CHAPTER 6
FUZZY WELFARE MEASURE*

6.0 Introduction

There have been several attempts to make a systematic study of interpersonal comparisons of welfare[11,61]. Paretian Welfare Economics rests on the assumed value judgment, that, if a particular change in the economy leaves at least one individual better off and no individual worse off, social welfare may be said to have increased[66].

Many researchers use current money-income as a proxy for a measure of economic welfare. The traditional money-revenue measure is obviously inadequate. Individuals at the same income level may vary substantially in their ability to consume goods and services. For population groups where earned income is a principal source of well being or where human capital is more important than physical assets held, more attention should be devoted to the specification of the earning component[50]. Income of a person is seemed to be exact. But the welfare that a person gets from the income is non-exact. For example, the welfare of a person having an income of Rs. 1000/-, when gets an additional income of Rs. 100/- is more than that another person of higher income, say Rs. 10,000/-.

* Some of the ideas presented in this chapter have been used in a paper communicated to ‘The Journal of Fuzzy Mathematics’, IFMI, California, U.S.A.
The comparisons of interpersonal welfare are made mainly by algebraic methods. But in many real definite applications there are some difficulties, mainly due to the rigidity of the methods. It is accepted that one's preferences and value judgments are inherently non-exact. Often in comparing the welfare of two persons say, \( i \) and \( j \), we have a mixed or divided feeling: we simultaneously feel that \( i \) is more well off than \( j \) to some extent and that \( j \) is also more well off than \( i \) to a certain extent. So it seems appropriate to model the concept of welfare in terms of Fuzzy Mathematical tool rather than in terms of Classical Mathematics.

6.1 Preliminaries

Sen\(^{[74]}\) has defined human welfare in terms of 'functioning' and 'capabilities'. Functioning denotes 'achievement of different attributes', and capability indicates the 'ability to achieve' these attributes. Towards comparison of individual welfare Sen\(^{[70]}\) put forward the 'Weak Equity Axiom'. Later Hammond\(^{[31]}\) presented the 'Equity Axiom', which is a modified version of the former.

Morris\(^{[31]}\) formed a composite index called the 'Physical Quality of Life Index' (PQLI), to measure the standard of living.

6.1.1 Non-money Income

There are many examples, which demonstrate absence of automatic link between income and human well-being\(^{[47]}\). According to Barter\(^{[8]}\), 'full income' \( Y_F \) of a person consists of the flow of services from all individual wealth.
i.e., \( Y_F = Y_M + Y_N \); where \( Y_M \) denotes the money income and \( Y_N \) denotes all forms of non-money income.

6.1.2 Human Development Index

United Nation’s Development Program (UNDP) in 1990 introduced an index to measure human well-being, called the ‘Human Development Index’ (HDI), which is an attempt to combine aspects related to human well-being with income.\(^{[47]}\). This index is meant for measuring welfare of individual countries. They identified three indicators of welfare of a country, namely, the mortality rate, the life expectancy and the education attainment of members of the country. The index is calculated using the formula,

\[
\text{HDI} = 1 - \frac{1}{3} \sum_{i=1}^{3} \left\{ \frac{\max_j X_{ij} - X_{ij}}{\max_j X_{ij} - \min_j X_{ij}} \right\} ; \quad j = 1, 2, \ldots, n
\]

where,

- \( X_{ij} \) is the value of the \( i^{th} \) indicator for the individual \( j \),
- \( \max_j X_{ij} \) is the maximum value of the \( i^{th} \) indicator in the country, and
- \( \min_j X_{ij} \) is the minimum value of the \( i^{th} \) indicator in the country.

As it takes only three variables as indicators of well-being, the major criticism against HDI is its incapability to express human well-being in its wide sense.
6.1.3 Achievement Index

In 1993, Kakwani[38] has introduced a class of ‘Achievement Indices’ to represent the actual levels of standard of living to measure the well-being of a country, over a period of time. The Achievement Index of well-being for a country ‘j’ at a point of time is given by:

\[
A_j = \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{X_{ij} - X_{imin}}{X_{imax} - X_{imin}} \right\},
\]

where,

- \(X_{ij}\) = the value of the \(i^{th}\) indicator
- \(X_{imin}\) = minimum value of the \(i^{th}\) indicator
- \(X_{imax}\) = maximum value of the \(i^{th}\) indicator, and

\(n = \) number of indicators.

This index is used by many investigators to compare well-being among people in a society as well as different countries under consideration.

6.1.4 Welfare in terms of Utility

Suppose there is a society of individuals, each of whom has a utility function, say \(U_i(\cdot)\), that represents \(i^{'}s\) personal welfare ordering \(R_i\) on the domain \(X\) of social states[32]. A comparison of utility levels between two individuals \(i\) and \(j\) in the
society is a statement such as \( u_i \geq u_j \), where \( u_i \) and \( u_j \) are two particular levels of the utility function \( U_i \) and \( U_j \) for persons \( i \) and \( j \).

Using individual welfare functions, the ordinal Social Welfare Function is defined as, \( W = W( U_1, U_2, \ldots, U_n ) \), where \( U_1, U_2, \ldots, U_n \) represent the level of welfare of each of the \( n \) individuals who collectively comprise the society under consideration. If any change in the allocation of resources increases the welfare of at least one individual without reducing the welfare of any other individual, then this change is treated as improving social welfare\(^{\text{[66, 45]}}\).

In other words, \( W \) should be a monotonically increasing function of any \( U \).

\[
\frac{\partial W}{\partial U_i} > 0 ; \quad 1 \leq i \leq n
\]

6.2 Fuzzy Welfare Function

A new term 'fuzzy income' is introduced in the following definition.

6.2.1 Definition

Let \( S = \{ 1, 2, \ldots, n \} \) be a finite set representing a society of \( n \) individuals and let an \( n \)-vector \( y = ( y_1, y_2, \ldots, y_n ) \) represents the income configuration of the individuals. Let \( l \) be the person in the society with largest income and let \( y_l \) be his (exact) income. Then the Fuzzy Income \( \bar{y}_i \) of any person \( i \) is defined as:
\[
\bar{y}_i = \begin{cases} 
\frac{1}{1+d_i} & ; \ y_i > 0 \\
0 & ; \ y_i = 0
\end{cases}
\]

where \( d_i = y_i - y_t \).

6.2.2 Note

This replacement of exact income by fuzzy income considerably reduces the over-importance of one's money income in measuring his welfare and thereby gives more meaning to the Welfare Measure. Using this notion of fuzzy income of an individual, a Fuzzy Welfare Function is defined which is used to compare the welfare of two individuals.

6.2.3 Definition

Let \( S = \{1, 2, \ldots, n\} \) be a finite set of individuals and let \( \bar{y}_i \) be the fuzzy income of the individual \( i \). Then a function,

\[ W_y : S \times S \rightarrow [0,1] \text{ defined by } \]

\[ W_y (i,j) = \max \{ \bar{y}_i - \bar{y}_j, 0 \} ; \ i, j \in S , \]

is called the Fuzzy Welfare Function on \( S \) and is denoted by FWF.
6.2.4 Remark

\( W \) for any pair \((i, j)\) gives a real number between 0 and 1. If \( W(i, j) \) is strictly greater than zero, \( i \) is more well off than \( j \) and if \( W(i, j) \) is equal to zero, \( i \) is not more well off than \( j \). But the other extremity namely 1 is not attained practically. For any two individuals, \( i, j \in S \), \( W(i, j) \) is regarded as the degree to which \( j \) is not dominated by \( i \) in the sense of welfare, or, equivalently the degree of well being of \( i \) over \( j \).

6.3 Properties of FWF

\( W \) satisfies the properties of irreflexivity, antisymmetry and transitivity.

6.3.1 Theorem

The FWF, \( W \) is irreflexive and antisymmetric.

Proof:

For every \( i \in S \), \( W(i, i) = 0 \).

Hence \( W \) is irreflexive.

Let \( W(i, j) > 0 \). Then from the definition it follows that \( \tilde{y}_i > \tilde{y}_j \).

This implies \( W(j, i) = 0 \).

Hence, \( \min\{ W(i, j), W(j, i) \} = 0 \); for every \( i, j \in S \).

This proves the antisymmetry of \( W \).

Before proving the transitivity, we give two more properties satisfied by \( W \).
6.3.2 Property-1

For any \( i, j \in S \), \( W_{\gamma}(i,j) \) depends solely on \( \tilde{y}_i \) and \( \tilde{y}_j \).

According to property-1, there exists a function,

\[
f : [0,1] \times [0,1] \rightarrow [0,1],
\]

defined by \( f(\tilde{y}_i, \tilde{y}_j) = W_{\gamma}(i,j) \).

6.3.3 Property-2

\( f \) is non-decreasing with respect to the first argument and non-increasing with respect to the second argument.

The following proposition helps us in proving the transitivity of \( W_{\gamma} \).

6.3.4 Proposition

Any FWF satisfying Property-1 and Property-2 is antisymmetric iff

\[
\tilde{y}_i < \tilde{y}_j \implies f(\tilde{y}_i, \tilde{y}_j) = 0, \ \forall \ i, j \in S.
\]

Proof:

Let \( W_{\gamma} \) is antisymmetric and let \( \tilde{y}_i < \tilde{y}_j \).

By Property-2,
\[ f(\bar{y}_i, \bar{y}_j) \leq f(\bar{y}_j, \bar{y}_i) \leq f(\bar{y}_i, \bar{y}_j). \]

If \( W_y \) is antisymmetric,

\[ 0 = \min\{ W_y(i, j), W_y(j, i) \} \]

\[ = \min\{ f(\bar{y}_i, \bar{y}_j), f(\bar{y}_j, \bar{y}_i) \} \]

\[ = f(\bar{y}_i, \bar{y}_j) \]

Conversely,

\[ \min\{ W_y(i, j), W_y(j, i) \} = \min\{ f(\bar{y}_i, \bar{y}_j), f(\bar{y}_j, \bar{y}_i) \} \]

\[ = 0, \]

\[ \therefore \text{ either } \bar{y}_i \leq \bar{y}_j \text{ or } \bar{y}_j \leq \bar{y}_i. \]

6.3.5 Theorem

The fuzzy welfare function \( W_y \) given in Definition-6.2.3 is transitive.

Proof:

We have to prove that for every \( i, j, k \in S, \)

\[ \min\{ W_y(i, j), W_y(j, k) \} \leq W_y(i, k). \]
We may take both \( W_y(i, j) \) and \( W_y(j, k) \) greater than zero. (Otherwise the inequality may get trivially true).

By the above proposition,

\[
W_y(i, j) > 0 \text{ implies } \tilde{y}_i > \tilde{y}_j \quad \text{ and }
\]

\[
W_y(j, k) > 0 \text{ implies } \tilde{y}_j > \tilde{y}_k.
\]

Consider,

\[
\min\{ W_y(i, j), W_y(j, k) \}
\]

\[
= \min\{ f(\tilde{y}_i, \tilde{y}_j), f(\tilde{y}_j, \tilde{y}_k) \}
\]

\[
\leq \min\{ f(\tilde{y}_i, \tilde{y}_j), f(\tilde{y}_i, \tilde{y}_k) \}.
\]

by property-2.

\[
\leq f(\tilde{y}_i, \min(\tilde{y}_j, \tilde{y}_k))
\]

\[
= f(\tilde{y}_i, \tilde{y}_k)
\]

\[
= W_y(i, k)
\]

This proves the transitivity property of \( W_y \).
6.3.6 Note

$W_y$ facilitates comparison of the interpersonal welfare of individuals in a society, in a fuzzy mathematical approach. But, since it had taken only one factor, namely the income of the individual as the parameter to measure the welfare, it is not sufficient to measure the individual's welfare as a whole. So a more general measure is needed. A generalized measure, which takes into account more parameters, is discussed in the next section. This may make the comparison of interpersonal welfare of individuals in a society in a better way.

6.4 Fuzzy Welfare Index

Sen\textsuperscript{[73]} says 'there are of course good resources to think that sometimes a richer person may have lower welfare than a poorer person, e.g., if he is a cripple'. This points out the relevance of taking more parameters along with the income to determine one's welfare. So it is very much needed to determine a Welfare Index of every individual of the society by taking one or more indicators, which contribute to the welfare of an individual along with income. Appropriate weights should be given to each indicator. Any factor which can contribute to the welfare of a person such as income, education, employment, leisure, social status, physical fitness, access to power etc., can be taken as an indicator for calculating the index by attributing appropriate weight for each indicator. The parameters may be either crisp (e.g. exact income) or fuzzy (e.g. social status).
6.4.1 Definition

Let $u$ be any parameter of the welfare of an individual in a society. Let $\alpha$ be any real number such that $0 \leq \alpha < 1$, and let $u_i$ be another real number such that $0 \leq u_i \leq 1$. Define a function,

$$u : S \rightarrow [0,1] \text{ by } u(i) = u_i.$$ 

The Fuzzy Welfare Index (FWI), of the person $i$ in the society $S$ is the real number,

$$\omega_i = (1-\alpha) \bar{y}_i + \alpha u_i$$

6.4.2 Remark

The value of $\omega_i$ lies between 0 and 1. Using this index, the FWF $W_y$ can be modified.

6.4.3 Definition

The modified FWF is defined as,

$$W_y' : S \times S \rightarrow [0,1] \text{ by }$$

$$W_y'(i, j) = \max \{ \omega_i - \omega_j, 0 \}; \quad i, j \in S.$$
6.4.4 Remark

The modified FWF, $W'_y$ takes into account one more parameter $u$ along with the income. More parameters can be included for finding the index. For example, if $x$ is any parameter other than income and $u$, then find another real number $\beta$ such that $0 \leq \beta < 1$ and a real number $x_i$ such that $0 \leq x_i \leq 1$. Define,

$$x : S \rightarrow [0, 1], \text{ by } x(i) = x_i.$$ 

The FWI, $\omega_i$ is then calculated using the formula,

$$\omega_i = (1 - \alpha - \beta) \bar{y}_i + \alpha u_i + \beta x_i.$$ 

Or, in general, if there are $m$ parameters, say $u_1, u_2, ..., u_m$ of an individual's welfare other than his income parameter $y_i$, then we define,

$$u_k : S \rightarrow [0, 1], \text{ by } u_k(i) = u_{ki}, \quad k = 1, 2, ..., m : i = 1, 2, ..., n.$$ 

Then $\omega_i$ is calculated using the formula,

$$\omega_i = (1 - \sum_{k=1}^{m} \alpha_k) \bar{y}_i + \sum_{k=1}^{m} \alpha_k u_{ki},$$

$$k = 1, 2, ..., m : i = 1, 2, ..., n.$$
6.4.5 Remark

Once we have identified the indicators and their respective weights, this modified fuzzy welfare function can measure welfare of individual or country in a better way and comparison of welfare between individuals or countries makes easy.

The next chapter shows how this fuzzy welfare measure can be used in constructing a fuzzy index of poverty.