Chapter 4

Generalized Pseudo Commutative Gamma Near-rings

S. Uma, R. Balakrishnan and T. Tamizh Chelvam [35] introduced the concept of Pseudo Commutative near-rings. Yong UK Cho[36] introduced the concepts of $\Gamma$-near-rings and obtained their properties on $\Gamma$-near-rings through regularity conditions. By analogy with these concepts, we introduce the notion of Generalized Pseudo Commutative(GPC) $\Gamma$-near-rings. We say that a $\Gamma$-near-ring $M$ is a Generalized Pseudo Commutative (GPC) $\Gamma$-near-ring if $a\Gamma M \Gamma b = b\Gamma M \Gamma a$ for all $a, b \in M$. In the first section of this chapter, we discuss some of their properties. We also obtain a characterization of GPC $\Gamma$-near-ring through Generalized Right Permutable (GRP) $\Gamma$-near-ring. In the second section of this chapter, we introduce $k$-pseudo commutative $\Gamma$-near-rings and discuss some of their properties and obtain a characterization of $k$-pseudo commutative $\Gamma$-near-ring.
4.1 GPC $\Gamma$-near-rings

In this section, we introduce Generalized Pseudo Commutative (GPC) $\Gamma$-near-rings and discuss some of their properties.

**Definition 4.1.1.** A $\Gamma$-near-ring $M$ is a Generalized Pseudo Commutative (GPC) $\Gamma$-near-ring if $a \Gamma M \Gamma b = b \Gamma M \Gamma a$ for all $a, b \in M$.

Having defined a new notion, let us see some examples.

**Example 4.1.2.** Consider the $\Gamma$-near-ring defined by the Klein’s four group $\{0, a, b, c\}$ with $\Gamma = \{\gamma_1, \gamma_2\}$ where $\gamma_1, \gamma_2$ are given by the schemes 1: $(0,13,0,13)$ and 12: $(0,13,0,0)$ (see p.408 [23, G. Pilz]).

\[
\begin{array}{cccc}
\gamma_1 & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & b & 0 & b \\
b & 0 & 0 & 0 & 0 \\
c & 0 & b & 0 & b \\
\end{array}
\]

\[
\begin{array}{cccc}
\gamma_2 & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & b & 0 & 0 \\
b & 0 & 0 & 0 & 0 \\
c & 0 & b & 0 & 0 \\
\end{array}
\]

This is a GPC $\Gamma$-near-ring.
Lemma 4.1.3. Every GPC $\Gamma$-near-ring is zero-symmetric.

Proof. For every $\gamma \in \Gamma$, $x \in M$, $x\gamma 0 = x\gamma \gamma_1 0 = 0\gamma_2 m\gamma_3 x = 0$ for some $m \in M$, $\gamma_2, \gamma_3 \in \Gamma$ and so $M$ is zero-symmetric. $\blacksquare$

Proposition 4.1.4. Homomorphic image of a GPC $\Gamma$-near-ring is a GPC $\Gamma$-near-ring.

Proof. Let $M$ be a GPC $\Gamma$-near-ring and $f : M \rightarrow M'$ a $\Gamma$-near-ring epimorphism. For any $x', y', m' \in M'$, and $\gamma_1, \gamma_2 \in \Gamma$, consider $x'\gamma_1 m' \gamma_2 y'$. Since $f$ is onto there exists $x, y, m \in M$ such that $f(x) = x'$, $f(y) = y'$ and $f(m) = m'$. Therefore $x'\gamma_1 m' \gamma_2 y' = f(x)\gamma_1 f(m) \gamma_2 f(y) = f(x\gamma_1 m \gamma_2 y)$. Since $M$ is a GPC $\Gamma$-near-ring, $x\gamma_1 m \gamma_2 y = y\gamma_3 m_1 \gamma_4 x$ for all $\gamma_3, \gamma_4 \in \Gamma$ and $m_1 \in M$. Therefore $x'\gamma_1 m' \gamma_2 y' = f(y\gamma_3 m_1 \gamma_4 x) = f(y)\gamma_3 f(m_1) \gamma_4 f(x) = y'\gamma_3 f(m_1) \gamma_4 x' \in y'\Gamma M' \Gamma x'$. Therefore $x'\Gamma M' \Gamma y' \subseteq y'\Gamma M' \Gamma x'$. Similarly one can prove the converse. Thus $M'$ is a GPC $\Gamma$-near-ring. $\blacksquare$

Proposition 4.1.5. Let $M$ be a GPC $\Gamma$-near-ring. Then $M$ is a left unital $\Gamma$-near-ring if and only if $M$ is a right unital $\Gamma$-near-ring.

Proof. Let $M$ be a left unital $\Gamma$-near-ring. Then $x \in x\Gamma M$ for every $x \in M$. Now $x = x\gamma m$ for some $\gamma \in \Gamma$ and $m \in M$. Since $M$ is left unital, $m = m\gamma_1 m_1$ for some $\gamma_1 \in \Gamma$ and $m_1 \in M$. Therefore $x = x\gamma m\gamma_1 m_1$. Since $M$ is a GPC $\Gamma$-near-ring, for some $m_2 \in M$ and
\[ \gamma_2, \gamma_3 \in \Gamma, \ x = m_1 \gamma_2 m_2 \gamma_3 x \in M\Gamma x. \] Hence \( M \) is a right unital GPC \( \Gamma \)-near-ring. Similarly one can prove the converse. \( \square \)

Hereafter in this section, we assume that \( M \) is a left unital \( \Gamma \)-near-ring.

**Proposition 4.1.6.** Let \( M \) be a GPC \( \Gamma \)-near-ring. Then the following are true:

(i) \( M \) is subcommutative;

(ii) \( M \) satisfies IFP;

(iii) For any \( s \in M \), the set \( I = (0 : s) = \{ x \in M / x \gamma s = 0 \text{ for all } \gamma \in \Gamma \} \) is an ideal of \( M \);

(iv) \( M \) satisfies strong IFP property.

**Proof.**

(i) Let \( y \in a\Gamma M \). Then \( y = a\gamma_1 m \) for some \( \gamma_1 \in \Gamma \) and \( m \in M \). Since \( M \) is left unital, \( a = a\gamma_2 m_1, \gamma_2 \in \Gamma \) and \( m_1 \in M \). Therefore \( y = a\gamma_2 m_1 \gamma_1 m \). Since \( M \) is a GPC \( \Gamma \)-near-ring, there exists \( \gamma_3, \gamma_4 \in \Gamma \) and \( m_2 \in M, y = m\gamma_3 m_2 \gamma_4 a \in M\Gamma a \). Therefore \( a\Gamma M \subseteq M\Gamma a \). Similarly one can prove that \( M\Gamma a \subseteq a\Gamma M \).

(ii) Let \( a, b \in M \) and \( a\gamma b = 0 \) for all \( \gamma \in \Gamma \). Since \( M \) is a left unital GPC \( \Gamma \)-near-ring, for any \( n \in M \) and \( \gamma_1, \gamma_2 \in \Gamma \), there exists \( \gamma_3, \gamma_4, \gamma_5 \in \Gamma \) and \( m_1, m_2 \in M \) such that \( a\gamma_1 n\gamma_2 b = a\gamma_1 n\gamma_3 m_1 \gamma_2 b = a\gamma_1 (b\gamma_4 m_2 \gamma_5 n) = (a\gamma_1 b)\gamma_4 m_2 \gamma_5 n = 0 \gamma_4 m_2 \gamma_5 n = 0 \).

(iii) For every \( \gamma \in \Gamma \) and \( s \in M \), \( 0 \gamma s = 0 \), and so \( I \) is non-empty. Let \( a, b \in I \). For any \( \gamma \in \Gamma, s \in M, (a + b) \gamma s = a\gamma s + b\gamma s = 0 \) which implies
that $a + b \in I$. For any $m \in M, (m + a - m)\gamma s = m\gamma s + a\gamma s - m\gamma s = m\gamma s + 0 - m\gamma s = 0$. Therefore $(m + a - m) \in I$. For every $x, y \in M, a \in I$ and $\gamma, \mu \in \Gamma, (x\gamma (a + y) - x\gamma y)\mu s = x\gamma (a + y)\mu s - x\gamma y\mu s = x\gamma (a\mu s + y\mu s) - x\gamma y\mu s = x\gamma (0 + y\mu s) - x\gamma y\mu s = x\gamma y\mu s - x\gamma y\mu s = 0$. This implies that $x\gamma (a + y) - x\gamma y \in I$. Hence $I$ is a left ideal of $M$. Let $a \in I$ and $x \in M$. Let $s \in M, \gamma_1 \in \Gamma$ and consider $(a\gamma x)\gamma_1 s$. Since $M$ is a left unital GPC $\Gamma$-near-ring, there exists $\gamma_3, \gamma_4 \in \Gamma$ and $m, m_1 \in M$ such that $(a\gamma x)\gamma_1 s = a\gamma (x\gamma_2 m)\gamma_1 s = (a\gamma x\gamma_2 m)\gamma_1 s = (m\gamma_3 m_1 \gamma_4 a)\gamma_1 s = m\gamma_3 m_1 \gamma_4 (a\gamma_1 s) = m\gamma_3 m_1 \gamma_4 0 = 0$. Hence $I$ is an ideal.

(iv) Let $I$ be an ideal of $M$ such that $a\gamma b \in I$ for every $a, b \in M$ and $\gamma \in \Gamma$. For any $n \in M, \gamma_1, \gamma_2 \in \Gamma, a\gamma_1 n\gamma_2 b = a\gamma_1 b\gamma_3 m$ for some $\gamma_3 \in \Gamma, m \in M$. Therefore $a\gamma_1 n\gamma_2 b = a\gamma_1 b\gamma_3 m \in IM \subseteq I$. \hfill \Box

Proposition 4.1.7. Let $M$ be a GPC $\Gamma$-near-ring and $A \subseteq M$. Then $A$ is a left $M$-$\Gamma$-subgroup of $M$ if and only if $A$ is a right $M$-$\Gamma$-subgroup of $M$.

Proof. Let $A$ be a left $M$-$\Gamma$-subgroup of $M$ and $x \in A\Gamma M$. Then $x = a\gamma_1 m$ for some $a \in A, m \in M$ and $\gamma_1 \in \Gamma$. Since $M$ is left unital, for some $\gamma_2 \in \Gamma$ and $m_1 \in M, x = a\gamma_2 m_1 \gamma_1 m = m\gamma_3 m_2 \gamma_4 a \in M\Gamma A \subseteq A$. Therefore $A\Gamma M \subseteq A$. Similarly one can prove the other inclusion. \hfill \Box

Proposition 4.1.8. Let $M$ be a GPC $\Gamma$-near-ring. If $M$ is regular, then $M$ has no non-zero nilpotent elements.
Proof. Since $M$ is regular, for any $a \in M$, there exists $x \in M$ such that $a = a\gamma_1 x \gamma_2 a$ for every pair of non-zero elements $\gamma_1, \gamma_2 \in \Gamma$. Let $a\gamma a = 0$ for every $\gamma \in \Gamma$. Since $M$ is left unital GPC $\Gamma$-near-ring, by (i) of Proposition 4.1.6, $M$ is subcommutative. Therefore $x\gamma_2 a = a\gamma_3 m$ for any $\gamma_3 \in \Gamma$ and $m \in M$. From this $a = a\gamma_1 x \gamma_2 a = a\gamma_1 a\gamma_3 m = 0$. Hence $M$ has no non-zero nilpotent elements. 

Proposition 4.1.9. In a GPC $\Gamma$-near-ring, every idempotent is central.

Proof. Let $e \in M$ be an idempotent element and $m \in M$. By (i) of Proposition 4.1.6, $M$ is subcommutative and so $e\Gamma M = M\Gamma e$. Now $e\Gamma M\Gamma e = e\Gamma (e\Gamma M) = (e\Gamma e)\Gamma M = e\Gamma M$. Therefore $M\Gamma e = e\Gamma M = e\Gamma M\Gamma e$. This implies that, for $m \in M$ there exists $u, v \in M$ such that $m\gamma_1 e = e\gamma_2 u \gamma_1 e$ and $e\gamma_2 m = e\gamma_2 v \gamma_1 e$ for every $\gamma_1, \gamma_2 \in \Gamma$. Now $e\gamma_2 m\gamma_1 e = (e\gamma_2 m)\gamma_1 e = (e\gamma_2 v \gamma_1 e)\gamma_1 e = e\gamma_2 v \gamma_1 (e\gamma_1 e) = e\gamma_2 v \gamma_1 e = e\gamma_2 m$. Also $e\gamma_2 m\gamma_1 e = e\gamma_2 (m\gamma_1 e) = e\gamma_2 (e\gamma_2 u \gamma_1 e) = (e\gamma_2 e)\gamma_2 u \gamma_1 e = e\gamma_2 u \gamma_1 e = m\gamma_1 e$. Therefore $e\gamma_2 m = m\gamma_1 e$ for every $m \in M$. i.e., $e$ is central.

Proposition 4.1.10. In a GPC $\Gamma$-near-ring $M$, the following are equivalent:

(i) $M$ is strongly regular;

(ii) $M$ is a $P(1,2)$ $\Gamma$-near-ring.
Proof. (i) ⇒ (ii) Since $M$ is strongly regular, for any $x \in M$ there exists $y \in M$ such that $x = x^2 \gamma y$ and $x = x \gamma_1 y \gamma_2 x$ for every $\gamma, \gamma_1, \gamma_2 \in \Gamma$. Let $y \in x\Gamma M$. Then, for every $\gamma_3 \in \Gamma$, $m \in M$, $y = x \gamma_3 m = (x^2 \gamma y) \gamma_3 m = m \gamma_2 m_1 \gamma_3 x^2$ for every $\gamma_2 \in \Gamma$ and $m_1 \in M$. i.e., $y \in M \Gamma x^2$. Conversely, let $y \in M \Gamma x^2$. Then for every $\gamma \in \Gamma$ and $m \in M$, $y = m \gamma x^2 = m \gamma x \gamma_1 x = x \gamma_2 m_1 \gamma_3 m$ for every $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ and $m_1 \in M$. Therefore $y \in x\Gamma M$. i.e., $M$ is a $P(1, 2)$-near-ring.

(ii) ⇒ (i) Let $x \in M$. Since $M$ is left unital, $x \in x\Gamma M$. Again by $M$ is a $P(1, 2)$-near-ring, $x \in M \Gamma x^2 = M \Gamma x \Gamma x$. Now $x \Gamma M = M \Gamma x$, $x \in x \Gamma M \Gamma x = x \Gamma x \Gamma M = x^2 \Gamma M$. Therefore $x = x^2 \gamma y$ for all $\gamma \in \Gamma$ and $y \in M$. Also $a \in a\Gamma M$ implies $a = a \gamma m$ for all $\gamma \in \Gamma$ and $m \in M$. Now $a = a^2 \gamma_1 y \gamma m$ for some $\gamma_1 \in \Gamma$. i.e., $a = (a \gamma_2 a) \gamma_1 y \gamma m = a \gamma_2 (a \gamma_1 y \gamma m) = a \gamma_2 (m \gamma_2 m_1 \gamma_3 a) = a \gamma_2 (m \gamma_2 m_1 \gamma_3 a) \in a \Gamma M \Gamma a$. Therefore for some $m \in M$ and $\gamma_1, \gamma_2 \in \Gamma$, $a = a \gamma_1 m \gamma_2 a$.

Proposition 4.1.11. Let $M$ be a GPC $\Gamma$-near-ring. Then $M$ is regular if and only if $M$ is a reduced left bi-potent $\Gamma$-near-ring.

Proof. Suppose $M$ is regular. By Proposition 4.1.8, $M$ has no non-zero nilpotent elements. For any $y \in M \Gamma a$, $y = m \gamma_1 a$ for some $\gamma_1 \in \Gamma$ and $m \in M$. Since $M$ is regular and subcommutative, $y = m \gamma_1 (a \gamma_2 m_1) \gamma_3 a = m \gamma_1 (m_2 \gamma_4 a) \gamma_3 a = (m \gamma_1 m_2) \gamma_4 a \gamma_3 a \in M \Gamma a^2$. Now, for any $y \in M \Gamma a^2$, $y = m \gamma a^2$ for every $\gamma \in \Gamma$ and $m \in M$. Hence $y = m \gamma (a \gamma_1 a) = (m \gamma a) \gamma_1 a \in M \Gamma a$ for $\gamma_1 \in \Gamma$ and so $M$ is left bi-potent. Conversely, let $a \in M$. Since $M$ is right unital and left bi-potent, $a \in M \Gamma a = M \Gamma a^2$. 

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Therefore $a = m\gamma a^2$ for some $\gamma \in \Gamma$ and $m \in M$. That is $a = m\gamma a\gamma_1 a = a\gamma_2 m_1 \gamma_3 a$ for some $m_1 \in M$ and all $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$. i.e., $M$ is regular. \(\square\)

**Proposition 4.1.12.** In a GPC $\Gamma$-near-ring $M$, the following are true:

(i) Every $M$-$\Gamma$-subgroup of $M$ is invariant;

(ii) Every left ideal of $M$ is an ideal;

(iii) For every $x, y \in M$ and $\alpha, \beta \in \Gamma$, $x\alpha y = 0$ implies $y\beta x = 0$.

**Proof.**

(i) Let $A$ be an $M$-$\Gamma$-subgroup of $M$. From we have $M\Gamma A \subseteq A$. For any $y \in A\Gamma M, y = a\gamma m$ for $\gamma \in \Gamma$ and $m \in M$. Since $M$ is left unital, $m = m\gamma_1 m_1$ for some $\gamma_1 \in \Gamma$ and $m_1 \in M$. Thus $y = a\gamma m\gamma_1 m_1 = m_1 \gamma_2 m_2 \gamma_3 a \in M\Gamma A \subseteq A$. Hence $A$ is invariant.

(ii) Let $A$ be a left ideal of $M$ and $y \in M\Gamma A$. Then, for some $\gamma \in \Gamma$ and $m \in M, y = m\gamma a$. From this $y = m\gamma(a + 0) - m\gamma 0 \subseteq A$. Therefore by (i), $A\Gamma M \subseteq A$ and hence $A$ is an ideal of $M$.

(iii) Since $M$ is left unital, $y\beta x = y\gamma_1 m\beta x = x\gamma_3 m_1 \gamma_4 y = x\gamma_3 m_1 \gamma_5 m_2 \gamma_4 y = x\gamma_3 y\gamma_6 m_3 \gamma_7 m_1 = 0\gamma_6 m_3 \gamma_7 m_1 = 0$. \(\square\)

**Proposition 4.1.13.** Every left unital GPC $\Gamma$-near-ring is a GRP $\Gamma$-near-ring.

**Proof.** By (i) of Proposition 4.1.6, $M$ is subcommutative. From this $M\Gamma a \Gamma b = a\Gamma M\Gamma b = b\Gamma M\Gamma a = M\Gamma b \Gamma a$. Thus $M$ is a GRP $\Gamma$-near-ring. \(\square\)
Proposition 4.1.14. Let $M$ be a subcommutative $\Gamma$-near-ring. Then $M$ is regular GRP $\Gamma$-near-ring if and only if $M$ is a left unital, left bi-potent GPC $\Gamma$-near-ring.

Proof. Since $M$ is regular, it is obviously left unital. Let $y \in M\Gamma a$. Then $y = m\gamma a$ for some $\gamma \in \Gamma$ and $m \in M$. Since $M$ is subcommutative, for any $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ and $m_1, m_2 \in M$ we have $y = m\gamma(a\gamma_1 m_1 \gamma_2 a) = m\gamma(a\gamma_1 m_1)\gamma_2 a = m\gamma(m_2 \gamma_3 a)\gamma_2 a = (m\gamma m_2)\gamma_3 (a\gamma_2 a) \in M\Gamma a^2$. Suppose $y \in M\Gamma a^2$. Then for some $\gamma, \gamma_1 \in \Gamma, y = m\gamma a\gamma_1 a = (m\gamma a)\gamma_1 a \in M\Gamma a$. Hence $M$ is left bipotent. Now for every $a, b \in M, a\Gamma M\Gamma b = (a\Gamma M)\Gamma b = (M\Gamma a)\Gamma b = M\Gamma b\Gamma a = b\Gamma M\Gamma a$.

For the converse, let $x \in M$. Since $M$ is a left unital and left bi-potent, $x \in M\Gamma x = M\Gamma x^2$. That is, for any $\gamma_1 \in \Gamma, x = m\gamma x^2 = m\gamma x\gamma_1 x$. Since $M$ is subcommutative, $x = x\gamma m'\gamma_1 x$ for some $m' \in M$. Therefore $M$ is regular. For every $a, b \in M, M\Gamma a\Gamma b = a\Gamma M\Gamma b = b\Gamma M\Gamma a = M\Gamma b\Gamma a$. Hence $M$ is a GRP $\Gamma$-near-ring. \[\square\]

4.2 $k$-pseudo commutative gamma near-ring

In this section, we introduce $k$-pseudo commutative $\Gamma$-near-ring and we study certain properties of $k$-pseudo commutative $\Gamma$-near-ring. Also we obtain some equivalent conditions in $k$-pseudo commutative $\Gamma$-near-rings.
Definition 4.2.1. Let $k \geq 1$ be a fixed integer. A $\Gamma$-near-ring $M$ is called a $k$-pseudo commutative $\Gamma$-near-ring if $x\gamma_1 y^k \gamma_2 z = z\gamma_3 y^k \gamma_4 x$ for every $x, y, z \in M$ and for binary operators $\gamma_1, \gamma_2$ and $\gamma_3, \gamma_4 \in \Gamma$.

Example 4.2.2. Consider the $\Gamma$-near-ring defined on the Klein’s four group $\{0, a, b, c\}$ with $\Gamma = \{\gamma_1, \gamma_2\}$ where $\gamma_1, \gamma_2$ are given by the schemes $1 : (0, 13, 0, 13)$ and $12 : (0, 13, 0, 0)$ (see p.408 [23, G. Pilz])

\[
\begin{array}{cccc}
\gamma_1 & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & b & 0 & b \\
b & 0 & 0 & 0 & 0 \\
c & 0 & b & 0 & b \\
\end{array}
\]

\[
\begin{array}{cccc}
\gamma_2 & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & b & 0 & 0 \\
b & 0 & 0 & 0 & 0 \\
c & 0 & b & 0 & 0 \\
\end{array}
\]

The above $\Gamma$-near-ring is a $k$-pseudo commutative $\Gamma$-near-ring with $k = 1$.

Definition 4.2.3. Let $k \geq 1$ be a fixed integer. A $\Gamma$-near-ring $M$ is said to be $k$-regular if, for each $a \in M$, there exists $b \in M$ such that $a = a\gamma_1 b^k \gamma_2 a$ for every pair of non zero elements $\gamma_1, \gamma_2$ of $\Gamma$. 
Proposition 4.2.4. Every $k$-pseudo commutative $\Gamma$-near-ring is zero symmetric.

Proof. Let $M$ be a $k$-pseudo commutative $\Gamma$-near-ring. For every $m \in M$ and $\gamma \in \Gamma$, $m\gamma 0 = m\gamma 0^{k+1} = m\gamma 0\gamma_1 0 = 0\gamma_2 0\gamma_3 m = 0$ where $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ i.e., for all $m \in M$ and $\gamma \in \Gamma$, $m\gamma 0 = 0$. Therefore $M$ is zero-symmetric.

Proposition 4.2.5. Let $M$ be a $k$-pseudo commutative $\Gamma$-near-ring. Then every idempotent element in $M$ is central. i.e., $E \subseteq C(M)$.

Proof. Let $e \in M$ be an idempotent element. Then $e^k = e$ for every $k \geq 2$. For all $\gamma \in \Gamma$ and $a \in M$, $a\gamma e = a\gamma e^{k+1} = a\gamma e\gamma_1 e = e\gamma_2 e\gamma_3 a = e^{k+1}\gamma_3 a = e\gamma_3 a$ for some $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$. (i.e) $a\gamma e = e\gamma_3 a$ for every $\gamma_3 \in \Gamma$. This implies $e \in C(M)$. Hence $E$ is central.

Proposition 4.2.6. Homomorphic image of a $k$-pseudo commutative $\Gamma$-near-ring is again a $k$-pseudo commutative $\Gamma$-near-ring.

Proof. One can draw the proof from the definition.

Corollary 4.2.7. Let $M$ be a $k$-pseudo commutative $\Gamma$-near-ring. If $I$ is an ideal of $M$, then $M/I$ is a $k$-pseudo commutative $\Gamma$-near-ring.

Proof. Since $M/I$ is a homomorphic image of $M$, the proof follows from Proposition 4.2.6.
Proposition 4.2.8. Any right permutable $\Gamma$-near-ring with left identity is a $k$-pseudo commutative $\Gamma$-near-ring.

Proof. Let $M$ be a right permutable $\Gamma$-near-ring. Then $x\gamma_1 y \gamma_2 z = x\gamma_1 z \gamma_2 y$ for all $x, y, z \in M$ and $\gamma_1, \gamma_2 \in \Gamma$. Let $e \in M$ be a left identity.

Now for all $a, b, c \in M, \gamma_1, \gamma_2 \in \Gamma, a\gamma_1 b^k \gamma_2 c = e\gamma(a\gamma_1 b^k \gamma_2 c) = e\gamma(a\gamma_1 c \gamma_2 b^k) = (e\gamma a \gamma_1 c) \gamma_2 b^k = (e\gamma c a \gamma_1) \gamma_2 b^k = c\gamma_1 a \gamma_2 b^k = c\gamma_1 b^k \gamma_2 a$. Hence $M$ is a $k$-pseudo commutative $\Gamma$-near-ring.

Lemma 4.2.9. In a $k$-regular $\Gamma$-near-ring $M$ Then, for every $a, x \in M, \gamma_1, \gamma_2 \in \Gamma$

(i) $a\gamma_1 x^k, x^k \gamma_2 a \in E$;

(ii) $\gamma_1 a x^k \gamma_2 M = a\gamma_1 M$ and $M \gamma_1 x^k \gamma_2 a = M \gamma_2 a$.

Proof. (i) Let $a \in M$. Since $M$ is $k$-regular, there exists $x \in M$ such that $a = a\gamma_1 x^k \gamma_2 a$ for every $\gamma_1, \gamma_2 \in \Gamma$.

Now, $(a\gamma_1 x^k)^2 = (a\gamma_1 x^k) \gamma_2 (a\gamma_1 x^k) = (a\gamma_1 x^k \gamma_2 a) \gamma_1 x^k = a\gamma_1 x^k$. Similarly $(x^k \gamma_2 a)^2 = (x^k \gamma_2 a) \gamma_1 (x^k \gamma_2 a) = x^k \gamma_2 (a\gamma_1 x^k \gamma_2 a) = x^k \gamma_2 a$. Therefore $a\gamma_1 x^k, x^k \gamma_2 a \in E$.

(ii) Let $y \in a\gamma_1 x^k \gamma_2 M$. Then $y = a\gamma_1 x^k \gamma_2 m \in a\gamma_1 M$ for some $m \in M$. Conversely, let $z \in a\gamma_1 M$. Then $z = a\gamma_1 m = (a\gamma_1 x^k \gamma_2 a) \gamma_1 m = a\gamma_1 x^k \gamma_2 (a\gamma_1 m) \in a\gamma_1 x^k \gamma_2 M$. Hence $a\gamma_1 x^k \gamma_2 M = a\gamma_1 M$. Similarly one can see the other part. □
Proposition 4.2.10. Let $M$ be a $k$-regular $k$-pseudo commutative $\Gamma$-near-ring. Then

(i) $A = \sqrt{A}$ for every $M$-$\Gamma$-subgroup $A$ of $M$;

(ii) $M$ is reduced;

(iii) $M$ has IFP.

Proof. (i) Let $A$ be a $M$-$\Gamma$-subgroup of $M$. Since $M$ is $k$-regular, for every $a \in M$ there exists $b \in M$ such that $a = a\gamma_1 b^k \gamma_2 a$ for every pair of binary operators $\gamma_1, \gamma_2 \in \Gamma$. By Lemma 4.2.9, $a\gamma_1 b^k, b^k \gamma_2 a \in E$ for every $a, b \in M$ and $\gamma_1, \gamma_2 \in \Gamma$. Since $E \subseteq C(M), a\gamma_1 b^k, b^k \gamma_2 a \in C(M)$.

Now $a = a\gamma_1 b^k \gamma_2 a = (a\gamma_1 b^k) \gamma_2 a = a\gamma_2 (a\gamma_1 b^k) = a^2 \gamma_1 b^k.$ Also, $a = a\gamma_1 (b^k \gamma_2 a) = (b^k \gamma_2 a) \gamma_1 a = b^k \gamma_2 a^2$. Let $a \in \sqrt{A}$, then there exists some positive integer $n$ such that $a^n \in A$. Now $a = b^k \gamma_2 a^2 = b^k \gamma_2 (a\gamma_3 a) = b^k \gamma_2 (b^k \gamma_2 a^2) \gamma_3 a = b^2 k \gamma_2 a^3 = b^2 \gamma_2 (a \gamma_4 a^2) = b^2 \gamma_2 (b^k \gamma_2 a^2) \gamma_4 a^2 = b^3 k \gamma_2 a^4$ for all $\gamma_2 \in \Gamma$. Proceeding like this we get, $a = b^{(n-1)k} \gamma_2 a^n$ for all $\gamma_2 \in \Gamma, a \in M\Gamma A \subseteq A$. Hence $\sqrt{A} \subseteq A$. Obviously $A \subseteq \sqrt{A}$.

Hence $A = \sqrt{A}$.

(ii) Let $a^2 = a\gamma a = 0$ for all $\gamma \in \Gamma$. We have $a = b^k \gamma a^2$ for all $\gamma \in \Gamma$.

Therefore $a = b^k \gamma 0 = 0$. Hence $M$ is reduced.

(iii) Let $a\gamma b = 0$ for all $\gamma \in \Gamma$ and $a, b \in M$. Now $(b\gamma a)^2 = (b\gamma a) (b\gamma a) = b\gamma (a\gamma b) a = b\gamma 0 a = 0$ for some $\gamma' \in \Gamma$. By (ii) $b\gamma a = 0$ for every $\gamma \in \Gamma$. Now for every $\gamma_1, \gamma_2 \in \Gamma$, and $m \in M$, $(a\gamma_1 m \gamma_2 b)^2 = (a\gamma_1 m \gamma_2 b) (a\gamma_1 m \gamma_2 b) = a\gamma_1 m \gamma_2 (b\gamma a) \gamma_1 m \gamma_2 b = a\gamma_1 m \gamma_2 0 \gamma_1 m \gamma_2 b = 0, \gamma \in \Gamma$. By (ii) $a\gamma_1 m \gamma_2 b = 0$. Hence $M$ has IFP. \qed
Theorem 4.2.11. Let $M$ be a $k$-regular $k$-pseudo commutative $\Gamma$-near-ring. Then every $M$-$\Gamma$-subgroup is an ideal of $M$.

Proof. Let $a \in M$. Since $M$ is $k$-regular for every $\gamma_1, \gamma_2 \in \Gamma$, there exists $b \in M$ such that $a = a\gamma_1b^k\gamma_2a$. By Lemma 4.2.9, $b^k\gamma a$ is an idempotent. Let $b^k\gamma a = e$. Again by Lemma 4.2.9, for any $\alpha \in \Gamma$, $Ma b^k\gamma a = M\gamma a$ for all $\gamma \in \Gamma$. Let $S = \{m - mae/m \in M, \alpha \in \Gamma\}$. Now $(0 : S) = \{y/s\gamma y = 0$ for every $\gamma \in \Gamma$ and for every $s \in S\}$. We claim that $(0 : S) = Mae$ for every $\alpha \in \Gamma$.

For all $m \in M, \alpha \in \Gamma, (m - mae)ae = mae - mae^2 = mae - mae = 0$. Since $M$ has IFP, we have $(m - mae)\alpha(Mae) = 0$ for all $\alpha \in \Gamma$. Therefore $Mae \subseteq (0 : S)$ for all $x \in \Gamma$. Let $y \in (0 : S)$ then $s\alpha y = 0$ for all $s \in S$ and $\alpha \in \Gamma$. Now $y \in M$ and since $M$ is $k$-regular there exists $x \in M$ such that $y = y\alpha x^k\beta y$ for all $\alpha, \beta \in \Gamma$. Since $y\alpha x^k - (y\alpha x^k)\alpha e \in S$ for all $\alpha \in \Gamma$. We have $(y\alpha x^k - (y\alpha x^k)\alpha e)\alpha y = 0$. (i.e) $y\alpha x^k\alpha y - y\alpha x^k\alpha e\alpha y = 0$. i.e., $y - y\alpha x^k\alpha e\alpha y = 0$. By Proposition 2.2.35, $y - y\alpha x^k\alpha e\alpha y = 0$. i.e., $y - (y\alpha x^k\alpha e)\alpha e = 0$. i.e., $y - yae = 0$. this implies $y = yae \in Mae$. (i.e) $(0 : S) \subseteq Mae$ for every $\alpha \in \Gamma$. Therefore $(0 : S) = Mae$ for every $\alpha \in \Gamma$. Since $(0 : S)$ is an ideal of $M, Mae$ is an ideal of $M$ for every $\alpha \in \Gamma$. But $Mae = Mab^k\gamma a = M\gamma a$ which implies that $M\gamma a$ is an ideal for every $a \in M$. Now if $N$ is any $M$-$\Gamma$-subgroup of $M$ then $N = \sum_{a \in M} M\gamma a$. Thus $N$ becomes an ideal of $M$. □

Proposition 4.2.12. If $M$ is a $k$-regular and $k$-pseudo commutative $\Gamma$-near-ring then
(i) any ideal of $M$ is completely semi prime;

(ii) if $I$ is an ideal of $M$ and if $a\gamma b \in I$ then $b\gamma a \in I$ for $a, b \in M$, $\gamma \in \Gamma$;

(iii) $M$ has strong IFP.

**Proof.** (i) Let $I$ be an ideal of $M$ and $a^2 \in I$. Since $M$ is $k$-regular, there exists $b \in M$ such that $a = a\gamma_1 b^k \gamma_2 a$ for every $\gamma_1, \gamma_2 \in \Gamma$. By Proposition 4.2.5, $E \subseteq C(M)$, and so $a = a\gamma_2 (a\gamma_1 b^k) = a^2 \gamma_1 b^k \in IM \subseteq I$. Therefore every ideal of $M$ is completely semi prime.

(ii) Let $I$ be an ideal of $M$ and $a\gamma b \in I$ for $a, b \in M$ and $\gamma \in \Gamma$. Now $(b\gamma a)^2 = (b\gamma a)\gamma_1 (b\gamma a) = b\gamma (a\gamma_1 b) \gamma a, \gamma_1 \in \Gamma \in M\Gamma IM \subseteq I$. By (i) $b\gamma a \in I$.

(iii) Let $I$ be an ideal of $M$. Let $a\gamma b \in I$. As discussed in the proof of Lemma 2.2.37, $M\Gamma I \subseteq I$. By Lemma 4.2.9, $a\gamma_1 x^k \gamma_2 M = a\gamma_1 M$ for every $a \in M$ and $\gamma_1, \gamma_2 \in \Gamma$. i.e., $a\gamma_1 m = a\gamma_1 x^k \gamma_2 n$ for some $m, n \in M$. Now, for $m \in M, \gamma_1, \gamma_3 \in \Gamma, a\gamma_1 m \gamma_3 b = (a\gamma_1 x^k \gamma_2 n) \gamma_3 b = (n\gamma_4 x^k \gamma_5 a) \gamma_3 b = n\gamma_4 x^k \gamma_5 (a\gamma_3 b) \in M\Gamma I \subseteq I$. Therefore $a\gamma_1 m \gamma_3 b \in I$ for every $m \in M$ and $\gamma_1, \gamma_3 \in \Gamma$. $\square$

**Proposition 4.2.13.** Let $M$ be a $k$-regular $k$-pseudo commutative $\Gamma$-near-ring. Then

(i) $P \cap Q = P\Gamma Q$ for any two $M\Gamma$-subgroups $P, Q$ of $M$;

(ii) $P = P^2$ for every $M\Gamma$-subgroup $P$ of $M$. 

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Proof. (i) Let $P, Q$ be two $M$-$\Gamma$-subgroups of $M$. By Theorem 4.2.11, $P$ and $Q$ are ideals. Hence $PTQ \subseteq P$ and $PTQ \subseteq Q$. Therefore $PTQ \subseteq P \cap Q$. Let $a \in P \cap Q$. Then $a \in P$ and $a \in Q$. Since $M$ is $k$-regular for $a \in M$, there exists $b \in M$ such that $a = a\gamma_1 b^k \gamma_2 a$ for every pair of non-zero elements $\gamma_1, \gamma_2 \in \Gamma$. $a \in (PTM)\Gamma Q \subseteq PTQ$. Hence $P \cap Q \subseteq PTQ$ and so $P \cap Q = PTQ$.

(ii) Taking $Q = P$ in (i), we get $P \cap P = PT^2$. i.e., $P = P^2$. □

Proposition 4.2.14. Let $M$ be a $k$-regular and $k$-pseudo commutative $\Gamma$-near-ring. Then $(M \gamma_1 a)(M \gamma_2 b) = (M \gamma_1 a) \cap (M \gamma_2 b) = M \gamma_1 a \gamma_2 b$ for every pair of non-zero elements $\gamma_1, \gamma_2 \in \Gamma$.

Proof. Since $M \gamma_1 a, M \gamma_2 b$ are $M$-$\Gamma$-subgroups of $M$, by Proposition 4.2.13, $(M \gamma_1 a) \cap (M \gamma_2 b) = (M \gamma_1 a) \Gamma (M \gamma_2 b)$. As $M \gamma_1 a \subseteq M$, $(M \gamma_1 a) \cap M = M \gamma_1 a = (M \gamma_1 a) \cap (M \gamma_1 a) = M \gamma_1 a \Gamma M \gamma_1 a \subseteq (M \gamma_1 a) \Gamma M$. Since $M \gamma_1 a$ is an ideal, $(M \gamma_1 a) \Gamma M \subseteq (M \gamma_1 a) = (M \gamma_1 a) \cap M$. Therefore $(M \gamma_1 a) \Gamma M \gamma_1 a \gamma_2 b = (M \gamma_1 a) \gamma_2 b = (M \gamma_1 a) \Gamma M \gamma_2 b = (M \gamma_1 a) \cap (M \gamma_2 b)$. □

Theorem 4.2.15. Let $M$ be a $k$-regular and $k$-pseudo commutative $\Gamma$-near-ring. Let $P$ be a proper $M$-$\Gamma$-subgroup of $M$. Then the following are equivalent:

(i) $P$ is a prime ideal;

(ii) $P$ is completely prime ideal.
Proof. (i) ⇒ (ii) Let \( a\gamma b \in P \) for every \( \gamma \in \Gamma \). By Proposition 4.2.14, \((M\gamma_1a)\Gamma(M\gamma_2b) = M\gamma_1a\gamma_2b \in M\Gamma P \subseteq P\). By Theorem 4.2.11, \( M\gamma_1a \) and \( M\gamma_2b \) are ideals of \( M \). Since \( P \) is prime, \( M\gamma_1a \subseteq P \) or \( M\gamma_2b \subseteq P \). Again by \( M \) is \( k \)-regular, for some \( x, y \in M \) and \( \gamma_1, \gamma_2 \in \Gamma \). \( a = a\gamma_1x^k\gamma_2a \in M\Gamma a \subseteq P \) and \( b = b\gamma_1x^k\gamma_2b \in M\Gamma b \subseteq P \). Therefore \( a \in P \) or \( b \in P \).

(ii) ⇒ (iii) Let \( A \) and \( B \) be ideals of \( M \) such that \( A\Gamma B \subseteq P \). (i.e) for any \( a, b \in M, \gamma \in \Gamma, a\gamma b \in A\Gamma B \subseteq P \). i.e., \( a\gamma b \in P \). Since \( P \) is completely prime ideal either \( a \in P \) or \( b \in P \). i.e., \( A \subseteq P \) or \( B \subseteq P \).

Theorem 4.2.16. Let \( M \) be a \( k \)-regular and \( k \)-pseudo commutative \( \Gamma \)-near-ring. For a proper \( M\Gamma \)-subgroup \( P \) of \( M \), the following are equivalent:

(i) \( P \) is prime;

(ii) \( P \) is a maximal ideal.

Proof. (i) ⇒ (ii) Let \( J \) be an ideal of \( M \) such that \( J \neq P \) and that \( P \subseteq J \subseteq M \). Let \( a \in J \setminus P \). Since \( M \) is \( k \)-regular there exists \( x \in M \) such that \( a = a\gamma_1x^k\gamma_2a \) for every \( \gamma_1, \gamma_2 \in \Gamma \). i.e., \( a = (x^k\gamma_2a)\gamma_1a \). Thus, for all \( m \in M, m\gamma a = m\gamma(x^k\gamma_2a^2), (m - m\gamma x^k\gamma_2a)\gamma a = 0 \). Since \( M \) has IFP, we get that \( (m - m\gamma x^k\gamma_2a)\gamma y\gamma_3a = 0 \) for all \( y \in M \). Consequently \( M\Gamma(M - M\gamma x^k\gamma_2a)\Gamma M\Gamma a = M\Gamma 0 = \{0\} \). Let \( b = M - M\gamma x^k\gamma_2a \). Then \( M\Gamma b\Gamma M\Gamma a = \{0\} \subseteq P \). Since \( P \) is prime, \( M\Gamma a \subseteq P \) or \( M\Gamma b \subseteq P \). If \( M\Gamma a \subseteq P \), then \( a = a\gamma_1x^k\gamma_2a \in P \) is a contradiction. Hence
$M\Gamma b \subseteq P$. This gives that $M\Gamma b \subseteq J$. Since $M$ is $k$-regular, there exists $y \in M$ such that $b = b\gamma_1 y^k \gamma_2 b, \gamma_1, \gamma_2 \in \Gamma, b \in M\Gamma b \subseteq J$. Therefore $b \in J$, i.e., $M - M \gamma_1 x^k \gamma_2 a \in J$. Since $a \in J, M \gamma x^k \gamma_2 a \in M\Gamma J \subseteq J$. Therefore $M \subseteq J$ and so $J = M$. Therefore $P$ is a maximal ideal.

(ii) $\Rightarrow$ (i) Proof is obvious.

Theorem 4.2.17. Let $M$ be a $k$-regular and $k$-pseudo commutative $\Gamma$-near-ring. Then the following are equivalent:

(i) $M$ is $P(1,2)$;

(ii) $M$ is subcommutative.

Proof. (i) $\Rightarrow$ (ii) Let $M$ be a $P(1,2)$ $\Gamma$-near-ring. Then $a\Gamma M = M\Gamma a^2$ for all $a \in M$. Let $x \in a\Gamma M = M\Gamma a^2$ for all $a \in M$. If $x \in a\Gamma M = M\Gamma a^2$, then $x = M\gamma a^2$ where $\gamma \in \Gamma, x = m\gamma a\gamma_1 a \in M\Gamma a$. Therefore $a\Gamma M \subseteq M\Gamma a$. Let $x \in M\Gamma a$. Then $x = m\gamma a, m \in M$ and $\gamma \in \Gamma$. As in Proposition 4.2.10, $a = b^k \gamma_1 a^2, \gamma_1 \in \Gamma$ and $b \in M$. Therefore $x = m\gamma (b^k \gamma_1 a^2) = (m\gamma b^k \gamma_1 a)\gamma_2 a, \gamma_2 \in \Gamma, x = (a\gamma_3 b^k \gamma_4 m)\gamma_2 a \in a\Gamma M$. From this $a\Gamma M = M\Gamma a$ and so $M$ is sub commutative.

(ii) $\Rightarrow$ (i) Let $a\Gamma M = M\Gamma a$ and $x \in a\Gamma M$. Then $x = a\gamma m, \gamma \in \Gamma, m \in M$, and $x = m_1 \gamma_1 a, \gamma_1 \in \Gamma, m \in M$. Hence $x = m_1 \gamma_1 (a\gamma_2 b^k \gamma_3 a)$ for some $\gamma_2, \gamma_3 \in \Gamma, b \in M$. Thus $x = m\gamma_1 (b^k \gamma_3 a)\gamma_2 a = m\gamma_1 b^k \gamma_3 a^2 \in M\Gamma a^2$. Therefore $M\Gamma a^2 = a\Gamma M$. i.e., $M$ is $P(1,2)$ $\Gamma$-near-ring. $\square$