}{L} \right)} \, dx - 1 \right) + \frac{1}{2} \rho ga^2
\]

(5.3.3)

The integral in equation (5.3.3) can be simplified using the substitution \( \theta = 2\pi mx/L \) to get

\[
\frac{1}{2} \rho_a U_a^2 a = \gamma \left( \frac{1}{2\pi} \int_0^{2\pi} \sqrt{1 + k^2 a^2 \cos^2 \theta} \, d\theta - 1 \right) + \frac{1}{2} \rho ga^2
\]

(5.3.4)

Equation (5.3.4) does not have dimensions of the pond in it. It can be applied to any patch of the pond where capillary waves prevail. The quantity \( ka \) is called steepness of the wave. We have seen above that lesser energy is needed to create waves with lesser number of crests, i.e. greater value of wavelength. We also know that the maximum wave number of capillary waves is \( \sqrt{\frac{\rho g}{\gamma}} \) [10] i.e. 366.899 \( m^{-1} \), where we have used the values \( \rho = 1000 \text{ kg/m}^3 \), \( \gamma = 72.8 \times 10^{-3} \text{ N/m} \) and
\( g = 9.8 \, m/s^2 \). If we further use \( \rho_a = 1.61 \, kg/m^3 \), assume a steepness of 0.1 and evaluate the integral numerically, we get \( U_a = 1.577 \, m/s \), little over Beaufort scale of 1 which is characterized by ripples on surface of water. Our simple analysis that capillary waves are excited by energy transfer between fluids appears to be correct. We also note that higher the density or the surface tension of the fluid in the pond, higher is the wind speed needed to excite waves on its surface.

2. Richardson number: While discussing instability of parallel flows, Drazin and Reid[7] (page 326) write, "The essential mechanism of the instability is the conversion of the available kinetic energy of relative motion of layers of the basic flow into kinetic energy of the disturbance". They go on to estimate the work done in exchanging fluid parcels moving at different speeds. They also assume a stable stratification of the fluid, that is, \( d\rho/dz < 0, \rho \) being the density and \( z \) being measured vertically. They derive a necessary condition for interchange of fluid parcels as

\[
\frac{-gD\rho}{\rho(Du)^2} \leq \frac{1}{4}
\]

(5.3.5)

where the operator \( D \equiv d/dz \). For a given value of \( Du \), a way to delay the onset of instability is to increase \( D\rho \). We will show in chapter 10 that applying an electric or magnetic field gradient effectively achieves the same goal.
5.4 Rotation (Taylor-Couette instability)

Flow driven by rotating coaxial cylinders is named after its investigator Maurice Marie Alfred Couette. We can observe the flow if the cylinders are transparent and the fluid has markers like fine aluminium powder dissolved in it. If the inner cylinder is at rest while the outer one is turned, the fluid begins a rotating motion. The motion continues to stable even at high rotational speed of the outer cylinder. If the outer cylinder is at rest and the rotation of the inner cylinder induces motion, the flow is steady up to a certain speed and but breaks into rotating wavy bands beyond. The appearance of bands was explained by G. I. Taylor [11] and they are called Taylor vortices in his honour. It is not immediately clear why the choice of driving cylinder should make such a marked difference to flow.

When the outer cylinder drives the flow, the no-slip condition guarantees that the velocity gradient points radially outwards. The centrifugal force experienced by outer layers of fluid is more than the inner layers. The flow merely alters the radial distribution of pressure and there is no cause for instability.

When the inner cylinder drives the flow, the inner fluid layers experience more centrifugal force than the outer ones. As they are pushed outwards, the mildly pushed outer layers come in their way. The flow then breaks into cells with the inner layer trying to flow down along the inner cylinder and flow up along the outer one. A secondary flow ensues showing up as Taylor vortices.

Mathematical analysis of Taylor-Couette instability is challenging even
in the linear case. Taylor simplified the problem by assuming the gap between cylinders to be narrow and the perturbations to be axi-symmetric with complex frequency \( \eta \). He further assumed that at the onset of instability, \( \text{Im}(\eta) = 0 \), that is the principle of exchange of stabilities is valid. He arrived at a dimensionless number, now called \textit{Taylor number}, and proved that the flow is unstable if it is greater than some critical value.

5.5 \textbf{Shock (Richtmyer-Meshkov instability)}

The Richtmyer-Meshkov instability occurs when a shock wave impinges an interface between two fluids, typically gases. The gases are of sufficiently different densities for their interface to be well differentiated. The physical mechanism of the instability is most clear if we consider the two fluids to be ideal, in which case they obey the Euler equation,

\[
\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p
\]  

(5.5.1)

We get the ideal vorticity equation if we operate both sides of this equation by the "curl" operator.

\[
\frac{\partial \mathbf{w}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{w} = \frac{1}{\rho^2} \nabla \rho \times \nabla p + \mathbf{w} \cdot \nabla \mathbf{u} - \mathbf{w} \nabla \cdot \mathbf{u},
\]  

(5.5.2)

where \( \mathbf{w} = \nabla \times \mathbf{u} \) is the vorticity field. This equation tells how vorticity evolves with time. One of the factors affecting the evolution is the baroclinic term \( \nabla \rho \times \nabla p \). Since shocks are characterized by a sharp discontinuity or gradient in pressure, the baroclinic term is most effective in creating vorticity.
when a shock wave impinges an interface. The pressure gradient, however high, is not effective in the bulk of the fluids because of the cross product with $\nabla \rho$. In their bulks, $\nabla \rho$ is negligibly small or even zero. Richtmyer-Meshkov instability is thus an interfacial phenomenon.

Before closing, we mention that there are several other causes of instability like, difference in viscosity (Saffman instability), intense gravity (Jeans' instability), surface tension (Plateau instability). However, we do not plan to study them in this thesis and therefore do not cover them in this chapter.
Bibliography


