Chapter 6

On Fuzzy Gamma Sub-near-Algebras

A new algebraic system $\Gamma$-near-algebra was introduced by T. Srinivas, P. Narashimha Swamy and K. Vijayakumar [31]. $\Gamma$-near-Algebra is generalization of both the concepts of near-algebra and $\Gamma$-near-ring. The theory of fuzzy sets was first inspired by Zadeh [38]. Fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians working in different fields of mathematics Wenxiang, Gu and Lu Tu [12] introduced the notion of fuzzy algebras over fuzzy fields and obtained some fundamental results pertaining to this notion. In the first section of this chapter, we introduce fuzzy-sub-near-algebras and obtain their properties. In the second section, we introduce anti-fuzzy gamma sub-near-algebras and discuss about their properties.
6.1 Fuzzy gamma sub-near-algebra

In this section, we introduce fuzzy gamma sub-near-algebras and study their properties.

**Definition 6.1.1.** Let \((F, X)\) be a fuzzy field of the field \(X, M\) be a \(\Gamma\)-near-algebra over \(X\) and \(\mu\) be a fuzzy set of \(M\). Then \((\mu, M)\) is called **fuzzy \(\Gamma\)-sub-near-algebra** of \(M\) over a fuzzy field \((F, X)\) if

(i) \(\mu(x - y) \geq \min\{\mu(x), \mu(y)\}\)

(ii) \(\mu(\lambda x) \geq \min\{F(\lambda), \mu(x)\}\)

(iii) \(\mu(x\gamma y) \geq \min\{\mu(x), \mu(y)\}\)

(iv) \(\mu((\lambda x)\alpha y) = \mu(\lambda(x\alpha y)) \geq \min\{F(\lambda), \mu(x), \mu(y)\}\) for all \(x, y \in M, \lambda \in F\) and \(\gamma, \alpha \in \Gamma\).

**Theorem 6.1.2.** Let \((\mu, M)\) and \((\sigma, M)\) be two fuzzy \(\Gamma\)-sub-near-algebra of a \(\Gamma\)-near-algebra \(M\) over a fuzzy field \((F, X)\). Then \((\mu, M) \cap (\sigma, M)\) is also a fuzzy \(\Gamma\)-sub-near-algebra over a fuzzy field \((F, X)\).

**Proof.** Let \(x, y \in M, \lambda \in X\) and \(\gamma, \alpha \in \Gamma\).

(i) \((\mu \cap \sigma)(x - y) = \min\{\mu(x - y), \sigma(x - y)\}\)

\[\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\sigma(x), \sigma(y)\}\}\]

\[= \min\{\min\{\mu(x), \sigma(x)\}, \min\{\mu(y), \sigma(y)\}\}\]

\[= \min\{(\mu \cap \sigma)(x), (\mu \cap \sigma)(y)\}\]
(ii) \((\mu \cap \sigma)\lambda(x) = \min\{\mu(\lambda x), \sigma(\lambda x)\}\)
\[\geq \min\{\min\{F(\lambda), \mu(x)\}, \min\{F(\lambda), \sigma(x)\}\}\]
\[= \min\{F(\lambda), \min\{\mu(x), \sigma(x)\}\}\]
\[= \min\{F(\lambda), \mu \cap \sigma(x)\}\]

(iii) \((\mu \cap \sigma)(x \gamma y) = \min\{\mu(x \gamma y), \sigma(x \gamma y)\}\)
\[\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\sigma(x), \sigma(y)\}\\}
\[= \min\{\min\{\mu(x), \sigma(x)\}, \min\{\mu(y), \sigma(y)\}\\}
\[= \min\{(\mu \cap \sigma)(x), (\mu \cap \sigma)(y)\}\]

(iv) \((\mu \cap \sigma)((\lambda x)\alpha y)\)
\[\geq \min\{\min\{F(\lambda), \mu(x), \mu(y)\}, \min\{F(\lambda), \sigma(x), \sigma(y)\}\}\]
\[= \min\{F(\lambda), \min\{\mu(x), \sigma(x)\}, \min\{\mu(y), \sigma(y)\}\}\]
\[= \min\{F(\lambda), (\mu \cap \sigma)(x), (\mu \cap \sigma)(y)\}\].

Hence \((\mu \cap \sigma)\) is a fuzzy \(\Gamma\)-sub-near-algebra of \(M\). 

\[\square\]

**Theorem 6.1.3.** Let \(M\) be a \(\Gamma\)-near-algebra over a field \(X\). Let \((\mu, M)\) be a fuzzy \(\Gamma\)-sub-near-algebra of \(M\) over the fuzzy field \((F, X)\). Let \(A\) be a subset of \(M\). Then \(A\) is a sub-\(\Gamma\)-near-algebra of \(M\) if and only if \(\chi_{\lambda A}\) is a fuzzy \(\Gamma\)-sub-near-algebra of \(M\) over a fuzzy field \((F, X)\).

**Proof.** Let \(A\) be a sub-algebra of \(M\)

\[\text{77}\]
Case(i) Let \( x, y \in A \). Then \( x - y, \lambda x \in A \) where \( \lambda \in X \). Therefore
\[
\chi_A(x - y) = 1;
\]
\[
\chi_A(\lambda x) = 1.
\]
\[
\chi_A(x - y) = 1 \geq \min\{\chi_A(x), \chi_A(y)\}.
\]
\[
\chi_A(x\gamma y) = 1 \geq \min\{\chi_A(x), \chi_A(y)\}
\]
\[
(\lambda x)\alpha y = \lambda (x\alpha y) \text{ implies}
\]
\[
\chi_A((\lambda x)\alpha y) = \chi_A(\lambda(x\alpha y)) \geq \min\{F(\lambda), \chi_A(x), \chi_A(y)\}.
\]
Case(ii) If \( x \in A, y \notin A \). Then \( x - y \notin A \).

Therefore, \( \chi_A(x - y) = 0 = \min\{\chi_A(x), \chi_A(y)\} \).
\[
\chi_A(\lambda x) = 1 \geq \min\{F(\lambda), \chi_A(x)\}.
\]
\[
\chi_A(x\gamma y) = 0 = \min\{\chi_A(x), \chi_A(y)\}
\]
\[
\chi_A((\lambda x)\alpha y) = 0 = \min\{F(\lambda), \chi_A(x), \chi_A(y)\}.
\]

In a similar manner, one can verify the result for other cases \( x \notin A, y \in A \) and \( x \notin A, y \notin A \). Thus \( \chi_A \) is a fuzzy \( \Gamma \)-sub-near-algebra of \( M \).

Conversely assume that \( \chi_A \) is a fuzzy \( \Gamma \)-sub-near-algebra of \( M \). We claim that \( A \) is a sub-\( \Gamma \)-near-algebra of \( M \).

(i)(a) If \( x, y \in A \) then \( \chi_A(x - y) \geq \min\{\chi_A(x), \chi_A(y)\} = \min\{1, 1\} = 1 \). This implies that \( x - y \in A \).

(b) If \( x \in A, y \notin A \) then \( \chi_A(x - y) \geq \min\{\chi_A(x), \chi_A(y)\} = \min\{1, 0\} = 0 \). This implies that \( x - y \notin A \).

(c) If \( x \notin A, y \in A \) then \( \chi_A(x - y) \geq \min\{\chi_A(x), \chi_A(y)\} = \min\{0, 1\} = 0 \). This implies that \( x - y \notin A \).

(d) If \( x \notin A, y \notin A \) then \( \chi_A(x - y) \geq \min\{\chi_A(x), \chi_A(y)\} = \min\{0, 0\} = 0 \). This implies that \( x - y \notin A \).
(ii) If \( x \in A, \chi_A(\lambda x) \geq \min\{F(\lambda), \chi_A(x)\} = \min\{F(\lambda), 1\} = 1 \). This implies that \( \lambda x \in A \). Thus \( A \) is a sub-\( \Gamma \)-near-algebra of \( M \). ☐

**Proposition 6.1.4.** Let \( M \) and \( M' \) be two \( \Gamma \)-near-algebras over field \( X \). If \( A \) and \( B \) are fuzzy \( \Gamma \)-sub-near-algebras of \( M \) and \( M' \) respectively then \( (A \times B) \) is a fuzzy \( \Gamma \)-sub-near-algebra of \( M \times M' \).

**Proof.** Let \( (x, y), (x', y') \in M \times M', \lambda \in X \) and \( \alpha, \gamma \in \Gamma \).

(i) \( (A \times B)(x, y) = (A \times B)(x' + y') = \min\{A(x), A(y)\} \)

\[ \geq \min\{\min\{A(x), A(y)\}, \min\{B(x), B(y)\}\} \]

\[ = \min\{\min\{A(x), B(x)\}, \min\{A(y), B(y)\}\} \]

\[ = \min\{(A \times B)(x, y), (A \times B)(x', y')\} \]

(ii) \( (A \times B)(\lambda x, \lambda y) = (A \times B)(\lambda x, \lambda y) = \min\{(A(\lambda x), B(\lambda y)\} \)

\[ \geq \min\{\min\{F(\lambda), A(x)\}, \min\{F(\lambda), B(y)\}\} \]

\[ = \min\{F(\lambda), \min\{A(x), B(y)\}\} \]

\[ = \min\{F(\lambda), (A \times B)(x, y)\} \]

(iii) \( (A \times B)(x, y) \alpha (x', y') = (A \times B)(x' \gamma x', y' \gamma y') \)

\[ = \min\{A(x' \gamma x'), B(y' \gamma y')\} \]

\[ \geq \min\{\min\{A(x), A(x)\}, \min\{B(y), B(y)\}\} \]

\[ = \min\{\min\{A(x), B(y)\}, \min\{A(x'), B(y')\}\} \]

\[ = \min\{(A \times B)(x, y), (A \times B)(x', y')\} \]

(iv) \( (A \times B)(\lambda x, \lambda y) \alpha (x', y') = (A \times B)((\lambda x, \lambda y) \alpha (x', y')) \)

\[ \geq \min\{(A \times B)(\lambda x, \lambda y), (A \times B)(x', y')\} \]
\[ \geq \min \{ F(\lambda), (A \times B)(x, y), (A \times B)(x', y') \}. \]

Hence \( A \times B \) is a fuzzy \( \Gamma \)-sub-near-algebra of \( M \times M' \). \( \square \)

**Proposition 6.1.5.** A fuzzy set \( \mu \) of \( M \) is a fuzzy \( \Gamma \)-sub-near-algebra of \( M \) if and only if the non-empty level subset \( U(\mu; t) = \{ x \in M / \mu(x) \geq t \} \) is a sub \( \Gamma \)-near-algebra of \( M \) for all \( t \in [0, 1] \).

**Proposition 6.1.6.** If \( \mu \) is a fuzzy \( \Gamma \)-sub-near-algebra of \( M \), then \( U(\mu; 1) = \{ x \in M / \mu(x) = 1 \} \) is either empty or is a sub \( \Gamma \)-near-algebra of \( M \).

**Theorem 6.1.7.** Let \( M \) and \( M' \) be \( \Gamma \)-near-algebras over a field \( X \). Let \( f : M \to M' \) be an onto homomorphism. If \( \mu \) is a fuzzy \( \Gamma \)-sub-near-algebra of \( M \) then the image \( f[\mu] \) of \( \mu \) under \( f \) is fuzzy \( \Gamma \)-sub-near-algebra of \( M' \).

**Proof.** We have \( f[\mu](y') = \sup_{x \in f^{-1}(y')} \mu(x) \). Let \( x', y' \in M' \). Let \( x_0 \in f^{-1}(x') \) and \( y_0 \in f^{-1}(y') \) be such that \( \mu(x_0) = \sup_{z \in f^{-1}(x')} \mu(z) \) and \( \mu(y_0) = \sup_{z \in f^{-1}(y')} \mu(z) \).

\[
(i) \quad f[\mu](x' - y') = \sup_{x - y \in f^{-1}(x' - y')} \mu(x - y) \\
= \sup_{x - y \in f^{-1}(x') - f^{-1}(y')} \mu(x - y) \\
= \sup_{z \in f^{-1}(x') - f^{-1}(y')} \mu(z) \geq \mu(x_0 - y_0) \\
\geq \min \{ \mu(x_0), \mu(y_0) \} \\
= \min \{ \sup_{z \in f^{-1}(x')} \mu(z), \sup_{z \in f^{-1}(y')} \mu(z) \} \\
= \min \{ f[\mu](x'), f[\mu](y') \}. 
\]
\( (ii) \ f[\mu](\lambda x') = \sup_{z \in f^{-1}(\lambda x') \mu(z)} \geq \mu(\lambda x_0) \geq \min\{F(\lambda), \mu(x_0)\} \)
\[= \min\{F(\lambda), \sup_{z \in f^{-1}(x')} \mu(z)\} = \min\{F(\lambda), f[\mu](x')\}.\]

\( (iii) \ f[\mu](x'\gamma y') = \sup_{z \in f^{-1}(x'\gamma y') \mu(z)} \)
\[= \sup_{z \in f^{-1}(x') \gamma f^{-1}(y')} \mu(z) \]
\[\geq \mu(x_0 \gamma y_0) \geq \min\{\mu(x_0), \mu(y_0)\} \]
\[= \min\{\sup_{z \in f^{-1}(x') \mu(z)}, \sup_{z \in f^{-1}(y') \mu(z)}\} \]
\[= \min\{f[\mu](x'), f[\mu](y')\}.\]

\( (iv) \ f[\mu][(\lambda x'\alpha y')] \geq \min\{f[\mu](\lambda x'), f[\mu](y')\} \)
\[\geq \min\{F(\lambda), f[\mu](x'), f[\mu](y')\}.\]

Thus \( f[\mu] \) is fuzzy \( \Gamma \)-sub-near-algebra of \( M' \).

\textbf{Theorem 6.1.8.} Let \( g : M \to M' \) be an onto homomorphism. If \( v \) is a fuzzy \( \Gamma \)-sub-near-algebra of \( M' \) then the pre-image \( g^{-1}[v] \) defined by \( g^{-1}[v](x) = \mu(x) = v(g(x)) \) for all \( x \in M \) is also a fuzzy \( \Gamma \)-sub-near-algebra of \( M \).

\textbf{Proof.} Let \( x, y \in M; \lambda \in X; \gamma, \alpha \in \Gamma \).

\( (i) \ g^{-1}[v](x - y) = v(g(x - y)) \)
\[= v(g(x) - g(y)) \]
\[\geq \min\{v(g(x), v(g(y))\} \]
\[= \min\{g^{-1}[v](x), g^{-1}[v](y)\}.\]

81
\[(ii) \quad g^{-1}[v](\lambda x) = v(g(\lambda x)) = v(\lambda(g(x))) \geq \min\{F(\lambda), v(g(x))\} = \min\{F(\lambda), g^{-1}[v](x)\}\]

\[(iii) \quad g^{-1}[v](x\alpha y) = v(g(x\alpha y)) = v(g(x)\alpha g(y)) \geq \min\{v(g(x)), v(g(y))\} = \min\{g^{-1}[v](x), g^{-1}[v](y)\}\]

\[(iv) \quad g^{-1}[v]((\lambda x)\alpha y) = v(g((\lambda x)\alpha y)) = v(g(\lambda x)\alpha g(y)) = v(\lambda g(x)\alpha g(y)) \geq \min\{F(\lambda), v(g(x)), v(g(y))\} = \min\{F(\lambda), g^{-1}[v](x), g^{-1}[v](y)\}.\]

Thus \(g^{-1}[v]\) is fuzzy \(\Gamma\)-sub-near-algebra of \(M\). \hfill \Box

**Theorem 6.1.9.** Let \(M\) be \(\Gamma\)-sub-near-algebra over field \(X\). Let \(I\) be an ideal of \(M\). If \(\mu\) is a fuzzy \(\Gamma\)-sub-near-algebra of \(M\) then the fuzzy set \(\mu'\) of \(M/I\) defined by \(\mu'(m + I) = \sup_{x \in I} \mu(m + x)\) is fuzzy \(\Gamma\)-sub-near-algebra of \(M/I\).
Proof. Let $m_1, m_2 \in M$ be such that $m_1 + I = m_2 + I$. Then $m_2 - m_1 \in I$ and so $m_2 - m_1 = m$ for some $m \in I$. Now $\mu'(m_2 + I) = \sup_{m' \in I} \mu(m_2 + m') = \sup_{m' \in I} \mu(m_1 + m + m') = \sup_{m + m' = t \in I} \mu(m_1 + t) = \mu'(m_1 + I)$. Therefore, $\mu'$ is well defined. Let $m_1 + I, m_2 + I \in M/I$.

(i) $\mu'((m_2 + I) - (m_1 + I)) = \mu'(m_2 - m_1 + I)$

$= \sup_{u-v \in I} \mu(m_2 - m_1 + u - v)$

$= \sup_{u,v \in I} \mu(m_2 + u) - (m_1 + v))$

$\geq \sup_{u,v \in I} \min\{\mu(m_2 + u), \mu(m_1 + v)\}$

$= \min\{\sup_{u \in I} \mu(m_2 + u), \sup_{v \in I} \mu(m_1 + v)\}$

$= \min\{\mu'(m_2 + I), \mu'(m_1 + I))\}$.

(ii) $\mu'(\lambda(m + I)) = \mu'(\lambda m + I)$

$= \sup_{t \in I} \mu(\lambda m + t)$

$\geq \sup_{t \in I} \min\{F(\lambda), \mu(m + t)\}$

$= \min\{F(\lambda), \sup_{t \in I} \mu(m + t)\}$

$= \min\{F(\lambda), \mu'(m + I)\}$.

(iii) Let $m_1 + I, m_2 + I \in M/I$ and $\gamma \in \Gamma$.

$\mu'((m_1 + I)\gamma(m_2 + I)) = \mu'(m_1 \gamma m_2 + I)$

$= \sup_{t \in I} \mu(m_1 \gamma m_2 + t)$

$= \sup_{t \in I} \mu((m_1 + t)\gamma(m_2 + t))$

$\geq \sup_{t \in I} \min\{\mu(m_1 + t), \mu(m_2 + t)\}$

$= \min\{\sup_{t \in I} \mu(m_1 + t), \sup_{t \in I} \mu(m_2 + t)\}$

$= \min\{\mu'(m_1 + I), \mu'(m_2 + I)\}$.

(iv) Let $m_1 + I, m_2 + I \in M/I, \gamma \in \Gamma$, and $\lambda \in X$. 

83
\[
\mu'(\lambda(m_1 + I)\gamma(m_2 + I)) = \mu'(\lambda((m_1 + I)\gamma(m_2 + I))) = \mu'(\lambda(m_1\gamma m_2 + I)) = \sup_{t \in I} \mu\lambda(m_1 + t)\gamma(m_2 + t) \\
\geq \sup_{t \in I} \min\{F(\lambda), \mu(m_1 + t), \mu(m_2 + t)\} = \min\{F(\lambda), \sup_{t \in I} \mu(m_1 + t), \sup_{t \in I} \mu(m_2 + t)\} = \min\{F(\lambda), \mu'(m_1 + I), \mu'(m_2 + I)\}.
\]

Thus \(\mu'\) is fuzzy \(\Gamma\)-sub-near-algebra of \(M/I\).

\section{Anti fuzzy gamma sub-near-algebra}

In this section we introduce Anti fuzzy gamma sub-near-algebras and discuss about their properties.

\textbf{Definition 6.2.1.} Let \(X\) be a field and \(F\) be a fuzzy subset on \(X\). \(F\) is called an Anti fuzzy field of \(X\) denoted by \((F, X)\) if for all \(x, y \in X\)

- (i) \(F(x - y) \leq F(x) \lor F(y) = \max\{F(x), F(y)\}\);
- (ii) \(F(-x) \leq F(x)\);
- (iii) \(F(xy) \leq F(x) \lor F(y) = \max\{F(x), F(y)\}\);
- (iv) \(F(x^{-1}) \leq F(x), x \neq 0 \in X\).

\textbf{Definition 6.2.2.} Let \((F, X)\) be an anti fuzzy field of the field \(X, M\) be a \(\Gamma\)-near-algebra over \(X\) and \(\mu\) be a fuzzy subset of \(M\). Then \((\mu, M)\)
is called a \textit{anti fuzzy }$\Gamma$-\textit{sub-near-algebra} of $M$ over an anti fuzzy field $(F, X)$ if

\begin{enumerate}[(i)]
  \item $\mu(x - y) \leq \mu(x) \lor \mu(y) = \max\{\mu(x), \mu(y)\}$;
  \item $\mu(\lambda x) \leq F(\lambda) \lor \mu(x) = \max\{F(\lambda), \mu(x)\}$;
  \item $\mu(x\gamma y) \leq \mu(x) \lor \mu(y) = \max\{\mu(x), \mu(y)\}$;
  \item $\mu((\lambda x)\alpha y) = \mu(\lambda(x\alpha y)) \leq \max\{F(\lambda); \mu(x), \mu(y)\}$ for all $x, y \in M, \lambda \in X$ and $\gamma, \alpha \in \Gamma$.
\end{enumerate}

\textbf{Proposition 6.2.3.} Let $X, Y$ be two fields. Let $f : X \to Y$ be an onto homomorphism. If $F$ is an anti fuzzy field of $X$ and $G$ be an anti fuzzy field of $Y$ then $f^{-1}(G)$ is an anti fuzzy field of $X$ and $f(F)$ is an anti fuzzy field of $Y$.

\textbf{Proof.} For any $x, y \in X$, we have

\begin{enumerate}[(i)]
  \item $f^{-1}(G)(x - y) = G(f(x - y))$
    \begin{align*}
    & = G(f(x) - f(y)) \\
    & \leq \max\{G(f(x)), G(f(y))\} \\
    & = \max\{f^{-1}(G)(x), f^{-1}(G)(y)\}
    \end{align*}
  \item $f^{-1}(G)(-x) = G(f(-x)) = G(-f(x)) \leq G(f(x)) = f^{-1}(G)(x)$.
  \item $f^{-1}(G)(x\gamma y) = G(f(x\gamma y)) = G(f(x)\gamma f(y))$
    \begin{align*}
    & \leq \max\{G(f(x)), G(f(y))\} \\
    & = \max\{f^{-1}(G)(x), f^{-1}(G)(y)\}.
    \end{align*}
\end{enumerate}
(iv) For every \( x \neq 0 \in X, f^{-1}(G)(x^{-1}) = G(f(x^{-1})) = G(f(x)^{-1}) \leq G(f(x)) = f^{-1}(G)(x) \). Hence \((f^{-1}(G), X)\) is an anti fuzzy field of \( X \).

In a similar manner, one can prove that \((f(F), Y)\) is an anti fuzzy field of \( Y \). \(\square\)

**Proposition 6.2.4.** Intersection of family of anti fuzzy \( \Gamma\)-sub-near-algebras is an anti fuzzy \( \Gamma\)-sub-near-algebra

**Proof.** Let \( \{\mu_i\}, i \in I \) be a family of anti fuzzy \( \Gamma\)-sub-near-algebras of \( M \) over an anti fuzzy field \((F, X)\). Let \( \mu(x) = \bigcap_{i \in I} \mu_i(x) = \inf_{i \in I} \mu_i(x) \).

For any \( x, y \in M, \gamma \in \Gamma \), we have

(i) \( \mu(x - y) = \inf_{i \in I} \mu_i(x - y) \)
\(\leq \inf_{i \in I} \max\{\mu_i(x), \mu_i(y)\} \)
\(\leq \max\{\inf_{i \in I} \mu_i(x), \inf_{i \in I} \mu_i(y)\} \)
\(= \max\{\mu(x), \mu(y)\} \).

(ii) \( \mu(\lambda x) = \inf_{i \in I} \mu_i(\lambda x) \)
\(\leq \inf_{i \in I} \max\{F(\lambda), \mu_i(x)\} \)
\(= \max\{F(\lambda), \inf_{i \in I} \mu_i(x)\} \)
\(= \max\{F(\lambda), \mu(x)\} \).

(iii) \( \mu(x \gamma y) = \inf_{i \in I} \mu_i(x \gamma y) \)
\(\leq \inf_{i \in I} \max\{\mu_i(x), \mu_i(y)\} \)
\(\leq \max\{\inf_{i \in I} \mu_i(x), \inf_{i \in I} \mu_i(y)\} \)
\(= \max\{\mu(x), \mu(y)\} \).
$$(iv) \, \mu((\lambda x)\alpha y) = \inf_{i \in I} \mu_i((\lambda x)\alpha y) = \inf_{i \in I} \mu_i(\lambda(x\alpha y))$$
\begin{align*}
&\leq \inf_{i \in I} \max\{F(\lambda), \mu_i(x), \mu_i(y)\} \\
&= \max\{F(\lambda), \inf_{i \in I} \mu_i(x), \inf i \in I \mu(y)\} \\
&= \max\{F(\lambda), \mu(x), \mu(y)\}.
\end{align*}

Hence \( \mu = \bigcap_{i \in I} \mu_i \) is an anti fuzzy \( \Gamma \)-sub-near-algebra. \( \square \)

Proposition 6.2.5. If \( \{\mu_i\}, i \in I \) is a family of anti fuzzy \( \Gamma \)-sub-near-algebras of \( M \) over an anti fuzzy field \( (F, X) \) then so is \( \bigcup_{i \in I} \mu_i \).

Proof. Straightforward. \( \square \)

Theorem 6.2.6. Let \( M \) and \( M' \) be \( \Gamma \)-near-algebras over a field \( X \). Let \( f : M \rightarrow M' \) be an onto homomorphism. If \( \mu \) is an anti fuzzy \( \Gamma \)-sub-near-algebra of \( M \) then the image \( f[\mu] \) of \( \mu \) under \( f \) is an anti fuzzy \( \Gamma \)-sub-near-algebra of \( M' \) over the anti fuzzy field \( (F, X) \).

Proof. We have \( f[\mu](y') = \sup_{x \in f^{-1}(y')} \mu(x) \). Let \( x', y' \in M' \). Let \( x_0 \in f^{-1}(x') \) and \( y_0 \in f^{-1}(y') \) be such that \( \mu(x_0) = \sup_{z \in f^{-1}(x')} \mu(z) \) and \( \mu(y_0) = \sup_{z \in f^{-1}(y')} \mu(z) \). Then
\begin{align*}
(i) \, f[\mu](x' - y') &= \sup_{x-y \in f^{-1}(x'-y')} \mu(x - y) \\
&= \sup_{z \in f^{-1}(x') - f^{-1}(y')} \mu(z) \\
&\leq \mu(x_0 - y_0) \\
&\leq \max\{\mu(x_0), \mu(y_0)\} \\
&= \max\{\sup_{z \in f^{-1}(x')} \mu(z), \sup_{z \in f^{-1}(y')} \mu(z)\} \\
&= \max\{f[\mu](x'), f[\mu](y')\}.
\end{align*}

87
(ii) \( f[\mu](\lambda x') = \sup_{z \in f^{-1}(\lambda x')} \mu(z) \)
\[ \leq \mu(\lambda x_0) \]
\[ \leq \max\{F(\lambda), \mu(x_0)\} \]
\[ = \max\{F(\lambda), \sup_{z \in f^{-1}(x')} \mu(z)\} \]
\[ = \max\{F(\lambda), f[\mu](x')\}. \]

(iii) \( f[\mu](x'\gamma y') = \sup_{z \in f^{-1}(x'\gamma y')} \mu(z) \)
\[ = \sup_{z \in f^{-1}(x') \cap f^{-1}(y')} \mu(z) \]
\[ \leq \mu(x_0 \gamma y_0) \]
\[ \leq \max\{\mu(x_0), \mu(y_0)\} \]
\[ = \max\{\sup_{z \in f^{-1}(x')} \mu(z), \sup_{z \in f^{-1}(y')} \mu(z)\} \]
\[ = \max\{f[\mu](x'), f[\mu](y')\}. \]

(iv) \( f[\mu]((\lambda x')\alpha y') \leq \max\{f[\mu](\lambda x'), f[\mu](y')\} \)
\[ \leq \max\{f(\lambda), f[\mu](x'), f[\mu](y')\}. \]

Thus \( f[\mu] \) is an anti fuzzy \( \Gamma \)-sub-near-algebra of \( M' \).

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**Theorem 6.2.7.** Let \( g : M \rightarrow M' \) be an onto homomorphism. If \( v \) is an anti fuzzy \( \Gamma \)-sub-near-algebra of \( M' \) then the pre-image \( g^{-1}[v] \) defined
by \( g^{-1}[v](x) = \mu(x) = v(g(x)) \) for all \( x \in M \) is also an anti fuzzy \( \Gamma \)-sub-near-algebra of \( M \).

**Proof.** Let \( x, y \in M, \lambda \in X, \gamma, \alpha \in \Gamma \).

(i) \( g^{-1}[v](x - y) = v(g(x - y)) \)

\[
= v(g(x) - g(y)) \\
\leq \max\{v(g(x)), v(g(y))\} \\
= \max\{g^{-1}[v](x), g^{-1}[v](y)\}.
\]

(ii) \( g^{-1}[v](\lambda x) = v(g(\lambda x)) \)

\[
= v(\lambda g(x)) \\
\leq \max\{F(\lambda), v(g(x))\} \\
= \max\{F(\lambda), g^{-1}[v](x)\}.
\]

(iii) \( g^{-1}[v](x\alpha y) = v(g(x\alpha y)) \)

\[
= v(g(x)\alpha g(y)) \\
\leq \max\{v(g(x)), v(g(y))\} \\
= \max\{g^{-1}[v](x), g^{-1}[v](y)\}.
\]

(iv) \( g^{-1}[v](\lambda x\alpha y) = v(g((\lambda x)\alpha y)) = v(g(\lambda x)\alpha g(y)) \)

\[
= v(\lambda g(x)\alpha g(y)) = v(\lambda (g(x)\alpha g(y))) \\
\leq \max\{F(\lambda), v(g(x)), v(g(y))\} \\
= \max\{F(\lambda), g^{-1}[v](x), g^{-1}[v](y)\}.
\]

Thus \( g^{-1}[v] \) is also an anti fuzzy \( \Gamma \)-sub-near-algebra of \( M \). \( \square \)

89
Proposition 6.2.8. Let $M$ be a $\Gamma$-near-algebra over a field $X$. Then the fuzzy subset $\mu$ is an anti fuzzy $\Gamma$-sub-near-algebra of $M$ over an anti fuzzy field $(F', X)$ if and only if $\mu' = 1 - \mu$ is an fuzzy $\Gamma$-sub-near-algebra of $M$ over an anti fuzzy field $(F, X)$.

Proof. Let $\mu$ be an anti fuzzy $\Gamma$-sub-near-algebra of $M$. Then for any $x, y \in M, \gamma \in \Gamma$, we have

(i) $\mu'(x - y) = 1 - \mu(x - y)$
\[
\geq 1 - \max\{\mu(x), \mu(y)\}
\]
\[
= \min\{1 - \mu(x), 1 - \mu(y)\}
\]
\[
= \min\{\mu'(x), \mu'(y)\}
\]

(ii) $\mu'(\lambda x) = 1 - \mu(\lambda x)$
\[
\geq 1 - \max\{F(\lambda), \mu(x)\}
\]
\[
= \min\{1 - F(\lambda), 1 - \mu(x)\}
\]
\[
= \min\{F'(\lambda), \mu'(x)\}
\]

(iii) $\mu'(x\gamma y) = 1 - \mu(x\gamma y) \geq 1 - \max\{\mu(x), \mu(y)\}$
\[
= \min\{1 - \mu(x), 1 - \mu(y)\}
\]
\[
= \min\{\mu'(x), \mu'(y)\}
\]

(iv) $\mu'((\lambda x)\alpha y) = 1 - \mu((\lambda x)\alpha y) \geq 1 - \max\{F(\lambda), \mu(x), \mu(y)\}$
\[
= \min\{1 - F(\lambda), 1 - \mu(x), 1 - \mu(y)\}
\]
\[
= \min\{F'(\lambda), \mu'(x), \mu'(y)\}
\]
Thus \( \mu' \) is a fuzzy \( \Gamma \)-sub-near-algebra of \( M \).

Conversely suppose that \( \mu' \) is a fuzzy \( \Gamma \)-sub-near-algebra of \( M \). Then

(i) \[
\mu(x - y) = 1 - \mu'(x - y) \\
\leq 1 - \min\{\mu'(x), \mu'(y)\} \\
= \max\{1 - \mu'(x), 1 - \mu'(y)\} \\
= \max\{\mu(x), \mu(y)\}
\]

(ii) \[
\mu(\lambda x) = 1 - \mu' (\lambda x) \\
\leq 1 - \min\{F'(\lambda), \mu'(x)\} \\
= \max\{1 - F'(\lambda), 1 - \mu'(x)\} \\
= \max\{F(\lambda), \mu(x)\}
\]

(iii) \[
\mu(x \gamma y) = 1 - \mu' (x \gamma y) \\
\leq 1 - \min\{\mu'(x), \mu'(y)\} \\
= \max\{1 - \mu'(x), 1 - \mu'(y)\} \\
= \max\{\mu(x), \mu(y)\}
\]

(iv) \[
\mu((\lambda x) \alpha y) = 1 - \mu'((\lambda x) \alpha y) \\
1 - \min\{F'(\lambda), \mu'(x), \mu'(y)\} \\
= \max\{1 - F'(\lambda), 1 - \mu'(x), 1 - \mu'(y)\} \\
= \max\{F(\lambda), \mu(x), \mu(y)\}
\]

Hence \( \mu \) is an anti fuzzy \( \Gamma \)-sub-near-algebra of \( M \) over an anti fuzzy field \( (F', X) \). □
Proposition 6.2.9. Let $M$ and $M'$ be $\Gamma$-near-algebras over a field $X$. If $\mu$ and $\sigma$ are anti fuzzy $\Gamma$-sub-near-algebras of $M$ and $M'$ respectively then $(\mu \times \sigma)$ is an anti fuzzy $\Gamma$-sub-near-algebra of $M \times M'$.

Proof. Let $(x, y), (x', y') \in M \times M', \lambda \in X$ and $\alpha, \gamma \in \Gamma$.

(i) $(\mu \times \sigma)((x, y) - (x', y')) = (\mu \times \sigma)(x - x', y - y')$

$= \min\{\mu(x - x'), \sigma(y - y')\}$

$\leq \min\{\max\{\mu(x), \mu(x')\}, \max\{\sigma(y), \sigma(y')\}\}$

$= \min\{\max\{\mu(x), \sigma(y)\}, \max\{\mu(x'), \sigma(y')\}\}$

$= \max\{\min\{\mu(x), \sigma(y)\}, \min\{\mu(x'), \sigma(y')\}\}$

$= \max\{(\mu \times \sigma)(x, y), (\mu \times \sigma)(x', y')\}$

(ii) $(\mu \times \sigma)(\lambda(x, y)) = (\mu \times \sigma)(\lambda x, \lambda y)$

$= \min\{(\mu(\lambda x), \sigma(\lambda y))\}$

$\leq \min\{\max\{F(\lambda), \mu(x)\}, \max\{F(\lambda), \sigma(y)\}\}$

$= \max\{F(\lambda), \min\{\mu(x), \sigma(y)\}\}$

$= \max\{F(\lambda), (\mu \times \sigma)(x, y)\}$

(iii) $(\mu \times \sigma)((x, y) \gamma(x', y')) = (\mu \times \sigma)(x \gamma x', y \gamma y')$

$= \min\{\mu(x \gamma x'), \sigma(y \gamma y')\}$

$\leq \min\{\max\{\mu(x), \mu(x')\}, \max\{\sigma(y), \sigma(y')\}\}$

$= \max\{\min\{\mu(x), \sigma(y)\}, \min\{\mu(x'), \sigma(y')\}\}$

$= \max\{(\mu \times \sigma)(x, y), (\mu \times \sigma)(x', y')\}$

(iv) $(\mu \times \sigma)(\lambda(x, y) \alpha (x', y')) = (\mu \times \sigma)((\lambda x, \lambda y) \alpha (x', y'))$

$\leq \max\{(\mu \times \sigma)\lambda x, \lambda y), (\mu \times \sigma)(x', y')\}$

92
\[ \leq \max\{F(\lambda), (\mu \times \sigma)(x, y), (\mu \times \sigma)(x', y')\} \]
\[ = \max\{(F(\lambda), (\mu \times \sigma)(x, y), (\mu \times \sigma)(x', y'))\}. \]

Hence \((\mu \times \sigma)\) is an anti fuzzy \(\Gamma\)-sub-near-algebra of \(M \times M'\). \(\square\)

**Definition 6.2.10.** Let \(M\) and \(M'\) be two \(\Gamma\)-near-algebras over a field \(X\). A mapping \(f : M \rightarrow M'\) is called an anti homomorphism if for all \(x, y \in M, \lambda \in X\) and \(\gamma \in \Gamma\),

(i) \(f(x + y) = f(x) + f(y)\)

(ii) \(f(\lambda x) = \lambda f(x)\)

(iii) \(f(x \gamma y) = f(y) \gamma f(x)\)

**Theorem 6.2.11.** Anti homomorphic image of a \(\Gamma\)-near-algebra is a left \(\Gamma\)-near-algebra.

**Proof.** Let \(f : M \rightarrow M'\) be a \(\Gamma\)-near-algebra anti homomorphism. The homomorphic image of \(M\) is \(f(M) = \{f(x) \in M' / x \in M\}\). Clearly \(f(0) = 0' \in f(M)\) where 0 and 0' are the additive identities in \(M\) and \(M'\) respectively. Thus \(f(M)\) is a non-empty subset of \(M'\).

Let \(x', y' \in f(M)\). Then there exists \(x, y \in M\) such that \(f(x) = x', f(y) = y'\). Now \(x - y, x \gamma y, y \gamma x, \lambda x \in M, (\lambda x) \gamma y = \lambda(x \gamma y)\) and so \(f(x - y), f(x \gamma y), f(y \gamma x)\) and \(f(\lambda x) \in f(M)\).

Then \(x' - y' = f(x) - f(y) = f(x - y) \in f(M), x' \gamma y' = f(x) \gamma f(y) = f(y \gamma x) \in f(M), \lambda x' = \lambda f(x) = f(\lambda x) \in f(M)\) and \(\lambda(x' \gamma y') = \lambda f(x \gamma y) \in f(M)\).
\[ \lambda(f(x) \gamma f(y)) = \lambda f(y \gamma x) = f((\lambda(y \gamma x)) = f((\lambda y) \gamma x) = f(x) \gamma f(\lambda y)) = f(x) \gamma (\lambda f(y)) = x' \gamma (\lambda y'). \]

Thus \( f(M) \) is a left \( \Gamma \)-near-algebra of \( M' \). \( \square \)