CHAPTER 7

FUZZY POVERTY INDEX

7.0 Introduction

Richard\textsuperscript{[62]} defined poverty as a condition of having insufficient funds to maintain an acceptable standard of living. The fact that being poor is different from being non-poor, and that this is associated with discrete changes in consumer behavior and possibly, utility. The extent of poverty is described by poverty indices.

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of individuals (society) in a population. The conventional way of constructing a poverty index is by identifying the poor among $S$ and defining a real valued function which enables us to determine whether an individual $x_i \in S$ is in the class of poor. This membership depends on some factors like poverty line fixed in $S$.

The delineating of the set of poor in terms of the classical (exact) measures as characteristic functions is rather unrealistic as the concept of poor is non-exact. That is, the boundary of the set of poor with regard to $S$ is rather sharp.

* Some of the ideas presented in this chapter have been used in a paper communicated to the ‘Journal of Fuzzy Mathematics’, IFMI, California, U.S.A.
In [62] an attempt to attach a poverty index for each \( x_i \in S \), instead of fixing an arbitrary poverty line was made. Zadeh's fuzzy mathematical methods seem to be more appropriate for describing poverty and defining poverty indices.

7.1 Preliminaries

There are several poverty indices available. The Head Count Ratio \( H \), the Income Gap Ratio \( I \), Sen's poverty index \( P \), Atkinson's Poverty index \( P(z) \), the FGT measures \( P_\alpha \) are popular among them. They are presented below\[^{14,77,78}\].

7.1.1 Definition

Let \( S = \{ x_1, x_2, \ldots, x_n \} \) be a society with \( n \) individuals. Let \( z \) be the level of income at which poverty begins (w.r.t a pre-determined poverty line). Let \( y_i : 1 \leq i \leq n \) be the income of the person \( x_i \). Let \( S(z) = \{ x_i : y_i < z \} \).

The poverty index \( H \), known as the 'Head Count Ratio' is given by,

\[
H = \frac{q}{n}
\]

where \( q \) is the number of elements in \( S(z) \).

7.1.2 Definition

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Let \( y_i : 1 \leq i \leq n \) be the income of the person \( x_i \). Let \( S(z) = \{ x_i : y_i < z \} \).
The index \( I \), which is called the ‘Income Gap Ratio’ is given by the formula,

\[
I = \sum_{i=1}^{g} \frac{g_i}{qz}
\]

where \( g_i = z - y_i \) is the income gap.

Sen\textsuperscript{[7]} had modified \( I \) using a normalized weighted sum of the gaps \( g_i \).

7.1.3 Definition

Sen’s poverty index \( P \) is given by,

\[
P = H [ I + (1 - I) G]
\]

where \( H \) is the head count ratio, \( I \) is the income gap ratio and \( G \) is the Gini coefficient of the income distribution of the poor.

Atkinson\textsuperscript{[3]} has pointed out that one approach to formulate a poverty index is to write it as an arithmetical average of the ‘deprivation function’ of all persons in a society.

7.1.4 Definition

Atkinson’s index \( P(z) \) is defined as,

\[
P(z) = \frac{\sum_{i=1}^{\Pi} \Pi(y_i)}{n}
\]

where \( \Pi(y_i) \) is the deprivation function of \( x_i \in S \) and \( \{ y_i : 1 \leq i \leq n \} \) is an income configuration of the society.
Foster, Greer and Thorbecke\textsuperscript{36} presented an entire class of relative poverty indices, popularly known as FGT-measures.

### 7.1.5 Definition

The FGT measures are parameterized in terms of the quantity $\alpha$ which can be interpreted as a measure of 'poverty aversion' given by,

$$p_a = \frac{\sum_{i=1}^{n} \Phi\left(\frac{y_i}{z}\right)}{n},$$

where $\Phi\left(\frac{y_i}{z}\right) = \left(\frac{z-y_i}{z}\right)^{\alpha}; \alpha \geq 0$

Several criteria have been proposed as desirable properties for a poverty measure to satisfy. They are Focus Axiom, Monotonicity Axiom, Symmetry, Transfer Axiom, Subgroup Consistency, Decomposability, Transfer Sensitivity etc\textsuperscript{1171}.

Let $\pi$ be any index of poverty w.r.t a poverty line $z$. Let $\pi^1$ be its value for an income configuration $y^1 = (y_1^1, y_2^1, ..., y_n^1)$ of a society $S$ and let $\pi^2$ be that of another income configuration $y^2 = (y_1^2, y_2^2, ..., y_n^2)$ of $S$. 


7.1.6 Focus Axiom

The axiom states that a poverty measure should be insensitive to an increase in the income of a non-poor person.

\[ g^1 = g^2, \text{ irrespective of } y^2_i < \text{ or } = \text{ or } > y^1_i; \forall i \text{ such that } y_i \geq z. \]

7.1.7 Monotonicity Axiom

It says that, other things being equal, a reduction in a poor person's income should increase the value of the poverty measure.

\[ g^1 < g^2, \text{ if } y^2_i < y^1_i; \text{ for some } i \text{ such that } y_i < z. \]

7.1.8 Weak Transfer Axiom

A regressive transfer of income from a poor person to a richer poor person should increase the value of the poverty index, provided the beneficiary of the transfer continues to remain poor after the transfer.

\[ g^1 < g^2 \text{ if } y^2_i = y^1_i - d, y^2_j = y^1_j + d; d > 0, \text{ with } y^1_i < y^1_j < z \text{ and } y^2_j < z. \]
7.1.9 Transfer Axiom

Other things being equal, a transfer of income from a poor person to a richer poor person should raise the value of the poverty index.

i.e., $\Psi^1 < \Psi^2$ if $y_i^2 = y_i^1 - d$, $y_j^2 = y_j^1 + d$; $d > 0$, with $y_i^1 < y_j^1 < z$.

7.1.10 Symmetry

Value of the poverty index should be invariant w. r. t a permutation of incomes across individuals.

7.1.11 Transfer Sensitivity

The increase in poverty arising from a regressive transfer between two poor persons, a fixed income apart should be greater the poorer is the pair of persons involved in the transfer.

Let $\Psi^3$ is also defined correspondingly.

Then, $\Psi^2 > \Psi^3$ if $y_i^2 = y_i^1 - d$, $y_j^2 = y_j^1 + d$; $d > 0$, with $y_i^1 < y_j^1 < z$ and $y_k^3 = y_k^1 - d$,

$y_i^3 = y_i^1 + d$, with $y_k^1 < y_i^1 < z$,

provided $y_j^1 - y_i^1 = y_l^1 - y_k^1$ and $y_i^1 < y_k^1$. 
7.1.12 Subgroup Consistency

Overall poverty should increase when, ceteris paribus, poverty in any subgroup increases.

i.e., let $S$ be divided into, say $m$ classes each with indices, $S_i$: $i = 1, 2, ..., m$. Let $S^1$ be the index of $S$ corresponds to a classification with indices $S_i^1$ and $S^2$ be that corresponds to $S_i^2$.

Then $S^1 < S^2$ if $S_i^2 > S_i^1$ for some $i \in \{1, 2, ..., m\}$.

7.1.13 Decomposability

The poverty index is amenable to being expressed as a weighted sum of subgroup poverty indices, the weights being the subgroup population shares.

i.e., $S = \sum_{i=1}^{m} w_i S_i$, where $w_i$ is the weight given to the $i^{th}$ class.

7.1.14 Remark

The Focus Axiom is satisfied by all of the indices that we have discussed earlier. The indices $P$ and $I$ satisfy monotonicity while $H$ is not. The Weak Transfer Axiom is satisfied by $P$ (not necessarily the Transfer Axiom), but not by $I$ or $H$. Even though $P$ is symmetric, it fails to satisfy Subgroup Consistency and therefore Decomposability. (Head Count Ratio satisfies Decomposability). $P_o$ satisfies almost
all the properties of \( P \) and in addition it satisfies Transfer Sensitivity and Decomposability. Kakwani\(^{[37]}\) has offered another index as a generalization of Sen's index which satisfies another aspect of Transfer Sensitivity.

Construction of a Poverty Index with all the desirable qualities is not feasible, however, the use of fuzzy notion can help us significantly in the formation of a good poverty index. The fuzzy set method can be used both in the identification and in the aggregation of poverty.

### 7.2 Fuzzy Notion of Poverty Measure

The fuzzy notion can be introduced in different ways. Two of them are discussed below.

#### 7.2.1 Method-1

In this method a pre-determined poverty line is not considered. Instead of that, this method gives grades to each \( i \in S \), using a membership function defined as follows:

#### 7.2.2 Definition

Let \( S = \{1,2,\ldots,n\} \) be a society with \( n \) individuals. Let \( y_i : 1 \leq i \leq n \), be the income of the person \( i \). Let \( y_i \leq y_{i} ; \ \forall i \in S \). A membership function,
\[ P_y : S \rightarrow [0,1] \] is defined by,

\[ P_y(i) = \frac{y_i - \bar{y}}{\bar{y}}. \]

\( P_y(i) \) is the 'grade' or 'intensity' of poverty of \( i \in S \).

This can be characterized in another way also. The following definition gives the characterization.

7.2.3 Definition

Let \( k = P_y(i) \) such that \( P_y(i) \leq P_y(j) \), \( \forall j \in S \). The real number \( 1-k \) is defined as the poverty span of the society.

As the 'poverty span' increases the inequality in the income distribution also increases. If it approaches 1, we can recognize some \( i \in S \) with acute poverty. So any antipoverty program for the society should aim at the minimization of the poverty span.

Method-1 has its own merits and limitations. Merit is mainly due to the fact that it does not put people in the society into two water-tight compartments. Since each person in the society has a grade of poverty with respect to the society it is easy to set up any benefitting program from the side of the government to uplift the poor.
This method has some limitations as well. It does not give a clear picture about the poverty of the society as a whole. The method fails to a certain extent if $y_i$ happens to be an income with which a person cannot achieve an acceptable standard of living.

7.2.4 Method - 2

In this method it is assumed that there is an accepted poverty line, say $z$. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a society with $n$ individuals. Let $y_i : 1 \leq i \leq n$ be the income of the person $x_i$. Here we take into consideration only those people whose income lies below the accepted poverty line and we define a Fuzzy Poverty Measure as a subset of $S(z)$, where $S(z) = \{x_i : y_i < z\}$.

7.2.5 Definition

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a society with $n$ individuals. Let $z$ be the level of income at which poverty begins (w.r.t a pre-determined poverty line). Let $y_i : 1 \leq i \leq n$ be the income of the person $x_i$. Let $S(z) = \{x_i : y_i < z\}$. The Fuzzy Poverty Measure is a fuzzy set with membership function,

$$P_y : S(z) \rightarrow [0, 1] , \text{ defined as}$$

$$P_y(i) = \frac{z - y_i}{z}.$$
The fuzzy poverty measure \( P_i \) gives a clear picture of the poverty of the individual \( i \). It measures the poverty of each individual and of the society as well and hence we can make use of the measure \( P_y \) to construct a poverty index, let us call it by the name 'Fuzzy Poverty Index'.

### 7.3 Fuzzy Poverty Index

#### 7.3.1 Definition

Let \( S = \{ x_1, x_2, \ldots, x_n \} \) be a society with \( n \) individuals. Let \( z \) be the level of income at which poverty begins (w.r.t a pre-determined poverty line).

Let \( y_i : 1 \leq i \leq n \), be the income of the person \( x_i \). Let \( S(z) = \{ x_i : y_i < z \} \).

The Fuzzy Poverty Index (FPI) is defined as,

\[
P_F = \frac{1}{q} \sum_{i=1}^{q} [P_y(i)],
\]

where \( q = \) number elements in the set \( S(z) \).

#### 7.3.2 Remark

\( P_F \) always lies in the closed interval \([0,1]\). If every one have income zero (theoretically assumed, even though it is not possible!), then \( P_F = 1 \). If nobody in the society is below the poverty line then \( P_F = 0 \).

Now we check how much this index fulfills the desirability criteria.
7.3.3 Proposition

The fuzzy poverty index $P_F$ satisfies the following axioms.

(a) Focus Axiom

(b) Monotonicity Axiom

(c) Weak Transfer Axiom

(d) Transfer Axiom

(e) Symmetry

(f) Subgroup Consistency and

(g) Decomposability.

Proof:

Any poverty index can be stated as a function of the income vector $y$, the poverty line $z$, the number of poor people $q$ and the total number of people in the society $n$.

\[ P_F = P_F(y_1, y_2, \ldots, y_n, \ldots, y_q, \ldots, y_n; z) \]

(a) Let $P'_F = P_F(y_1', y_2', \ldots, y_i', \ldots, y_q', \ldots, y_n'; z)$,

where $y_i = y_i'$ for $y_i < z$. 
Then $P_F = P_F'$, irrespective of $y_i >$ or $= or < y_i'$, for $y_i \geq z$.

Hence $P_F$ satisfies Focus axiom.

(b) Let $P_F' = P_F(y_1, y_2, \ldots, y_i', \ldots, y_q, \ldots, y_n; z)$, where $y_i' < y_i$.

Then $P_F < P_F'$, since $z - y_i < z - y_i'$.

Thus $P_F$ satisfies Monotonicity Axiom.

(c) Let $P_F' = P_F(y_1, y_2, \ldots, y_i', \ldots, y_q, \ldots, y_n; z)$,

where $y_i' = y_i - d; d > 0$ and $y_j' = y_j + d < z$.

Then $P_F' = \frac{1}{q} \left[ \left( \frac{z - y_1}{z} \right)^2 + \ldots + \left( \frac{z - y_i'}{z} \right)^2 + \ldots + \left( \frac{z - y_q}{z} \right)^2 \right]$

$= P_F + \left( \frac{d}{z} \right)^2 > P_F$

Hence $P_F$ satisfies Weak Transfer Axiom.

(d) Let $P_F' = P_F(y_1, y_2, \ldots, y_i', \ldots, y_j', \ldots, y_q, \ldots, y_n; z)$,

where $y_i' = y_i - d; d > 0$ and $y_j' = y_j + d \geq z$. 
\[ P_F = \frac{1}{q} \left[ \left( \frac{z-y_1}{z} \right)^2 + \cdots + \left( \frac{z-y_t}{z} \right)^2 + \cdots + \left( \frac{z-y_t}{z} \right)^2 \right] \]

\[ = \frac{1}{q} \left[ K + \left( \frac{z-y_j}{z} \right)^2 \right], \quad \text{where } K > 0, \text{ and} \]

\[ P_{F'} = \frac{1}{q-1} \left[ K + \left( \frac{d}{z} \right)^2 + 2d \left( \frac{z-y_j}{z} \right) \right] \]

Then, \[ P_{F'} - P_F = \frac{1}{q(q-1)} \left[ K + q \left( \frac{d}{z} \right)^2 + 2qd \left( \frac{z-y_j}{z} \right) - (q-1) \left( \frac{z-y_j}{z} \right)^2 \right] \]

\[ : P_{F'} - P_F > 0 \text{ since } \frac{z-y_j}{z} > \frac{z-y_j}{z} \text{ and } \frac{z-y_j}{z} \in [0,1]. \]

i.e., \[ P_{F'} > P_F, \] which shows that \( P_F \) satisfies the Transfer Axiom.

(e) Since \( P_F \) is invariant w.r.t. a permutation of incomes across individuals, it is clearly symmetric.

(f) Suppose that the individuals in \( S \) are classified into \( m \) subgroups with income vectors, say, \( y^1, y^2, \ldots, y^m \). Let the poverty index of \( S \) be \( P_F \) and those of the subgroups be \( P_{F1}, P_{F2}, \ldots, P_{Fm} \) respectively. Let the poverty in any of the subgroup increases. That is, let us assume that, \( P_{F1} \) has increased to \( P_{F1} + d, \quad d > 0 \) and \( P_{F'} \) be the new poverty index for \( S \). This implies that for some \( i \in y^1 \), the gap
$z - y_i$ has increased. Since $y^1 \subset S$, this implies $P_{F'} > P_F$. Hence $P_F$ is Subgroup Consistent.

\[(g)\] As in \((f)\) let $P_{F_1}, P_{F_2}, \ldots, P_{F_m}$ be the indices of the subgroup and let $n_i = \text{the number of individuals in the } i^{th} \text{ subgroup. Then } P_F \text{ can be additively decomposed as,}

$$P_F = \sum_{i=1}^{m} w_i P_{F_i}, \quad \text{where} \quad w_i = \frac{n_i}{q}.$$  

This proves that $P_F$ is Decomposable.

### 7.4 Modification of $P_F$

If we think poverty as the lack of welfare instead of lack of income, one may construct Welfare-Poverty measures rather than Income-Poverty measures. Here using the individual welfare measure FWI, which was defined in the previous chapter we make the modification (c.f. Definition-6.4.1).

The modification is done as follows:

#### 7.4.1 Definition

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a society with $n$ individuals. Let $z$ be the level of income at which poverty begins (w.r.t a pre-determined poverty line).
Let $y_i : 1 \leq i \leq n$, be the income of the person $x_i$. Let $S(z) = \{ x_i : y_i < z \}$.

The modified Fuzzy Poverty Measure $\tilde{P}_w$ is defined as a fuzzy subset of $S(z)$ as,

$$\tilde{P}_w : S(z) \rightarrow [0,1] ; \text{ defined by}$$

$$\tilde{P}_w(i) = 1 - \omega_i$$

The modified version of Fuzzy Poverty Index is defined as,

$$\tilde{P}_F = \frac{1}{q} \sum_{i=1}^{q} [\tilde{P}_w(i)]^2$$

### 7.4.2 Remark

This modified formula satisfies all properties of $P_F$ except that of Symmetry.

A summary of the findings is presented in the next chapter.