Chapter 3

Generalized Permutable Gamma Near-rings

Throughout this chapter $M$ stands for a $\Gamma$-near-ring. $\Gamma$-near-rings were defined by Bh. Satyanarayana[26] and the left and right operator near-rings $L$ and $R$ respectively of a $\Gamma$-near-ring $M$ were constructed by G.L. Booth[6] using operator theoretic approach to $\Gamma$-near-rings. Actually with operator theory approach, G.L. Booth[6] provided examples of $\Gamma$-near-rings by taking $\Gamma$ to be collection of operators. Now we construct examples of $\Gamma$-near-rings by taking $\Gamma$ to be collection of different multiplicative operators on a group.

R. Balakrishnan, S. Silviya and T. Tamizh Chelvam[3] introduced the concept of $B_1$ near-rings. By analogy with these concepts, we introduce the notion of Generalized Right Permutable(GRP) $\Gamma$-near-rings. We say that a $\Gamma$-near-ring $M$ is a GRP $\Gamma$-near-ring if $MTa\Gamma b = MTb\Gamma a$ for all $a, b \in M$. In the first section of this chapter, we discuss some of
their properties and obtain a characterization of GRP $\Gamma$-near-ring. In the second section of this chapter, we introduce the notion of Generalized Left Permutable $\Gamma$-near-rings and discuss some of their properties and also obtain a characterization of generalized left permutable $\Gamma$-near-rings. Certain results in the first section of this chapter are published as a paper entitled “Generalized Right Permutable $\Gamma$-near-rings” and appeared in the “International Journal of Mathematical Sciences”.

### 3.1 GRP gamma near-rings

In this section, we introduce the notion of GRP $\Gamma$-near-ring and study some of its important properties. Also we obtain a characterization theorem for GRP $\Gamma$-near-ring.

**Definition 3.1.1.** A $\Gamma$-near-ring $M$ is said to be a **GRP $\Gamma$-near-ring** if $M\Gamma a\Gamma b = M\Gamma b\Gamma a$ for all $a, b \in M$.

Let us provide an example to substantiate our new notion. Up course one can construct many examples using the near-rings constructed on groups of small order available in [23, G. Pilz].

**Example 3.1.2.** Consider the $\Gamma$-near-ring defined on the Klein’s four group $\{0, a, b, c\}$ with $\Gamma = \{\gamma_1, \gamma_2\}$ where $\gamma_1, \gamma_2$ are given by the schemes $7: (0, 7, 11, 1)$ and $12: (0, 7, 0, 7)$ (see p.408 [23, G. Pilz]).
This is a GRP $\Gamma$-near-ring.

**Example 3.1.3.** Consider the $\Gamma$-near-ring $M = \{0, 1, 2, 3\}$ defined on the cyclic group $(\mathbb{Z}_4, +_4)$ with $\Gamma = \{\gamma_1, \gamma_2\}$ where $\gamma_1, \gamma_2$ are given by the schemes 11: $(0,1,3,2)$ and 12: $(0,3,0,3)$ (see p. 407 [23, G. Pilz]).
Proposition 3.1.4. Any homomorphic image of a GRP $\Gamma$-near-ring is again a GRP $\Gamma$-near-ring.

Proof. Follows from straightforward verification. □

Proposition 3.1.5. Any right permutable $\Gamma$-near-ring $M$ is a GRP $\Gamma$-near-ring.

Proof. Let $a, b \in M$. For $y \in M\Gamma a\Gamma b$, we have $y = m\gamma_1a\gamma_2b$ for some $m \in M$ and for every pair of non-zero elements $\gamma_1, \gamma_2 \in \Gamma$. Since $M$ is right permutable, $y = m\gamma_1b\gamma_2a \in M\Gamma b\Gamma a$. Therefore $M\Gamma a\Gamma b \subseteq M\Gamma b\Gamma a$. Similarly $M\Gamma b\Gamma a \subseteq M\Gamma a\Gamma b$. Hence $M\Gamma a\Gamma b = M\Gamma b\Gamma a$ for all $a, b \in M$. Thus $M$ is a GRP $\Gamma$-near-ring. □

Theorem 3.1.6. Let $M$ be a GRP $\Gamma$-near-ring. If $M$ is regular, then we have the following:

(i) for every $a \in M$, there exists $x \in M$ such that $a = a^2\Gamma x$;
(ii) $M$ has no non-zero nilpotent elements;

(iii) any two principal $M$-$\Gamma$-sub groups of $M$ commute with each other;

(iv) $a\Gamma M\Gamma a = a\Gamma M$ for every $a \in M$;

(v) $M\Gamma a = M\Gamma a^2$ for all $a \in M$.

**Proof.** (i) Since $M$ is a regular GRP $\Gamma$-near-ring, $a = a\gamma_1 x \gamma_2 a \in M\Gamma x \Gamma a = M\Gamma a\Gamma x$ for every pair of non-zero elements $\gamma_1, \gamma_2 \in \Gamma$. Therefore $a = m\Gamma a\Gamma x$ for some $m \in M$. This gives that $m\Gamma a = m\Gamma (a\Gamma x \Gamma a) = (m\Gamma a\Gamma x)\Gamma a = a\Gamma a = a^2$. In conclusion we have $a = m\Gamma a\Gamma x = a^2\Gamma x$.

(ii) Let $a \in M$. Suppose $a\Gamma a = a^2 = 0$. By (i), there exists $x \in M$ such that $a = a^2\Gamma x$ and therefore $a = 0$. By Proposition 2.2.19, $M$ has no non-zero nilpotent elements.

(iii) Let $y \in M\Gamma a\Gamma M$. Then $y = m\Gamma a\Gamma m'$ for some $m, m' \in M$. Since $M$ is a GRP $\Gamma$-near-ring, $m\Gamma a\Gamma m' = z\Gamma m' \Gamma a$ for some $z \in M$. Hence $y = z\Gamma m' \Gamma a = (z\Gamma m') \Gamma a \in M\Gamma a$. Also $M\Gamma a = M\Gamma a\Gamma x\Gamma a = M\Gamma a\Gamma (x\Gamma a) \subseteq M\Gamma a\Gamma M$. Therefore $M\Gamma a\Gamma M = M\Gamma a$. Let $b, c \in M$.

Now $M\Gamma b\Gamma M\Gamma c = (M\Gamma b \Gamma M)\Gamma c = (M\Gamma b)\Gamma c = M\Gamma c\Gamma b = (M\Gamma c)\Gamma b = (M\Gamma c\Gamma M)\Gamma b = M\Gamma c\Gamma M\Gamma b$. That is $M\Gamma b\Gamma M\Gamma c = M\Gamma c\Gamma M\Gamma b$ and (iii) follows.

(iv) Let $a \in M$. For any $y \in a\Gamma M$, there exists $z \in M$ such that $y = a\Gamma z = (a\Gamma x\Gamma a)\Gamma z = a\Gamma (x\Gamma a\Gamma z)$. Since $M$ is a GRP $\Gamma$-near-ring, $x\Gamma a\Gamma z = m\Gamma z\Gamma a$ for some $m \in M$. Hence $y = a\Gamma (m\Gamma z\Gamma a) = a\Gamma (m\Gamma z)\Gamma a \in a\Gamma M\Gamma a$ which implies that $a\Gamma M \subseteq a\Gamma M\Gamma a$. Clearly $a\Gamma M\Gamma a \subseteq a\Gamma M$ and hence $a\Gamma M\Gamma a = a\Gamma M$.  

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Theorem 3.1.7. Let $M$ be a GRP $\Gamma$-near-ring. If $M$ is Boolean, then the following are true:

(i) $M\Gamma a M\Gamma b = M\Gamma a b$ for all $a, b \in M$;

(ii) all principal $M$-$\Gamma$-sub groups of $M$ commute with one another;

(iii) every ideal of $M$ is a GRP $\Gamma$-near-ring;

(iv) every $M$-$\Gamma$-sub group of $M$ is a GRP $\Gamma$-near-ring;

(v) every $M$-$\Gamma$-sub group of $M$ is an invariant $M$-$\Gamma$-sub group of $M$.

Proof. (i) Let $a, b \in M$. Since $M$ is Boolean, $a = a\Gamma a \in a\Gamma M$ which implies that $M\Gamma a \subseteq M\Gamma a\Gamma M$. Hence $M\Gamma a b \subseteq M\Gamma a\Gamma M\Gamma b$. Let $y \in M\Gamma a \Gamma M\Gamma b$. Then $y = m\Gamma a \Gamma m' \Gamma b$ for some $m, m' \in M$. Since $M$ is a GRP $\Gamma$-near-ring, we get that $m\Gamma a \Gamma m' = z \Gamma m' \Gamma a$ for some $z \in M$. Thus $y = (z \Gamma m' \Gamma a) \Gamma b = (z \Gamma m') \Gamma a \Gamma b \in M\Gamma a \Gamma b$.

(ii) For $a, b \in M$, we have $M\Gamma a \Gamma M\Gamma b = M\Gamma a b = M\Gamma b \Gamma a = M\Gamma b \Gamma M\Gamma a$.

(iii) Let $I$ be any ideal of $M$ and $a, b \in I$. Now $I\Gamma a \Gamma b = I\Gamma a^2 \Gamma b = (I\Gamma a) \Gamma (a \Gamma b) \subseteq I\Gamma (M\Gamma a \Gamma b) = I\Gamma (M\Gamma b \Gamma a) \subseteq I\Gamma b \Gamma a$. That is $I\Gamma a \Gamma b \subseteq I\Gamma b \Gamma a$. Similarly we get $I\Gamma b \Gamma a \subseteq I\Gamma a \Gamma b$ and therefore $I$ is a GRP $\Gamma$-near-ring.
(iv) For any $M$-$\Gamma$-sub group of $N$, we have $M\Gamma N \subseteq N$. Let $x, y \in N$. Now $N\Gamma x\Gamma y \subseteq M\Gamma x\Gamma y = M\Gamma y\Gamma x = (M\Gamma y)\Gamma y\Gamma x \subseteq (M\Gamma N)\Gamma y\Gamma x \subseteq N\Gamma y\Gamma x$. Similarly we get $N\Gamma y\Gamma x \subseteq N\Gamma x\Gamma y$. Consequently $N$ is a GRP $\Gamma$-near-ring.

(v) Let $N$ be an $M$-$\Gamma$-sub group of $M$. For $z \in N\Gamma M$, $z = n\Gamma m$ for some $n \in N, m \in M$ which implies that $z = n^2\Gamma m = n\Gamma n\Gamma m = m\Gamma m\Gamma n$ for some $m' \in M$. Therefore $z = m\Gamma m\Gamma n \in M\Gamma N \subseteq N$ and hence $N\Gamma M \subseteq N$. i.e., $N$ is an invariant $M$-$\Gamma$-sub group of $M$. \hfill $\square$

**Proposition 3.1.8.** Let $M$ be a Boolean regular GRP $\Gamma$-near-ring and $A$ an $M$-$\Gamma$-subgroup of $M$. Then $A = \sqrt{A}$ where $\sqrt{A}$ is radical of $A$.

**Proof.** Let $a \in \sqrt{A}$. Then there exists some positive integer $k$ such that $a^k \in A$. By (i) of Theorem 3.1.6, $a = a^2\Gamma x$ for every $x \in M$. Now $a = a^2\Gamma x = a\Gamma a\Gamma x = a\Gamma (a^2\Gamma x)\Gamma x = a^3\Gamma x^2 = \ldots = a^k\Gamma x^{k-1} \in A\Gamma M \subset A$ (By (v) of Theorem 3.1.7). Therefore $\sqrt{A} \subset A$. Obviously $A \subset \sqrt{A}$. Hence $A = \sqrt{A}$. \hfill $\square$

**Proposition 3.1.9.** Let $M$ be a zero symmetric regular GRP $\Gamma$-near-ring. Then the following are true:

(i) $M$ has IFP;

(ii) any ideal of $M$ is semi completely prime;

(iii) for any ideal $I$ of $M, M$ has the property that for $a, b \in M$, if $a\Gamma b \subseteq I$ then $b\Gamma a \subseteq I$;
(iv) for every ideal $I$ of $M$, and $x_1, x_2, \ldots, x_n \in M$ if $x_1 \Gamma x_2 \ldots \Gamma x_n \subseteq I$ then $< x_1 > \Gamma < x_2 > \ldots \Gamma < x_{n-1} > \Gamma < x_n > \subseteq I$.

Proof. (i) Proof follows from (ii) of Theorem 3.1.6 and Proposition 2.2.31.

(ii) Let $I$ be an ideal of $M$ and $a^2 \in I$ for $a \in M$. By (i) of Theorem 3.1.6, for $a \in M$, there exists $x \in M$ such that $a = a^2 \Gamma x$. From this we have $a = a^2 \Gamma x \in I \Gamma M \subseteq I$. Therefore $I$ is semi completely prime.

(iii) Let $I$ be an ideal of $M$ and $a \Gamma b \subseteq I$ for $a, b \in M$. From this $(b \Gamma a)^2 = (b \Gamma a)(b \Gamma a) = b \Gamma (a \Gamma b) \Gamma a \subseteq M \Gamma I \Gamma M$. By Lemma 2.2.37, $M \Gamma I \Gamma M \subseteq I$ and hence by (ii), $b \Gamma a \subseteq I$.

(iv) Let $x_1 \Gamma x_2 \ldots \Gamma x_n \subseteq I$. It can be easily verified that $(I : S)_\Gamma$ is an ideal for any subset $S$ of $M$. Since $x_1 \in (I : x_2 \Gamma \ldots \Gamma x_n)_\Gamma$ we have $< x_1 > \subseteq (I : x_2 \Gamma \ldots \Gamma x_n)_\Gamma$ so that $< x_1 > \Gamma x_2 \ldots \Gamma x_n \subseteq I$. By (iii) we have $x_2 \Gamma \ldots \Gamma x_n \Gamma < x_1 > \subseteq I$. Now $x_2 \in (I : x_2 \Gamma \ldots \Gamma x_n \Gamma < x_1 >)_\Gamma$ which implies that $< x_2 > \subseteq (I : x_2 \Gamma \ldots \Gamma x_n \Gamma < x_1 >)_\Gamma$ and hence $< x_2 > \Gamma x_3 \Gamma \ldots \Gamma x_n \Gamma < x_1 > \subseteq I$ which implies that $x_3 \Gamma \ldots \Gamma x_n \Gamma < x_1 > \Gamma < x_2 > \ldots \Gamma < x_n > \subseteq I$. Continuing this process, we get $< x_1 > \Gamma < x_2 > \ldots \Gamma < x_n > \subseteq I$. \qed

Lemma 3.1.10. Let $M$ be a zero-symmetric reduced $\Gamma$-near-ring. If $M$ is regular, then every $M$-$\Gamma$-subgroup of $M$ is an ideal.

Proof. Let $a \in M$. Since $M$ is regular, for each $a \in M$ and for every pair of non-zero elements $\gamma_1, \gamma_2 \in \Gamma$, $a = a^2 \Gamma b \gamma_2 a$ for some $b \in M$. 29
By (i) of Proposition 2.2.34, $b\gamma_2 a$ is an idempotent. Let $b\gamma_2 a = e$. By (ii) of Proposition 2.2.34, $M\gamma_1 e = M\gamma_1 b\gamma_2 a = M\gamma_2 a$ for all $\gamma_1, \gamma_2 \in \Gamma$. Therefore $M\Gamma e = M\Gamma a$. Let $S = \{m - m\gamma e/m \in M, \gamma \in \Gamma\}$.

Since $(m - m\gamma_1 e)\gamma_1 e = 0$ for all $m \in M$, $(m - m\gamma_1 e)\gamma_1 M\gamma_2 e = 0$ [by Proposition 2.2.31]. This implies that $M\gamma_2 e \subseteq (0 : S)$ for all $\gamma_2 \in \Gamma$ and so $M\Gamma e \subseteq (0 : S)$. Now let $y \in (0 : S)$. Since $M$ is regular, $y = y\gamma_1 x\gamma_2 y$ for some $x \in M, \gamma_1, \gamma_2 \in \Gamma$ and so $y\gamma_1 x - (y\gamma_1 x)\gamma_2 e \in S$.

From this we have $(y\gamma_1 x - (y\gamma_1 x)\gamma_2 e)\gamma_2 y = 0$ and hence $y\gamma_1 x\gamma_2 y - y\gamma_1 (x\gamma_2 e\gamma_2 y) = 0$. i.e., $y - y\gamma_1 (x\gamma_2 e\gamma_2 y) = 0$ and by Proposition 2.2.35, $y - y\gamma_1 (x\gamma_2 y\gamma_2 e) = 0$ implies $y - (y\gamma_1 x\gamma_2 y)\gamma_2 e = 0$ implies $y - y\gamma_2 e = 0$. Hence $y = y\gamma_2 e \in M\Gamma e$. It follows that $(0 : S) \subseteq M\Gamma e$ and therefore $(0 : S) = M\Gamma e = M\Gamma a$. By Proposition 2.2.33, we get $M\Gamma a$ is an ideal of $M$. Now if $N$ is any $M$-$\Gamma$-subgroup of $M$ then $N = \sum_{a \in N} M\Gamma a$. Thus $N$ is an ideal of $M$. $\Box$

**Theorem 3.1.11.** Let $M$ be a zero-symmetric, Boolean, regular GRP $\Gamma$-near-ring and $P$ be a proper $M$-$\Gamma$-sub group of $M$. Then the following are equivalent:

(i) $P$ is a prime ideal;

(ii) $P$ is a completely prime ideal;

(iii) $P$ is a primary ideal;

(iv) $P$ is a maximal ideal.
Proof. (i) $\Rightarrow$ (ii) For $a, b \in M$, let $a \Gamma b \in P$. Then $M \Gamma a \Gamma M \Gamma b = M \Gamma a \Gamma b \subseteq M \Gamma P \subseteq P$. As argued in Lemma 3.1.10, $M \Gamma a$ and $M \Gamma b$ are ideals in $M$. Since $P$ is prime, $M \Gamma a \Gamma M \Gamma b \subseteq P \Rightarrow M \Gamma a \subseteq P$ or $M \Gamma b \subseteq P$. Since $M$ is regular, we get that, for some $x, y \in M, a = a \Gamma x \Gamma a \subseteq M \Gamma a \subseteq P$ or $b = b \Gamma y \Gamma b \subseteq M \Gamma b \subseteq P$. Therefore either $a \in P$ or $b \in P$.

(ii) $\Rightarrow$ (i) Obvious.

(ii) $\Rightarrow$ (iii) Since $M$ is a GRP $\Gamma$-near-ring, $M \Gamma a \Gamma b = M \Gamma b \Gamma a$ for all $a, b \in M$. For $a, b, c \in M, M \Gamma a \Gamma b \Gamma c = M \Gamma b \Gamma c \Gamma a = M \Gamma c \Gamma a \Gamma b = M \Gamma a \Gamma c \Gamma b = M \Gamma b \Gamma a \Gamma c = M \Gamma c \Gamma b \Gamma a$. If $a \Gamma b \Gamma c \subseteq P$ and $a \Gamma b \not\subseteq P$ then $c \in P$. On the other hand, if $a \Gamma b \Gamma c \subseteq P$ and $a \Gamma c \not\subseteq P$. Since $M$ is regular, $a \Gamma c \Gamma b \subseteq M \Gamma a \Gamma c \Gamma b = M \Gamma a \Gamma b \Gamma c \subseteq M \Gamma P \subseteq P$. Thus $a \Gamma c \Gamma b = (a \Gamma c) \Gamma b \subseteq P \Rightarrow b \in P$. Hence $P$ is primary.

(iii)$\Rightarrow$ (ii) For $a, b \in M$, let $a \Gamma b \subseteq P$ and $a \not\in P$. Since $M$ is regular, $a = a \Gamma x \Gamma a$ for some $x \in M$. First we claim that $x \Gamma a \not\subseteq P$. Suppose $x \Gamma a \subseteq P$ then $a = a \Gamma (x \Gamma a) \subseteq M \Gamma P \subseteq P$ which is a contradiction. Therefore $x \Gamma a \not\subseteq P$. Also $x \Gamma (a \Gamma b) \subseteq M \Gamma P \subseteq P$. Thus $x \Gamma a \Gamma b \subseteq P$ and $x \Gamma a \not\subseteq P$. As $P$ is primary ideal of $M, b^k \in P$ for some integer $k$. Now $b^k \in P \Rightarrow b \in \sqrt{P}$ and $\sqrt{P} = P$ which implies $b \in P$. Hence (ii) follows.

(i) $\Rightarrow$ (iv) Let $J$ be an ideal of $M$ such that $J \neq P$ and that $P \subseteq J \subseteq M$. Let $a \in J \setminus P$. Since $M$ is regular, by (i) of Theorem 3.1.6, there exists $x \in M$ such that $a = a^2 \Gamma x$. From this we have $m \Gamma a = m \Gamma a^2 \Gamma x = m \Gamma a \Gamma a \Gamma x = m \Gamma a \Gamma x \Gamma a$ which in turn gives that $(m - m \Gamma a \Gamma x) \Gamma a = 0 \Rightarrow (m - m \Gamma a \Gamma x) \Gamma y \Gamma a = 0 \Rightarrow M \Gamma (m - m \Gamma a \Gamma x) \Gamma M \Gamma a = 0$. Let $b =$
\[ m - m \Gamma a' \Gamma x. \] Then \( M \Gamma b \Gamma M a = 0 \subseteq P \Rightarrow M \Gamma a \subseteq P \) or \( M \Gamma b \subseteq P. \)

If \( M \Gamma a \subseteq P \) then \( M \Gamma b \subseteq J \Rightarrow b \in J \). i.e., \( m - m \Gamma a' \Gamma x \subseteq J. \) Since \( a' \in J, m \Gamma a' \Gamma x \subseteq J \Rightarrow m \in J. \) Therefore \( J = M. \)

(iv) ⇒ (i) Obvious.

We conclude this section with the following characterization theorem.

**Theorem 3.1.12.** Let \( M \) be a zero symmetric regular \( \Gamma \)-near-ring. Then the following are equivalent:

(i) \( M \) is a GRP \( \Gamma \)-near-ring;

(ii) \( M \) is reduced;

(iii) \( M \Gamma a \cap M \Gamma b = M \Gamma a \Gamma b \) for all \( a, b \in M. \)

**Proof.** (i) ⇒ (ii) Let \( M \) be a GRP \( \Gamma \)-near-ring. Since \( M \) is regular by (ii) of Theorem 3.1.6, \( M \) has no non-zero nilpotent elements and hence \( M \) is reduced.

(ii) ⇒ (iii) Let \( M \) be a reduced zero-symmetric regular \( \Gamma \)-near-ring. Let \( M \Gamma a \) and \( M \Gamma b \) be two \( M \)-\( \Gamma \)-subgroups of \( M \). By Lemma 3.1.10, \( M \Gamma a \) and \( M \Gamma b \) are ideals. Hence \( M \Gamma a \Gamma M \Gamma b \subseteq M \Gamma a \) and \( M \Gamma a \Gamma M \Gamma b \subseteq M \Gamma b. \) Therefore \( M \Gamma a \Gamma M \Gamma b \subseteq M \Gamma a \cap M \Gamma b. \) Let \( x \in M \Gamma a \cap M \Gamma b. \) Since \( M \) is regular, \( x = (x \Gamma y) \Gamma x \in (M \Gamma a \Gamma M) \Gamma M \Gamma b \subseteq M \Gamma a \Gamma M \Gamma b. \) Hence \( M \Gamma a \cap M \Gamma b \subseteq M \Gamma a \Gamma M \Gamma b \) and so \( M \Gamma a \cap M \Gamma b = M \Gamma a \Gamma M \Gamma b. \) Since \( M \Gamma a \subseteq M, M \Gamma a \cap M = M \Gamma a = M \Gamma a \cap M \Gamma a = M \Gamma a \Gamma M \Gamma a \subseteq (M \Gamma a) \Gamma M \subseteq M \Gamma a = M \Gamma a \cap M. \) Therefore \( M \Gamma a = M \Gamma a \cap M = \)
This implies that $M\Gamma a\Gamma b = (M\Gamma a)\Gamma b = (M\Gamma a\Gamma M)\Gamma b = M\Gamma a\Gamma M\Gamma b = M\Gamma a \cap M\Gamma b$.

(iii)$\Rightarrow$(i) Assume $M\Gamma a \cap M\Gamma b = M\Gamma a\Gamma b$ for all $a, b \in M$. Let $a, b \in M$. Now $M\Gamma a\Gamma b = M\Gamma a \cap M\Gamma b = M\Gamma b \cap M\Gamma a = M\Gamma b\Gamma a$. Thus $M$ is a GRP $\Gamma$-near-ring.

\[ \blacksquare \]

### 3.2 GLP gamma near-rings

In this section, we introduce GLP $\Gamma$-near-ring and study some of it’s important properties. Also we obtain a characterization theorem for GLP $\Gamma$-near-ring.

**Definition 3.2.1.** A $\Gamma$-near-ring $M$ is said to be a *Generalized Left Permutable* (GLP) $\Gamma$-near-ring if $a\Gamma b\Gamma M = b\Gamma a\Gamma M$ for all $a, b \in M$.

Let us give some examples for the new notion defined above.

**Example 3.2.2.** Consider the $\Gamma$-near-ring defined by the Klein’s four group $\{0, a, b, c\}$ with $\Gamma = \{\gamma_1, \gamma_2\}$ where $\gamma_1, \gamma_2$ are given by the schemes. $1 : (0,13,0,13)$ and $12 : (0,13,0,0)$ (see p. 408 [23, G. Pilz]).
This is a GLP $\Gamma$-near-ring.

The following proposition is immediate from the definition.

**Proposition 3.2.3.** Any homomorphic image of a GLP $\Gamma$-near-ring is a GLP $\Gamma$-near-ring.

**Theorem 3.2.4.** Let $M$ be a regular GLP $\Gamma$-near-ring. Then the following hold good.

(i) For every $a \in M$, there exists $x \in M$ such that $a = x\gamma a^2$ for all $\gamma \in \Gamma$;

(ii) For every $a, b \in M$, $a\Gamma M \Gamma b \Gamma M = b \Gamma M \Gamma a \Gamma M$;

(iii) $a\Gamma M \Gamma a = M \Gamma a$ for all $a \in M$;

(iv) $a\Gamma M = a^2 \Gamma M$ for all $a \in M$.

**Proof.** (i) Let $M$ be a regular GLP $\Gamma$-near-ring. Let $a \in M$. Since $M$ is regular, for every pair of elements $\gamma_1, \gamma_2 \in \Gamma$, there exists an element $x \in M$ such that $a = a\gamma_1 x\gamma_2 a = a\gamma_2 x\gamma_1 a$. Since $M$ is a GLP $\Gamma$-near-ring, there exists $m \in M$ such that $a = a\gamma_1 x\gamma_2 a = x\gamma_2 a \gamma_1 m$. 

\[
\begin{array}{cccc}
\gamma_2 & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & b & 0 & 0 \\
b & 0 & 0 & 0 & 0 \\
c & 0 & b & 0 & 0 \\
\end{array}
\]
Now $a^2 = a\gamma_1 a = a\gamma_1(x\gamma_2 a\gamma_1 m) = (a\gamma_1 x\gamma_2 a)\gamma_1 m = a\gamma_1 m$. Also $a^2 = a\gamma_2 a = a\gamma_2(x\gamma_1 a\gamma_2 m) = (a\gamma_2 x\gamma_1 a)\gamma_2 m = a\gamma_2 m$. (i.e) $a\gamma m = a^2$ for all $\gamma \in \Gamma$. Hence $a = x\gamma_1 a^2$ for all $\gamma_1 \in \Gamma$. i.e., $a = x\gamma a^2$ for all $\gamma \in \Gamma$.

(ii) Let $a, b \in M$. Now $y \in M\gamma \Gamma M$ implies $y = m\gamma_1 a\gamma_2 m'$ for some $m, m' \in M$ and for every pair of elements $\gamma_1, \gamma_2 \in \Gamma$. Since $M$ is a GLP $\Gamma$-near-ring, $m\gamma_1 a\gamma_2 m' = a\gamma_1 m\gamma_2 z$ for some $z \in M$ and for every pair of elements $\gamma_1$ and $\gamma_2 \in \Gamma$. Hence $y = m\gamma_1 a\gamma_2 m' = a\gamma_1 m\gamma_2 z = a\gamma_1 (m\gamma_2 z) \in a\gamma M$ and so $M\gamma a\gamma M \subseteq a\gamma M$. Since $M$ is regular, there exists an element $x \in M$ such that $a = a\gamma_1 x\gamma_2 a$ and this gives that $a\gamma M = (a\gamma_1 x\gamma_2 a)\gamma M = (a\gamma_1 x)\gamma_2 a\gamma M \subseteq M\gamma a\gamma M$. Therefore $a\gamma M = M\gamma a\gamma M$. For any $a, b \in M, a\gamma M b\gamma M = a\Gamma (M\gamma b\gamma M) = a\Gamma b\gamma M = b\Gamma a\gamma M = b\Gamma (a\gamma M) = b\Gamma M a\gamma M$. Thus $a\Gamma M b\gamma M = b\Gamma M a\gamma M$.

(iii) Let $a \in M$ and $y \in M\gamma a$. Then for any $\gamma \in \Gamma$, there exists $z \in m$ such that $y = z\gamma a$. Since $M$ is regular, there exists an $x \in M$ such that $a = a\gamma_1 x\gamma_2 a$ for all $\gamma_1, \gamma_2 \in \Gamma$. Now $y = z\gamma a = z\gamma (a\gamma_1 x\gamma_2 a)$ for any $\gamma_1, \gamma_2 \in \Gamma$. $y = (z\gamma a\gamma_1 x)\gamma_2 a$. That is, $y = (a\gamma z\gamma_1 m)\gamma_2 a = a\gamma (z\gamma_1 m)\gamma_2 a \in a\gamma M \gamma a$. Therefore $M\gamma a \subseteq a\gamma M \gamma a$. Clearly $a\gamma M \gamma a \subseteq M\gamma a$. Hence $a\gamma M \gamma a = M\gamma a$.

(iv) Let $a \in M$. By (i), there exists $x \in M$ such that $a = x\gamma a^2$ for all $\gamma \in \Gamma$. i.e., $a = x\Gamma a^2$. Now $a\gamma M = x\Gamma a^2 \gamma M = a^2 \Gamma x\gamma M \subseteq a^2 \gamma M$ and hence $a\gamma M = a^2 \gamma M$. \qed

**Proposition 3.2.5.** Every GLP $\Gamma$-near-ring $M$ is zero-symmetric.
Proof. For every $m \in M$ and $\gamma \in \Gamma$, $m\gamma 0 = m\gamma 0 \gamma 0 \in m\Gamma 0 \Gamma M = 0 \Gamma m \Gamma M = 0 \Gamma M = 0 \in \{0\}$. Hence $m\gamma 0 = 0$ for every $m \in M$ and $\gamma \in \Gamma$. \hfill \Box

Theorem 3.2.6. Let $M$ be a GLP $\Gamma$-near-ring.

(i) If $M$ is regular, then $M$ has no non-zero nilpotent elements.

(ii) If $M$ is Boolean, then every ideal of $M$ is a GLP $\Gamma$-near-ring.

Proof. (i) Suppose there exists an element $a \in M$ such that $a^2 = a\gamma a = 0$ for every $\gamma \in \Gamma$. By (i) of Theorem 3.2.4, there exists $x \in M$ such that $a = x\gamma a^2$ for all $\gamma \in \Gamma$. Therefore $a = x\gamma 0 = 0$. i.e., $M$ has no non-zero nilpotent elements.

(ii) Let $I$ be an ideal of $M$. Let $a, b \in I$. Now $a\Gamma b\Gamma I = a\Gamma (b\Gamma b)\Gamma I = a\Gamma b\Gamma b\Gamma I$. Therefore $a\Gamma b\Gamma I \subseteq a\Gamma b\Gamma M\Gamma I = (a\Gamma b\Gamma M)\Gamma I = (b\Gamma a\Gamma M)\Gamma I = b\Gamma a\Gamma (M\Gamma I) \subseteq b\Gamma a\Gamma I$. Similarly we get $b\Gamma a\Gamma I \subseteq a\Gamma b\Gamma I$. Consequently $I$ is a GLP $\Gamma$-near-ring. \hfill \Box

Proposition 3.2.7. If $M$ is a regular GLP $\Gamma$-near-ring, then $M$ has IFP.

Proof. Let $a\gamma b = 0$ for every $a, b \in M$ and for every $\gamma \in \Gamma$. Now $(b\gamma a)^2 = (b\gamma a)(b\gamma a) = b\gamma (a\gamma b)\gamma a = b\gamma 0\gamma a = b\gamma 0 = 0$. By (i) of Theorem 3.2.6, $b\gamma a = 0$ for every $\gamma \in \Gamma$. From this $(a\gamma_1 m\gamma_2 b)^2 = (a\gamma_1 m\gamma_2 b)(a\gamma_1 m\gamma_2 b) = a\gamma_1 m\gamma_2 (b\gamma a)\gamma_1 m\gamma_2 b = 0$. Again by (i) of Theorem 3.2.6, $a\gamma_1 m\gamma_2 b = 0$. Hence $M$ has IFP. \hfill \Box
Proposition 3.2.8. Let \( M \) be a GLP \( \Gamma \)-near-ring and \( S \subseteq M \) be any non-empty subset. Then \( I = (0 : S) = \{ x \in M : x \gamma S = \{ 0 \} \} \) is an ideal for all \( \gamma \in \Gamma \).

Proof. For any \( x, y \in I \), \((x + y) \Gamma S = x \Gamma S + y \Gamma S = \{ 0 \}\) implies that \( x + y \in I \). For \( x \in I \) and \( n \in M \), \((n + x - n) \gamma S = n \gamma S + x \gamma S - n \gamma S = n \gamma S - n \gamma S = 0\) implies \( n + x - n \in I \). Hence \( I \) is a normal subgroup. For \( x \in I \) and \( n \in M \), \((n \gamma(x + n') - n \gamma n') \gamma S = n \gamma(x + n') \gamma S - n \gamma n' \gamma S = n \gamma(x \gamma S + n' \gamma S) - n \gamma n' \gamma S = n \gamma(0 + n' \gamma S) - n \gamma n' \gamma S = \{ 0 \}\) implies that \( n \gamma(x + n') - n \gamma n' \in I \). Thus \( I \) is an ideal. \( \square \)

Proposition 3.2.9. Any right permutable \( \Gamma \)-near-ring \( M \) with left identity is a GLP \( \Gamma \)-near-ring.

Proof. Let \( a, b \in M \) and \( e \) be a left identity. Let \( y \in a \Gamma b \Gamma M \). Then \( y = a \gamma_1 b \gamma_2 m \) for every pair of non-zero elements \( \gamma_1, \gamma_2 \in \Gamma \). For every \( \gamma \in \Gamma \), \( y = e \gamma y = e \gamma a \gamma_1 b \gamma_2 m = e \gamma(a \gamma_1 m \gamma_2 b) = (e \gamma a \gamma_1 m) \gamma_2 b = (e \gamma m \gamma_1 a) \gamma_2 b = e \gamma(m \gamma_1 a \gamma_2 b) = e \gamma(m \gamma_1 b \gamma_2 a) = (e \gamma m \gamma_1 b) \gamma_2 a = (e \gamma b \gamma_1 m) \gamma_2 a = e \gamma(b \gamma_1 m \gamma_2 a) = e \gamma(b \gamma_1 a \gamma_2 m) = b \gamma_1 a \gamma_2 m \in b \Gamma a \Gamma M \). Hence \( a \Gamma b \Gamma M \subseteq b \Gamma a \Gamma M \). Similarly one can prove \( b \Gamma a \Gamma M \subseteq a \Gamma b \Gamma M \). Therefore \( M \) is a GLP \( \Gamma \)-near-ring. \( \square \)

Proposition 3.2.10. If \( M \) is a regular GLP \( \Gamma \)-near-ring then \( A = \sqrt{A} \) for every \( M \)-\( \Gamma \)-subgroup \( A \) of \( M \).
Proof. Let \( A \) be an \( M\)-\( \Gamma \)-subgroup of \( M \) and \( a \in \sqrt{A} \). Then there exists some positive integer \( k \) such that \( a^k \in A \). Since \( M \) is a regular GLP \( \Gamma \)-near-ring, by Theorem 3.2.4, there exists \( x \in M \) such that \( a = x\gamma a^2 \) for all \( \gamma \in \Gamma \). Hence \( a = x\gamma a\gamma a = x\gamma(x\gamma a^2)\gamma a = (x\gamma x)\gamma(a^2\gamma a) = x^2\gamma a^3 = \ldots = x^{(k-1)}\gamma a^k \in M\Gamma A \subseteq A \). Therefore \( \sqrt{A} \subseteq A \). Obviously \( A \subseteq \sqrt{A} \). 

\[ \square \]

Theorem 3.2.11. Let \( M \) be a GLP \( \Gamma \)-near-ring. If \( M \) is Boolean, then the following are true:

(i) \( a\Gamma M\Gamma b\Gamma M = a\Gamma b\Gamma M \) for all \( a, b \in M \);

(ii) \( a\Gamma M\Gamma b\Gamma M = b\Gamma M\Gamma a\Gamma M \) for all \( a, b \in M \);

(iii) Every invariant \( M\)-\( \Gamma \)-sub group of \( M \) is a GLP \( \Gamma \)-near-ring;

(iv) Every \( M\)-\( \Gamma \)-sub group of \( M \) is an invariant \( M\)-\( \Gamma \)-sub group of \( M \) if \( M \) is subcommutative.

Proof. (i) Let \( a, b \in M \). Since \( M \) is Boolean, \( b = b\Gamma b \in M\Gamma b \) implies \( b\Gamma M \subseteq M\Gamma b\Gamma M \) which in turn implies \( a\Gamma b\Gamma M \subseteq a\Gamma M\Gamma b\Gamma M \). Let \( y \in a\Gamma M\Gamma b\Gamma M \). Then \( y = a\gamma_1 m\gamma_2 b\gamma_3 m' \) for some \( m, m' \in M \) and for every \( \gamma_1, \gamma_2, \gamma_3 \in \Gamma \). Since \( M \) is a GLP \( \Gamma \)-near-ring, there exists an element \( z \in M \) such that \( m\gamma_2 b\gamma_3 m' = b\gamma_2 m\gamma_3 z \) for every pair of elements \( \gamma_2, \gamma_3 \in \Gamma \). Therefore, \( y = a\gamma_1 m\gamma_2 b\gamma_3 m' = a\gamma_1 b\gamma_2 m\gamma_3 z \in a\Gamma b\Gamma M \).

(ii) For \( a, b \in M \), \( a\Gamma M\Gamma b\Gamma M = a\Gamma b\Gamma M = b\Gamma a\Gamma M = b\Gamma M\Gamma a\Gamma M \).

(iii) Let \( N \) be an invariant \( M\)-\( \Gamma \)-sub group of \( M \). Therefore \( N\Gamma M \subseteq N \). Let \( a, b \in N \). Consider \( a\Gamma b\Gamma N \subseteq a\Gamma b\Gamma M = b\Gamma a\Gamma M = b\Gamma a^2\Gamma M =
\[(b \Gamma a) \Gamma (a \Gamma M) \subseteq b \Gamma a \Gamma (N \Gamma M) \subseteq b \Gamma a \Gamma N.\] Similarly we get \[b \Gamma a \Gamma N \subseteq a \Gamma b \Gamma N.\] Consequently \(N\) is a \(GLP \Gamma\)-near-ring.

(iv) Let \(N\) be an \(M-\Gamma\)-sub group of \(M\). Let \(z \in N \Gamma M, z = n \gamma m = x \gamma n^2 \gamma m = x \gamma n \gamma m\) for some \(n \in N, \gamma \in \Gamma\) and \(x, m \in M\). Since \(M\) is a subcommutative \(\Gamma\)-near-ring, we have \(z = x \gamma n \gamma m = x \gamma m' \gamma n\) where \(m' \in M\). Therefore \(z \in M \Gamma N \subseteq N\) and hence \(N \Gamma M \subseteq N\). i.e., \(N\) is an invariant \(M-\Gamma\)-sub group of \(M\).

We conclude this section, with the following characterization theorem.

**Theorem 3.2.12.** Let \(M\) be a Boolean \(\Gamma\)-near-ring. Then \(M\) is a \(GLP \Gamma\)-near-ring if and only if \(a \Gamma M \cap b \Gamma M = a \Gamma b \Gamma M\) for every \(a, b \in M\).

**Proof.** For the 'if' part, let \(a, b \in M\). Now \(a \Gamma b \Gamma M = a \Gamma M \cap b \Gamma M = b \Gamma M \cap a \Gamma M = b \Gamma a \Gamma M\). Thus \(M\) is a \(GLP \Gamma\)-near-ring.

For the only if part, let \(y \in a \Gamma M \cap b \Gamma M\). Then \(y = a \gamma_1 m = b \gamma_2 m'\) for some \(m, m' \in M\) and for all \(\gamma_1, \gamma_2 \in \Gamma\). Since \(M\) is a \(GLP \Gamma\)-near-ring, there exists \(z \in M\) such that \(m \gamma_1 b \gamma_2 m' = b \gamma_1 m \gamma_2 z\). Then \(y^2 = y \gamma y = (a \gamma_1 m) \gamma (b \gamma_2 m') = (a \gamma_1 m \gamma b) \gamma_2 m' = a \gamma_1 (m \gamma b \gamma_2 m') = a \gamma_1 (b \gamma m \gamma_2 z') = (a \gamma_1 b) \gamma m \gamma_2 z' \in a \Gamma b \Gamma M\). Thus \(a \Gamma M \cap b \Gamma M \subseteq a \Gamma b \Gamma M\). Since \(M\) is a \(GLP \Gamma\)-near-ring, \(a \Gamma b \Gamma M = b \Gamma a \Gamma M\). Therefore \(a \Gamma b \Gamma M = b \Gamma a \Gamma M \subseteq b \Gamma M\) and \(b \Gamma a \Gamma M = a \Gamma b \Gamma M \subseteq a \Gamma M\). Hence \(a \Gamma b \Gamma M \subseteq a \Gamma M \cap b \Gamma M\). i.e., \(a \Gamma M \cap b \Gamma M = a \Gamma b \Gamma M\) for every \(a, b \in M\). \(\Box\)