Chapter 2

Review of Literature

In this chapter, we put forth all developments in the past related to problems attempted in this dissertation.

2.1 Notations

First let us list out the notations used in this dissertation.

1. $M$ stands for a $\Gamma$-near-ring, where $\Gamma$ is a non-empty set of binary operators on $M$

2. $M_0$ denotes zero-symmetric part of $M$

3. $E$ denotes the set of all idempotents in $M$

4. $L$ denotes the set of all nilpotent elements of $M$

5. $C(M)$ denotes the center of $M$

6. $\langle A \rangle$ denotes the ideal generated by $A$

7. $\langle x \rangle$ denotes the ideal generated by $x$
8. $\sqrt{A}$ stands for the radical of $A$

9. $x^m$ means $x\gamma_1 x\gamma_2 \ldots x\gamma_{m-1} x$ where $\gamma_i \in \Gamma, 1 \leq i \leq m - 1$

10. $(A : B)_{\gamma} = \{x \in M/x\gamma B \subseteq A\}$

11. $A_s(M)$ denotes the right annihilator of $M$

12. $r(x) = \{y \in M/x\gamma y = 0 \text{ for all } \gamma \in \Gamma\}$

### 2.2 Basic definitions and results

In this section, we give all basic definitions and results which are used in subsequent chapters. We collect all the basic concepts and results in near-rings from [23, G. Pilz] and in $\Gamma$-near-rings from [26, Bh. Satyanarayana] and [6, G.L. Booth]. We also furnish proofs for certain results, which are frequently used in our work.

**Definition 2.2.1.** [23, G. Pilz] A right near-ring is a set $N$ together with two binary operations ‘+’ and ‘.’ such that

(i) $(N, +)$ is a group (not necessarily abelian);

(ii) $(N, .)$ is a semi-group;

(iii) For all $x, y, z$ in $N$, $(x + y).z = x.z + y.z$.

**Remark 2.2.2.** Throughout this dissertation, by a near-ring, we mean only a right near-ring. We write $xy$ for $x.y$ for any two elements $x, y \in N$. It can easily proved that $0a = 0$ and $(-a)b = -(ab)$ for all $a, b \in N$. 
Definition 2.2.3. [26, Bh. Satyanarayana] A \( \Gamma \)-near-ring is a *triple* \( (M, +, \Gamma) \) where

(i) \( (M, +) \) is a (not necessarily abelian) group;

(ii) \( \Gamma \) is a non-empty set of binary operators on \( M \) such that for each \( \alpha \in \Gamma \), \( (M, +, \alpha) \) is a right near-ring;

(iii) \( x\alpha(y\beta z) = (x\alpha y)\beta z \) for all \( x, y, z \in M \) and \( \alpha, \beta \in \Gamma \).

Example 2.2.4. Let \( M = \{0, a, b, c\} \). Let \( \Gamma = \{\gamma_1, \gamma_2\} \) where \( \gamma_1, \gamma_2 \) are defined by the schemes 10: \( (0, 0, 0, 1) \) and 3: \( (0, 0, 1, 1) \)(see p.408 [23]).

\[
\begin{array}{cccc}
\gamma_1 & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & b \\
c & 0 & 0 & 0 & c \\
\gamma_2 & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & a & a \\
b & 0 & 0 & b & b \\
c & 0 & 0 & c & c \\
\end{array}
\]

Then \( M \) is a \( \Gamma \)-near-ring.

**Definition 2.2.5.** A \( \Gamma \)-near-ring \( M \) is said to be *zero-symmetric* if \( a\gamma_0 = 0 \) for all \( a \in M \) and for all \( \gamma \in \Gamma \).
Definition 2.2.6. An element \( a \in M \) is called Boolean if \( a\gamma a = a \) for all \( \gamma \in \Gamma \).

Notation 2.2.7. For \( x \in M \) and a positive integer \( m \), by \( x^m \) we mean \( x\gamma_1 x\gamma_2 \ldots x\gamma_{m-1} x \) where \( \gamma_i \in \Gamma \) for \( 1 \leq i \leq m - 1 \).

Definition 2.2.8. An element \( 0 \neq a \in M \) is called nilpotent if there exists a positive integer \( n \geq 1 \) such that \( (a\gamma)^n a = 0 \) for each \( \gamma \in \Gamma \).

Definition 2.2.9. A \( \Gamma \)-near-ring \( M \) is called reduced if it has no non-zero nilpotent elements.

Definition 2.2.10. \( \Gamma \)-near-ring \( M \) is called regular if for each \( a \in M \) and for every pair of non-zero elements \( \gamma_1, \gamma_2 \in \Gamma \), \( a = a\gamma_1 b\gamma_2 a \) for some \( b \in M \).

Definition 2.2.11. A \( \Gamma \)-near-ring \( M \) is said to fulfill the insertion-of-factors property (IFP) provided that for all \( a, b, n \in M \), \( a\gamma b = 0 \) for all \( \gamma \in \Gamma \) implies \( a\gamma_1 n\gamma_2 b = 0 \) for every pair of non-zero elements \( \gamma_1, \gamma_2 \) of \( \Gamma \).

Definition 2.2.12. A \( \Gamma \)-near-ring \( M \) is said to be

(i) right permutable if \( x\gamma_1 (y\gamma_2 z) = x\gamma_1 (z\gamma_2 y) \) for all \( x, y, z \in M \) and for every pair of non-zero elements \( \gamma_1, \gamma_2 \) of \( \Gamma \);
(ii) left permutable if \((x\gamma_1 y)\gamma_2 z = (y\gamma_1 x)\gamma_2 z\) for all \(x, y, z \in M\) and for every pair of non-zero elements \(\gamma_1, \gamma_2\) of \(\Gamma\).

**Definition 2.2.13.** A non empty subset \(I\) of \(M\) is called

(i) a left ideal of \(M\) if \(I\) is normal subgroup of \((M, +)\) and for all \(a \in I, x, y \in M\) and \(\gamma \in \Gamma, x\gamma(a + y) - x\gamma y \in I;\)

(ii) a right ideal of \(M\) if \(I\) is a normal subgroup of \((M, +)\) and for all \(a \in I, x \in M\) and \(\gamma \in \Gamma, a\gamma x \in I;\)

(iii) an ideal of \(M\) if \(I\) is both left and right ideal of \(M\).

**Definition 2.2.14.** [6, G.L. Booth] Let \(P\) be an ideal of \(M\). Then \(P\) is called a prime ideal if for all ideals \(A, B\) of \(M, A\Gamma B \subseteq P\) implies \(A \subseteq P\) or \(B \subseteq P\).

**Definition 2.2.15.** An ideal \(P\) of \(M\) is called

(i) a completely prime ideal if for any \(a, b \in M, a\Gamma b \subseteq P\) implies either \(a \in P\) or \(b \in P;\)

(ii) a completely semi prime if \(a^2 (= a\Gamma a) \subseteq P\) implies \(a \in P.\)

**Definition 2.2.16.** An ideal \(P\) of \(M\) is called a primary ideal if \(a\gamma_1 b\gamma_2 c \in P\) and \(a\gamma_1 c \notin P\) for every pair of non-zero elements \(\gamma_1, \gamma_2\) of \(\Gamma\) there exists a \(k > 0\) such that \(b^k \in P.\)
Definition 2.2.17. Let $P$ be an ideal of $M$. Then $P$ is called a \textit{maximal ideal} (\textit{minimal ideal}) if it is maximal (minimal) in the set of all non-zero ideals of $M$.

Definition 2.2.18. An element $x \in M$ is said to be \textit{central} if $x \gamma y = y \gamma x$ for all $y \in M$.

Proposition 2.2.19. [22, N.H. McCoy] $M$ has no non-zero nilpotent elements if and only if $a \gamma a = 0$ implies $a = 0$ for all $\gamma \in \Gamma$.

Definition 2.2.20. A non-empty subset $A$ of $M$ is called a \textit{left $M$-$\Gamma$-subgroup} (simply an $M$-$\Gamma$-subgroup) of $M$ if $A$ is a subgroup of $(M, +)$ and $M \Gamma A \subseteq A$.

Definition 2.2.21. An $M$-$\Gamma$-subgroup $A$ of $(M, +)$ is called \textit{invariant} if $A \Gamma M \subseteq A (A$ is a right $M$-$\Gamma$-subgroup).

Definition 2.2.22. $M$ is said to have \textit{strong IFP} if for all ideals $I$ of $M$ and for all $a, b, n \in M, a \gamma b \in I$ for all $\gamma \in \Gamma$ implies $a \gamma_1 n \gamma_2 b \in I$ for every pair of non-zero elements $\gamma_1, \gamma_2 \in \Gamma$.

Definition 2.2.23. Let $P$ be an ideal of $M$. If $P$ has strong IFP then the ideal $P$ is called \textit{IFP-ideal of $M$}.
Definition 2.2.24. $M$ is said to have *IFP if $M$ has IFP and $x\gamma y = 0$ implies $y\gamma x = 0$ for every $x, y \in M$ and $\gamma \in \Gamma$.

Definition 2.2.25. An element $a \in M$ is called idempotent if $a\gamma a = a$ for all $\gamma \in \Gamma$.

Definition 2.2.26. (i) If $A \subseteq M$, then the smallest ideal containing $A$ is said to be an ideal generated by $A$ and is denoted by $< A >$.

(ii) If $x \in M$, the ideal generated by $x$ is the smallest ideal of $M$ containing $x$. (i.e.), the intersection of all ideals of $M$ containing $x$ and will be denoted by $< x >$.

Definition 2.2.27. For $A \subseteq M$, we define the radical $\sqrt{A}$ of $A$ to be $\{x \in M/x^k \in A \text{ for some positive integer } k\}$.

Definition 2.2.28. Let $A$ and $B$ be any two subsets of $M$. Let $\gamma \in \Gamma$, then $(A : B)_\gamma = \{x \in M/x\gamma B \subseteq A\}$.

Definition 2.2.29. Let $M$ and $M'$ be $\Gamma$-near-rings (for the same $\Gamma$) and let $f : M \rightarrow M'$ be a group homomorphism. Then if $f(x\gamma y) = f(x)\gamma f(y)$ for all $x, y \in M$ and $\gamma \in \Gamma$, $f$ is called a $\Gamma$-near-ring homomorphism.
Remark 2.2.30. The concepts of $\Gamma$-near-ring epimorphism, monomorphism and isomorphism etc are defined in the usual way.

Proposition 2.2.31. [33, T. Tamizh Chelvam and N. Meenakumari] Let $M$ be a zero-symmetric $\Gamma$-near-ring without non-zero nilpotent elements. Then $M$ has IFP.

Proof. If $x\gamma y = 0$ for $x, y \in M$ and $\gamma \in \Gamma$, then $(y\gamma x)^2 = y\gamma 0 = 0$. This implies that $y\gamma x = 0$. For every pair of non-zero elements $\gamma_1, \gamma_2 \in \Gamma$ and $n \in M$, $(x\gamma_1 n\gamma_2 y)^2 = (x\gamma_1 n\gamma_2 y)(x\gamma_1 n\gamma_2 y) = x\gamma_1 n\gamma_2 (y\gamma x)\gamma_1 n\gamma_2 y = x\gamma_1 n\gamma_2 0\gamma_1 n\gamma_2 y = x\gamma_1 n\gamma_2 0 = 0$ and hence $x\gamma_1 n\gamma_2 y = 0$. Therefore $M$ has IFP. $\square$

Proposition 2.2.32. [20, N. Meenakumari] Let $M$ be a zero-symmetric $\Gamma$-near-ring. If $M$ is Boolean, then $M$ has IFP.

Proposition 2.2.33. [20, N. Meenakumari] Let $M$ be a zero-symmetric $\Gamma$-near-ring with IFP property. Then $(0 : S) \Gamma = \{x \in M/x\Gamma S = \{0\}\}$ is an ideal where $S$ is any non-empty subset of $M$.

Proof. Let $I = (0 : S) \Gamma = \{x \in M/x\Gamma S = \{0\}\}$. For $x, y \in I$, $(x + y)\Gamma S = x\Gamma S + y\Gamma S = \{0\}$ implies that $x + y \in I$. For $x \in I, n \in M, (n + x - n)\Gamma S = n\Gamma S + x\Gamma S - n\Gamma S = \{0\}$ implies $n + x - n \in I$. Hence $I$ is a normal subgroup. For $x \in I$ and $n \in M$, since $M$ has IFP property, $x\Gamma n\Gamma S = \{0\}$ implies $x\Gamma n \subseteq I$. For $x \in I, n, n' \in M, (n\Gamma(n' +
\( x - n \Gamma n') \Gamma S = n \Gamma (n' + x) \Gamma S - n \Gamma n' \Gamma S = n \Gamma (n' \Gamma S + x \Gamma S) - n \Gamma n' \Gamma S = n \Gamma (n' \Gamma S + 0) - n \Gamma n' \Gamma S = n \Gamma n' \Gamma S - n \Gamma n' \Gamma S = \{0\} \) which implies that \( n \Gamma (n' + x) - n \Gamma n' \in I \). Thus \( I \) is an ideal. \( \square \)

**Proposition 2.2.34.** [20, N. Meenakumari] Let \( M \) be a regular \( \Gamma \)-near-ring, \( a \in M \) and \( a = a \gamma_1 x \gamma_2 a \) for \( x \in M \) and for every pair of non-zero elements \( \gamma_1, \gamma_2 \in \Gamma \). Then for all \( \gamma \in \Gamma \),

(i) \( a \gamma_1 x, x \gamma_2 a \) are idempotent elements in \( M \) for every pair of \( \gamma_1, \gamma_2 \in \Gamma \);

(ii) \( a \gamma_1 x \gamma_2 M = a \gamma_1 M \) and \( M \gamma_1 x \gamma_2 a = M \gamma_2 a \) for every pair of non-zero elements \( \gamma_1, \gamma_2 \in \Gamma \).

**Proof.** (i) \((x \gamma_2 a)^2 = (x \gamma_2 a) \gamma_1 (x \gamma_2 a) = x \gamma_2 (a \gamma_1 x \gamma_2 a) = x \gamma_2 a\). Similarly, one can prove that \( a \gamma_1 x \) is an idempotent for all \( \gamma_1 \in \Gamma \).

(ii) Trivially \( M \gamma_2 a = M \gamma_2 a \gamma_1 x \gamma_2 a \subseteq M \gamma_1 x \gamma_2 a \subseteq M \gamma_2 a \) and hence \( M \gamma_1 x \gamma_2 a = M \gamma_2 a \). Similarly one can prove that \( a \gamma_1 x \gamma_2 M = a \gamma_1 M \). \( \square \)

**Proposition 2.2.35.** [20, N. Meenakumari] Let \( M \) be a zero-symmetric reduced \( \Gamma \)-near-ring. For any \( a, b \in M \), and \( e \) an idempotent, \( a \gamma_1 b \gamma_2 e = a \gamma_2 e \gamma_1 b \) for every pair of non-zero elements \( \gamma_1, \gamma_2 \in \Gamma \).

**Proof.** Let \( e \) be an idempotent in \( M \), \( a, b \in M \) and \( \gamma_1, \gamma_2 \) a pair of non-zero elements in \( \Gamma \). Since \((a - a \gamma_2 e) \gamma_2 e = 0\), we have \((a - a \gamma_2 e) \gamma_1 b \gamma_2 e = 0\) so that \( a \gamma_1 b \gamma_2 e = a \gamma_2 e \gamma_1 b \gamma_2 e \).
Since \((e\gamma_1b - e\gamma_1b\gamma_2e)\gamma_2e = 0\), we have \(e\gamma_1b\gamma_2(e\gamma_1b - e\gamma_1b\gamma_2e)\gamma_2e = 0\) and \(e\gamma_1b\gamma_2\gamma_2(e\gamma_1b - e\gamma_1b\gamma_2e) = 0\) so that \((e\gamma_1b - e\gamma_1b\gamma_2e)^2 = 0\). Hence \(e\gamma_1b = e\gamma_1b\gamma_2e\). Thus \(a\gamma_1b\gamma_2e = a\gamma_2e\gamma_1b\). \(\square\)

**Proposition 2.2.36.** [20, N. Meenakumari] If \(M\) is a zero-symmetric \(\Gamma\)-near-ring, then every left ideal of \(M\) is a \(M\)-\(\Gamma\)-subgroup of \(M\).

**Proof.** Since \(M\) is zero-symmetric, \(M = M_0\). Let \(L\) be a left ideal of \(M\). For any \(m_0 \in M_0, l \in L\) and \(\gamma \in \Gamma, m_0\gamma l = m_0\gamma(l + 0) - m_0\gamma0 \in L\). Therefore \(M\Gamma L \subseteq L\) and hence \(M\) is a \(M\)-\(\Gamma\)-subgroup of \(M\). \(\square\)

**Lemma 2.2.37.** If \(M\) is a zero-symmetric \(\Gamma\)-near-ring, then for any ideal \(I\) of \(M, M\Gamma I \Gamma M \subseteq I\).

**Proof.** For any \(i \in I, m, m' \in M\) and \(\gamma \in \Gamma, m\gamma(i + m') - m\gamma m' \in I\). Substituting \(m' = 0\) we get \(M\Gamma I \subseteq I\). Also \(I\Gamma M \subseteq I\). Hence \(M\Gamma I \Gamma M \subseteq I\Gamma M \subseteq I\). \(\square\)

**Definition 2.2.38.** A near-ring \(N\) is said to be strong \(B_1\) near-ring if \(Nab = Nba\) for all \(a, b \in N\).

**Theorem 2.2.39.** [3, R. Balakrishnan, S. Silviya and T. Tamizh Chelvam] Let \(N\) be a Boolean near-ring. Then \(N\) is a strong \(B_1\) near-ring if and only if \(Na \cap Nb = Nab\) for all \(a, b \in N\).
2.3 Special properties of gamma near-rings

In this section, we discuss some special properties of gamma near-rings.

**Definition 2.3.1.** A $\Gamma$-near-ring $M$ is said to be *subcommutative* if $x\Gamma M = M\Gamma x$ for all $x \in M$.

**Definition 2.3.2.** [37, Young UK Cho]

(i) A $\Gamma$-near-ring $M$ is called *left unital $\Gamma$-near-ring* if $x \in x\Gamma M$ and *right unital $\Gamma$-near-ring* if $x \in M\Gamma x$ for all $x \in M$.

(ii) A $\Gamma$-near-ring $M$ is said to be *unital* if it is both left unital as well as right unital.

**Definition 2.3.3.** [36, Yong UK Cho, et. al] A $\Gamma$-near-ring $M$ is said to be a $P(1, 2)$ $\Gamma$-near-ring if $x\Gamma M = M\Gamma x^2$ for all $x \in M$.

**Definition 2.3.4.** A $\Gamma$-near-ring $M$ is said to be *left bi-potent* if $M\Gamma a = M\Gamma a^2$ for all $a \in M$.

**Definition 2.3.5.** [18, G. Mason] A $\Gamma$-near-ring $M$ is said to be a

(i) *left strongly regular* (simply strongly regular) if for every $a \in M$, $a = x\gamma a^2$ for some $x \in M$ and for every pair of non-zero elements $\gamma_1, \gamma_2 \in \Gamma$. 
(ii) right strongly regular if for every $a \in M$, $a = a^2 \gamma x$ for some $x \in M$
and for every pair of non-zero elements $\gamma_1, \gamma_2 \in \Gamma$.

Definition 2.3.6. [19, G. Mason] A near-ring $N$ is said to be strongly regular if any of the following equivalent conditions is satisfied:

(i) For any $a \in N$, there exists $x \in N$ such that $a = xa^2$;

(ii) For any $a \in N$, there exists $x \in N$ such that $a = xa^2$ and $a = axa$;

(iii) For any $a \in N$, there exists $y \in N$ such that $a = a^2 y$ and $a = aya$;

(iv) For any $a \in N$, there exists $x, y \in N$ such that $a^2 y = xa^2$.

Definition 2.3.7. A near-ring $N$ is said to be pseudo commutative near-ring if $xyz = zyx$ for all $x, y, z \in N$.

Theorem 2.3.8. [35, S. Uma, R. Balakrishnan and T. Tamizh Chelvam] Let $N$ be a regular pseudo commutative near-ring and $P$ a proper $N$-subgroup of $N$. Then the following are equivalent:

(i) $P$ is a prime ideal;

(ii) $P$ is a completely prime ideal;

(iii) $P$ is primary ideal;

(iv) $P$ is a maximal ideal.
Definition 2.3.9. A \( \Gamma \)-near-ring \( M \) is called \( b \)-regular if, for every \( a \in M \), \( a \in (a)_r \Gamma M (a)_l \) where \( (a)_r \) and \( (a)_l \) are the right and left \( M-\Gamma \)-subgroups respectively generated by \( a \in M \).

Definition 2.3.10. Let \( N \) be a near-ring with unity and \( P \) a right ideal of \( N \). Then the near-ring \( N \) is said to be a \( P \)-regular near-ring if, for each \( a \in N \), there exists \( x \in N \) such that \( axa - a \in P \) and \( ax P \subseteq P \).

Definition 2.3.11. An element \( e \in M \) is called left (right) identity in \( M \) if \( e \gamma m = m (m \gamma e = m) \) for all \( m \in M \) and for all \( \gamma \in \Gamma \).

Definition 2.3.12. (i) An element \( a \in M \) is called a left zero-divisor if \( a \gamma b = 0 \) for some \( \gamma \in \Gamma \) implies \( a = 0 \);

(ii) An element \( b \in M \) is called a right zero-divisor if \( a \gamma b = 0 \) for some \( \gamma \in \Gamma \) implies \( b = 0 \).

Definition 2.3.13. A \( \Gamma \)-near-ring \( M \) is said to be without zero divisors if \( a \gamma b = 0 \) for some \( \gamma \in \Gamma \), then either \( a = 0 \) or \( b = 0 \).

Lemma 2.3.14. [25, Y.V. Reddy & C.V.L.N. Murty] Suppose that \( N \) is a regular IFP near-ring. Then the following are true:

(i) \( a \in Na^2 \cap a^2 N \) for all \( a \in N \);

(ii) \( N \) has no non-zero nilpotent elements;

(iii) \( N \) has the strong IFP.
Definition 2.3.15. A $\Gamma$-near-ring $M$ is called $\alpha_1\Gamma$-near-ring if for every $a \in M$ and for every pair of non-zero elements $\gamma_1, \gamma_2 \in \Gamma$ there exists $x \in M$ such that $a = x\gamma_1a\gamma_2x$.

Definition 2.3.16. A $\Gamma$-near-ring $M$ is called irreducible (simple) if it contains only the trivial left $M\Gamma$-subgroups (ideals) $(0)$ and $M$.

Definition 2.3.17. An ideal $P$ of $M$ is called strictly prime if for any two left $M\Gamma$-subgroups $A$ and $B$ of $M$ such that $A\Gamma B \subseteq P$, then $A \subseteq P$ or $B \subseteq P$.

Definition 2.3.18. A $\Gamma$-near-ring $M$ is called weakly semiprime if $K\Gamma H = \{0\}$ implies either $K = \{0\}$ or $H = \{0\}$ where $K$ and $H$ are principal $M\Gamma$-subgroups.

Definition 2.3.19. $A_s(M) = \{x \in M/M\Gamma x = \{0\}\}$ is called right annihilator of $M$.

Definition 2.3.20. An $M\Gamma$-subgroup $I$ is called essential if $I \cap J \neq \{0\}$ for every $M\Gamma$-subgroup $J$.

Definition 2.3.21. A $\Gamma$-near-ring $M$ is said to be strictly semiprime if $K^2 = \{0\}$ implies $K = \{0\}$ for every $M\Gamma$-subgroup $K$. 
Definition 2.3.22. A family $F$ of subsets of $M$ is said to fulfill descending chain condition (dcc) if every chain $A_1 \supseteq A_2 \supseteq \ldots \supseteq \ldots$ of members of $F$ terminates after finitely many steps.

Definition 2.3.23. A family $F$ of subsets of $M$ is said to fulfill ascending chain condition (acc) if every chain $A_1 \subseteq A_2 \subseteq \ldots \subseteq \ldots$ of members of $F$ terminates after finitely many steps.

Definition 2.3.24. [31, T. Srinivas, P. Narasimha and K. Vijayakumar]
Let $M$ be a linear space over a field $X$ and $\Gamma$ be a non-empty set. Then $M$ is said to be a $\Gamma$-near-algebra (right $\Gamma$-near-algebra) over $X$ if there exists a mapping $M \times \Gamma \times M \to M$ (the image of $(a, \alpha, b)$ is denoted by $a \alpha b$) satisfying the following conditions:

(i) $(a \alpha b) \beta c = a \alpha (b \beta c)$;

(ii) $(a + b) \alpha c = a \alpha c + b \alpha c$;

(iii) $(\lambda a) \alpha b = \lambda (a \alpha b)$ for all $a, b, c \in M, \alpha, \beta \in \Gamma$ and $\lambda \in X$.

Definition 2.3.25. Let $M$ be a linear space over the field $X$ and $\Gamma$ be a non-empty set. Then $M$ is said to be a left $\Gamma$-near-algebra over $X$ if there exists a mapping $M \times \Gamma \times M \to M$ (the image of $(a, \alpha, b)$ is denoted by $a \alpha b$) satisfying the following conditions:

(i) $(a \alpha b) \beta c = a \alpha (b \beta c)$; (ii) $a \alpha (b + c) = a \alpha b + a \alpha c$

(iii) $\lambda (a \alpha b) = a \alpha (\lambda b)$ for all $a, b, c \in M, \alpha, \beta \in \Gamma$ and $\lambda \in X$. 

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Definition 2.3.26. Let $M$ be a $\Gamma$-near-algebra over a field $X$. A linear subspace $L$ of $M$ is said to be a sub $\Gamma$-near-algebra over the field $X$ if there exists a mapping $L \times \Gamma \times L \to L$ satisfying the following conditions:

(i) $(a\alpha b)\beta c = a\alpha(b\beta c)$;

(ii) $(a + b)\alpha c = a\alpha c + b\alpha c$;

(iii) $(\lambda a)\alpha b = \lambda(a\alpha b)$ for all $a, b, c \in L$, $\alpha, \beta \in \Gamma$ and $\lambda \in X$.

Theorem 2.3.27. A non-empty subset $L$ of a $\Gamma$-near-algebra $M$ over a field $X$ is a sub $\Gamma$-near-algebra of $M$ if and only if the following conditions hold:

(i) $x - y, \lambda x \in L$;

(ii) $x\alpha y \in L$;

(iii) $(\lambda x)\alpha y = \lambda(x\alpha y)$ for all $x, y \in L$, $\lambda \in X$ and $\alpha \in \Gamma$.

Theorem 2.3.28. Let $I$ be an ideal of a $\Gamma$-near-algebra $M$ over a field $X$. Then the set $M/I$ is a $\Gamma$-near-algebra over $X$ with respect to the operations defined by

(i) $(x + I) + (y + I) = (x + y) + I$;

(ii) $\lambda(x + I) = \lambda x + I$;

(iii) $(x + I)\alpha(y + I) = x\alpha y + I$ for every $x, y \in M$, $\lambda \in X$ and $\alpha \in \Gamma$. 

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Definition 2.3.29. [38, L.A. Zadeh] A fuzzy set in a set $M$ is a function $\mu : M \to [0, 1]$.

Definition 2.3.30. If $\mu$ is a fuzzy set in $M$ and $f$ is a function defined on $M$ then the fuzzy set $f[\mu](y) = \sup_{x \in f^{-1}(y)} \mu(x)$ for all $y \in f(M)$ is called the image of $\mu$ under $f$.

Definition 2.3.31. If $v$ is a fuzzy set in $f(M)$, then the fuzzy set $\mu$ defined by $\mu(x) = v(f(x))$ for all $x \in M$. That is $\mu = v \circ f$ in $M$ and is called the pre-image of $v$ under $f$.

Definition 2.3.32. Let $\mu$ be a fuzzy set in a set $M$. Then $\mu_t = \{x \in M/\mu(x) \geq t\}$ where $t \in [0, 1]$ is called a level subset of $M$.

Definition 2.3.33. The complement of a fuzzy set $\mu$, denoted by $\mu'$ is the fuzzy set in $M$ defined by $\mu'(x) = 1 - \mu(x)$ for all $x \in M$.

Definition 2.3.34. Let $X$ be a field and $F$ be a fuzzy set on $X$. $F$ is called a fuzzy field of $X$ denoted by $(F, X)$ if

(i) $F(x - y) \geq F(x) \wedge F(y), x, y \in X$;

(ii) $F(-x)) \geq F(x), x \in X$;

(iii) $F(xy)) \geq F(x) \wedge F(y), x, y \in X$;

(iv) $F(x^{-1}) \geq F(x), x \neq 0 \in X$.  

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