CHAPTER - 3

FUZZY CONGRUENCE RELATION ON THE SEMIGROUP OF ALTERNATIVES

3.0 Introduction

The theory of ‘fuzzy semigroup’ is a part of fuzzy algebra, which is an important branch of fuzzy mathematics. Many important results in crisp algebra are not yet generalized to fuzzy algebra, not because of the impossibility or the difficulty of such tasks, but because of the insufficiency of fuzzy algebraic tool. In [64], Rosenfeld formulated the notion of fuzzy subgroups and showed how some basic notions of group theory can be extended in an elementary manner to fuzzy subgroups. Since then, fuzzy sets with many other kinds of algebraic structure have received attention in the literature\textsuperscript{[2,39,83]}. These have included ‘Semigroups’ also.

The notion of a fuzzy congruence on a semigroup is studied by many authors\textsuperscript{[21,37,44,48,59,60]}. The set of alternatives $X$ can be realized as a semigroup, by introducing a semigroup structure in $X$. We make use of the fuzzy weak preference relation $R$ on $X$ for this purpose. A quotient semigroup of $X$, induced by the fuzzy indifference relation $I$ is introduced, and the Homomorphism Theorem with respect to the relation is presented.
3.1 Preliminaries

3.1.1 Definition

A semigroup is a nonempty set with an associative binary operation defined on it.

3.1.2 Definition

An element $x$ of a semigroup is called regular if there exists an element $e$ of the semigroup such that $x e x = x$.

If all the elements of a semigroup are regular, then the semigroup is called a regular semigroup.

3.1.3 Definition

If an element $x$ of a semigroup is regular, an element $x'$ of the semigroup satisfying the equations,

\[
x x' x = x \quad \text{and} \quad x' x x' = x'
\]

is called an inverse of $x$.

If every element of a semigroup has a unique inverse in the semigroup, we call the semigroup an inverse semigroup.

3.1.4 Definition

An element $i$ of a semigroup is called an idempotent, if $i^2 = i$. 
If every element of a semigroup is an idempotent, then the semigroup is called a 'band'.

3.1.5 Definition

A relation $R$ on a semigroup $G$ is

i. left compatible with respect to the operation in $G$, if for all $s, t, a \in G$,

$$(s, t) \in R \implies (as, at) \in R$$

ii. right compatible if, for all $s, t, a \in G$,

$$(s, t) \in R \implies (sa, ta) \in R$$

iii. compatible if it is both left and right compatible

3.1.6 Definition

A compatible equivalence relation is called a congruence $^{30, 39}$. 

3.1.7 Definition

Let $G$ be a semigroup. A fuzzy binary relation $R$ on $G$ is said to be

i. fuzzy left compatible if and only if

$$R(x, y) \leq R(tx, ty), \quad \text{for every } x, y, t \in G$$
ii. fuzzy right compatible if and only if
\[ R(x, y) \leq R(x_t, y_t), \quad \text{for every } x, y, t \in G \]

iii. fuzzy compatible if and only if
\[
\min\{ R(u, v), R(x, y) \} \leq R(ux, vy), \quad \text{for every } u, v, x, y \in G.
\]

3.1.8 Definition

A fuzzy compatible similarity relation on a semigroup \( G \) is called a fuzzy congruence.

3.1.9 Lemma\(^{[43]} \)

A binary relation \( R \) on a semigroup \( G \) is a congruence on \( G \) if and only if its characteristic function is a fuzzy congruence on \( G \).

3.1.10 Proposition\(^{[39]} \)

A fuzzy binary relation \( R \) on a semigroup \( G \) is a fuzzy congruence if and only if it is both a fuzzy left and a fuzzy right compatible similarity relation.

3.1.11 Definition\(^{[38,29]} \)

Let \( A \) and \( B \) are two fuzzy subsets of \( S \). Then the product \( A \circ B \) of \( A \) and \( B \) is defined as follows:
\[
A \cdot B (x) = \begin{cases} 
\sup_{x = y, z} \{ \min A(y), B(z) \}, & \text{if } x \text{ is factorisable in } S \\
0, & \text{otherwise}
\end{cases}
\]

This product is associative and monotone in both the factors.

### 3.2 \(X\) as a Semigroup

Let \(R\) be the fuzzy weak preference relation on the set \(X\) of alternatives. Define an operation \(\bullet\) on \(X\) by:

\[
x \cdot y = \begin{cases} 
x, & \text{if } R(x, y) \geq R(y, x) \\
y, & \text{otherwise,}
\end{cases}
\]

for every \(x, y \in X\).

Clearly \(\bullet\) is an associative, binary operation on \(X\). Hence \(X\) is a semigroup under this operation.

#### 3.2.1 Note

The binary operation \(\bullet\) infact gives an agent's (consumer's) preference on a set of alternatives. In theoretical discussions we simply denote \(x \cdot y\) by \(xy\) except when more binary operations are involved.
3.2.2 Remark

Consider a fuzzy preference structure on the set $X$ of alternatives, without incomparability (c.f. section-2.4).

Then, for any $x, y \in X$,

- either $R(x, y) < R(y, x)$,
- or $R(x, y) = R(y, x)$,
- or $R(x, y) > R(y, x)$.

We write $x \geq y$ if

$$R(x, y) > R(y, x), \text{ or } R(x, y) = R(y, x).$$

Then, ' $\geq$ ' is a partial order in $X$.

Let $e \in X$ be such that $R(x, e) \geq R(e, x)$, for every $x \in X$.

i.e., $x \geq e$, for every $x \in X$.

Let $y \in X$. Then,

$$y e y = y (e y)$$

$$= y y$$

$$= y$$
Therefore, $y$ is a regular element of $X$. Since $y$ is arbitrary, every element of $X$ is regular and hence $X$ is a regular semigroup.

For any $y \in X$, let $y' = e y e$

Then,

$$y' y' y = y' (e y e) y$$

$$= (y e) y e y$$

$$= yy e y$$

$$= (yy) e y$$

$$= y e y$$

$$= (ye) y$$

$$= yy$$

$$= y$$

But,

$$y' y y = \begin{cases} 
  y, & \text{if } R(y, e) > R(e, y) \\
  e, & \text{if } R(y, e) = R(e, y) 
\end{cases}$$
Hence elements of $X$ have only partial inverses. Since each element of $X$ is an idempotent, $X$ is a band. Clearly $X$ is not commutative, since, $x \cdot y \neq y \cdot x$ in general in $X$.

### 3.2.3 Remark

As a convention, for $x, y, z \in X$, we take

$$R(x, z) \geq R(x, y), \text{ if } x \geq y \geq z.$$

i.e., the degree by which the alternative ‘$x$ is at least as good as $z$’ is greater than the degree by which ‘$x$ is at least as good as $y$’.

### 3.3 The Indifference Relation on $X$

Consider a fuzzy preference structure $(P, I, J)$ on the set $X$ of alternatives.

For any $(a, b) \in X^2$, we have defined the fuzzy indifference relation $I$ on $X$ as

$$I(a, b) = \min\{ R(a, b), R(b, a) \}$$

$I$ is a similarity relation on $X$ if it is derived from a reflexive and transitive fuzzy weak preference relation (c.f. Theorem-2.3.6 and Definition-1.4.2).

For each $a \in X$, define a fuzzy subset $I_a$ of $X$ such that

$$I_a(x) = I(a, x), \text{ for every } x \in X.$$

### 3.3.1 Lemma

Let $a, b \in X$. Then, $I_a = I_b$ if and only if $I(a, b) = 1$.
Proof:

Let $I_a = I_b$.

Then,

$$I(a, b) = I_{a}(b)$$

$$= I_b(b)$$

$$= I(b, b) = 1, \text{ by the reflexivity of } I.$$

Conversely, assume that $I(a, b) = 1$.

For any $x \in X$,

$$I_a(x) = I(a, x)$$

$$\geq \min \{ I(a, b), I(b, x) \}, \text{ by the transitivity of } I.$$

i.e.,

$$I_a(x) = \min \{ 1, I(b, x) \}$$

$$= I(b, x)$$

$$= I_b(x)$$

Hence $I_a \supseteq I_b$.

By symmetry of $I$ we also have $I_b \supseteq I_a$.

$\therefore \quad I_a = I_b$
3.3.2 Note

The fuzzy subset $I_a$ of $X$ defined by,

$$I_a(x) = I(a, x),$$

for every $x \in X$ is called the fuzzy similarity class of $I$ containing $a \in X$.

3.3.3 Proposition

The indifference relation $I$ defined on the semigroup $X$ of alternatives is a fuzzy congruence relation on $X$.

Proof:

Let $x, y, z \in X$.

Consider the following cases.

Case-1 $x \leq y \leq z$ or $y \leq x \leq z$

Then, $I(x, y) \leq I$

$$= I(z, z)$$

$$= I(xz, yz).$$

Case-2 $z \leq x \leq y$ or $z \leq y \leq x$

Then, $I(x, y) = I(xz, yz)$. 
Case-3 \[ x \leq z \leq y \]

Then, \[ I(x, y) = R(x, y) \]

\[ \leq R(z, y), \text{ by Remark-3.2.2} \]

\[ = \min \{ R(z, y), R(y, z) \} \]

\[ = I(z, y) \]

\[ = I(xz, yz). \]

Case-4 \[ y \leq z \leq x \]

Then, \[ I(x, y) = R(y, x) \leq R(z, x) \]

\[ \therefore I(x, y) \leq \min \{ R(z, x), R(x, z) \} \]

\[ \leq I(x, z) \]

\[ \leq I(xz, yz). \]

Thus for any \( x, y, z \in X, \quad I(x, y) \leq I(xz, yz). \)

Hence \( I \) is fuzzy right compatible.

Similarly we can prove that \( I \) is fuzzy left compatible.

As \( I \) is a similarity relation on \( X \), this proves that \( I \) is a congruence relation on \( X \).
3.3.4 Remark

Since I is a fuzzy congruence relation on $X$, the fuzzy subset $I_a$ of $X$ is a fuzzy congruence class if I is containing $a \in X$.

3.4 Quotient Semigroup of $X$ induced by I

Let $X$ be the semigroup of alternatives and I be the fuzzy congruence relation of indifference on $X$.

We put $\frac{X}{I} = \{ I_a : a \in X \}$.

3.4.1 Lemma

The operation `*' defined by, $I_a * I_b = I_{a \cdot b}$ is a binary operation in $\frac{X}{I}$, where $a, b \in X$, and $I_a$ and $I_b$ are fuzzy congruence classes of I containing $a$ and $b$ respectively.

Proof:

Let $x \in X$.

Put $X' = \{ \ yb : y \in X, \ yz = x \}$.

Then, $I_a * I_b (x) = \sup_{x=yz} \{ \min I(a, y), I(b, z) \}$

$\leq \sup_{x=yz} \{ \min I(a \cdot b, yb), I(yb, yz) \}$, since I is fuzzy compatible
i.e., \( I_a \ast I_b (x) = \text{Sup} \{ \text{min} \{ I(a, b, t), I(t, x) \} \} \leq \text{Sup} \{ \text{min} \{ I(a, b, t), I(t, x) \} \} \leq I(a \ast b, x) \), by the transitivity of \( \ast \)

\[ = I_{a \ast b}(x) \]

i.e., \( I_a \ast I_b \subseteq I_{a \ast b} \).

Hence the operation \( \ast \) defined by, \( I_a \ast I_b = I_{a \ast b} \) is a binary operation in \( X_f \).

3.4.2 Lemma

The binary operation \( \ast \) defined in \( X_f \) is well defined.

Proof:

Suppose that \( I_a = I_b \) and \( I_c = I_d \), for \( a, b, c, d \in X \).

By Lemma-3.3.1, \( I(a, b) = 1 \) and \( I(c, d) = 1 \).

By the transitivity of \( \ast \),

\[ I(a \ast c, b \ast d) \geq \text{min} \{ I(a \ast c, b \ast c), I(b \ast c, b \ast d) \} \geq \text{min} \{ I(a, b) \ast I(c, d) \}, \text{ as } \ast \text{ is fuzzy compatible} \]

\[ = \text{min} \{ I, I \} = 1. \]
i.e., \[ I(ac, bd) = 1 \]

i.e., \[ I_{ac} = I_{bd} \]

i.e., \[ I_{a} * I_{c} = I_{b} * I_{d} \]

Hence the binary operation '•' defined in \( X/1 \) is well defined.

### 3.4.3 Theorem

For a semigroup of alternatives \( X \), the set \( X/1 \) is the quotient semigroup induced by the fuzzy congruence relation of indifference \( I \) on \( X \).

**Proof:**

\[ X/1 = \{ I_{a} : a \in X \}, \text{ where } I_{a}(x) = I(a, x), \]

for every \( x \in X \).

Define '•' on \( X/1 \) by

\[ I_{a} * I_{b} = I_{ab}, \text{ where } a, b \in X. \]

The operation is clearly associative.

Then by Lemma-3.4.1 and by Lemma-3.4.2, \( X/1 \) is a semigroup and is the quotient semigroup of \( X \) induced by \( I \).
3.4.4 Theorem

If $I$ is the fuzzy congruence relation of indifference on the semigroup $X$ of alternatives, then,

$$I' = \{ (a, b) \in X \times X : I(a, b) = 1 \}$$

is a congruence relation on $X$.

Proof:

Since $I$ is reflexive and symmetric so is $I'$.

For transitivity, Let $(a, b), (b, c) \in I'$.

Then, $I(a, b) = 1$ and $I(b, c) = 1$.

By the transitivity of $I$,

$$I(a, c) \geq \min \{ I(a, b), I(b, c) \}$$

$$= \min \{ 1, 1 \} = 1$$

i.e., $I(a, c) = 1$

$\therefore (a, c) \in I'$.

Hence $I'$ is transitive and is an equivalence relation on $X$.

Let $(a, b) \in I'$ and $x \in X$.

Then, $I(a, b) = 1$. 
By the fuzzy compatibility of \( I \),

\[
I(a, b) \geq I(a, b) = 1
\]

i.e., \( I(a, b) = 1 \)

Therefore, \( (a, b) \in I' \).

Similarly, we get \( (a, b) \in I' \). Thus \( I' \) is a congruence relation on \( X \).

### 3.5 Homomorphism

Let \( X \) be the semigroup of alternatives. Let \( e \in X \) is such that

\[
R(a, e) \geq R(e, a), \quad \text{for every } a \in X.
\]

Consider the unit interval \([0,1]\). Define an operation \('V'\) on \([0,1]\) by

\[
t_1 \ V \ t_2 = \max \{ t_1, t_2 \}, \quad t_1, t_2 \in [0,1].
\]

The operation, \('V'\) is an associative, binary operation on \([0,1]\).

Let \( f : X \to [0,1] \) defined by \( f(a) = R(a, e) \), \( a \in X \), be a homomorphism of \( X \) to \([0,1]\). Then, the relation, \( k(f) \), defined by

\[
k(f) = \{(a, b) \in X \times X : f(a) = f(b)\}
\]

is a congruence relation on \( X \).

#### 3.5.1 Remark

By the Lemma-3.1.9, the characteristic function of \( k(f) \) is a fuzzy congruence on \( X \). We denote it by \( k(f) \), and call it the fuzzy kernel, given by,
\[
k(f)(a,b) = \begin{cases} 1, & \text{if } f(a) = f(b) \\ 0, & \text{otherwise} \end{cases}
\]

3.5.2 Lemma

The mapping \( I^# : X \to X/I \), defined by

\[
I^#(a) = I_a : a \in X, \quad \text{is a homomorphism.}
\]

Proof:

Let \( a, b \in X \).

\[
I^#(a \cdot b) = I_{a \cdot b} = I_a \cdot I_b = I^#(a) \cdot I^#(b)
\]

\( \therefore I^# \) is a homomorphism.

Next we give the Homomorphism Theorem.

3.5.3 Theorem

Define, \( f : X \to [0,1] \), by \( f(a) = R(a,e) \), \( a \in X \).

Let \( e \in X \) be such that, \( R(x,e) \geq R(e,x) \), for every \( x \in X \). Then the fuzzy
relation \( k(f) \) is a fuzzy congruence on \( X \), and there is a monomorphism,

\[
g : X/k(f) \to [0,1], \text{ such that the following diagram (Diagram-3.1) commutes.}
\]

![Diagram-3.1](attachment:image.png)

Proof:

The first part of the theorem is already proved.

Define, \( g : X/k(f) \to [0,1] \), by

\[
g\{ [k(f)]_a \} = f(a) = R(a, e).
\]

i. \( g \) is well-defined

Let \( [k(f)]_a = [k(f)]_b \)

Then, by Lemma-3.3.1,

\[
k(f)(a, b) = 1
\]
This implies, \( f(a) = f(b) \)

\[ \therefore g \{ [k(f)]_a \} = g \{ [k(f)]_b \}. \]

ii. **\( g \) is one-one**

Let \( f(a) = f(b) \).

Then, \( (a, b) \in k(f) \), which

\[ \Rightarrow k(f)(a, b) = 1 \]

\[ \Rightarrow [k(f)]_a = [k(f)]_b. \]

iii. **\( g \) is a homomorphism**

Let \( a, b \in X \).

\[ g \{ [k(f)]_a \ast [k(f)]_b \} \]

\[ = g \{ [k(f)]_{a \ast b} \} \]

\[ = f(a \ast b) \]

\[ = f(a) \ast f(b), \]

since \( f \) is a homomorphism.
i.e., \( g\{[k(f)]_a \ast [k(f)]_b \} = g\{[k(f)]_a \} \lor g\{[k(f)]_b \} \).

iv. the diagram commutes

\[
g\{[k(f)]^*\}(a) = g\{[k(f)]^*(a)\}
\]

\[
= g\{[k(f)]_a\} = f(a).
\]

Hence the theorem.

3.5.4 Note

So far we have discussed about fuzzy preference relations and their behavior in some spaces. As we know, preference leads us to choice. The next chapter deals with choice functions that are based on fuzzy preferences.